Problems in Introductory Physics, Instructor's Edition

B. Crowell and B. Shotwell

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1 Measurement

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

1.1 The SI

The Système International (SI) is a system of measurement in which mechanical quantities are expressed in terms of three basic units: the meter (m), the kilogram (kg), and the second (s). Other units can be built out of these. For example, the SI unit to measure the flow of water through a pipe would be kg/s.

To modify the units there is a consistent set of prefixes. The following are common and should be memorized:

prefix		meaning
nano-	n	10^{-9}
micro-	μ	10^{-6}
milli-	m	10^{-3}
kilo-	k	10^{3}
mega-	M	10^{6}

The symbol μ, for micro-, is Greek lowercase mu, which is equivalent to the Latin "m." There is also centi-, 10^{-2}, which is only used in the centimeter.

1.2 Significant figures

The international governing body for football ("soccer" in the US) says the ball should have a circumference of 68 to 70 cm. Taking the middle of this range and dividing by π gives a diameter of approximately 21.96338214668155633610595934540698196 cm. The digits after the first few are completely meaningless. Since the circumference could have varied by about a centimeter in either direction, the diameter is fuzzy by something like a third of a centimeter. We say that the additional, random digits are not *significant figures.* If you write down a number with a lot of gratuitous insignificant figures, it shows a lack of scientific literacy and imples to other people a greater precision than you really have.

As a rule of thumb, the result of a calculation has as many significant figures, or "sig figs," as the least accurate piece of data that went in. In the example with the soccer ball, it didn't do us any good to know π to dozens of digits, because the bottleneck in the precision of the result was the figure for the circumference, which was two sig figs. The result is 22 cm. The rule of thumb works best for multiplication and division.

The numbers 13 and 13.0 mean different things, because the latter implies higher precision. The number 0.0037 is two significant figures, not four, because the zeroes after the decimal place are placeholders. A number like 530 could be either two sig figs or three; if we wanted to remove the ambiguity, we could write it in scientific notation as 5.3×10^2 or 5.30×10^2.

1.3 Proportionalities

Often it is more convenient to reason about the ratios of quantities rather than their actual values. For example, suppose we want to know what happens to the area of a circle when we triple its radius. We know that $A = \pi r^2$, but the factor of π is not of interest here because it's present in both cases, the small circle and the large one. Throwing away the constant of proportionality, we can write $A \propto r^2$, where the proportionality symbol $\propto$, read "is proportional to," says that the left-hand side doesn't necessarily equal the right-hand side, but it does equal the right-hand side multiplied by a constant.

Any proportionality can be interpreted as a statement about ratios. For example, the statement $A \propto r^2$ is exactly equivalent to the statement that $A_1/A_2 = (r_1/r_2)^2$, where the subscripts 1 and 2 refer to any two circles. This in our example, the given information that $r_1/r_2 = 3$ tells us that $A_1/A_2 = 9$.

In geometrical applications, areas are always proportional to the square of the linear dimensions, while volumes go like the cube.

1.4 Estimation

It is useful to be able to make rough estimates, e.g., how many bags of gravel will I need to fill my driveway? Sometimes all we need is an estimate so rough that we only care about getting the result to about the nearest factor of ten, i.e., to within an order of magnitude. For example, anyone with a basic knowledge of US geography can tell that the distance from New Haven to New York is probably something like 100 km, not 10 km or 1000 km. When making estimates of physical quantities, the following guidelines are helpful:

1. Don't even attempt more than one significant figure of precision.
2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them. Mass is often best found by estimating linear dimensions and density.
3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.
4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield 0.01 m^2 of leather, then you have probably made a mistake with conversion factors somewhere.

Problems

1-a1 Convert 134 mg to units of kg, writing your answer in scientific notation.

▷ Solution, p. 231

1-a2 Express each of the following quantities in micrograms:
(a) 10 mg, (b) 10^4 g, (c) 10 kg, (d) 100×10^3 g, (e) 1000 ng. √

1-a3 In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implanted steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas? √

1-d1 The usual definition of the mean (average) of two numbers a and b is $(a+b)/2$. This is called the arithmetic mean. The geometric mean, however, is defined as $(ab)^{1/2}$ (i.e., the square root of ab). For the sake of definiteness, let's say both numbers have units of mass. (a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If a and b both have units of grams, what should we call the units of ab? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as $(ab)^{1/3}$. Is this reasonable?

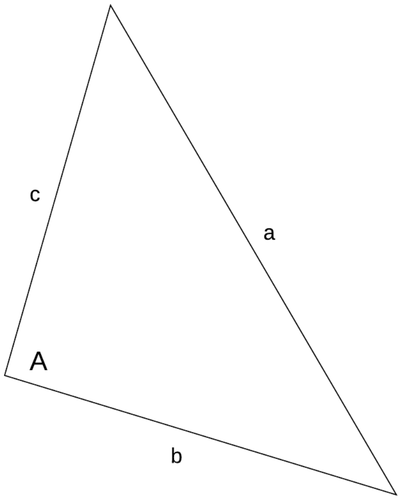
▷ Solution, p. 231

1-d2 (a) Based on the definitions of the sine, cosine, and tangent, what units must they have? (b) A cute formula from trigonometry lets you find any angle of a triangle if you know the lengths of its sides. Using the notation shown in the figure, and letting $s = (a+b+c)/2$ be half the perimeter, we have

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Show that the units of this equation make sense. In other words, check that the units of the right-hand side are the same as your answer to part a of the question.

▷ Solution, p. 231



Problem 1-d2.

1-d3 Jae starts from the formula $V = \frac{1}{3}Ah$ for the volume of a cone, where A is the area of its base, and h is its height. He wants to find an equation that will tell him how tall a conical tent has to be in order to have a certain volume, given its radius. His algebra goes like this:

$$
\begin{aligned}
V &= \frac{1}{3}Ah \\
A &= \pi r^2 \\
V &= \frac{1}{3}\pi r^2 h \\
h &= \frac{\pi r^2}{3V}
\end{aligned}
$$

Use units to check whether the final result makes sense. If it doesn't, use units to locate the line of algebra where the mistake happened.

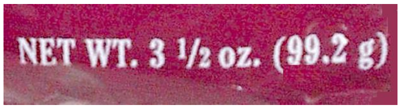
▷ Solution, p. 231

1-d4 The distance to the horizon is given by the expression $\sqrt{2rh}$, where r is the radius of the Earth, and h is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. Don't try to prove the result, just check its units. (For an example of how to do this, see problem 1-d3 on p. 9, which has a solution given in the back of the book.)

1-d5 Let the function x be defined by $x(t) = Ae^{bt}$, where t has units of seconds and x has units of meters. (For $b < 0$, this could be a fairly accurate model of the motion of a bullet shot into a tank of oil.) Show that the Taylor series of this function makes sense if and only if A and b have certain units.

1-g1 In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

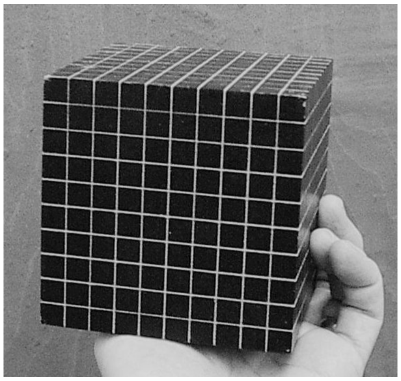
1-g2 The photo shows the corner of a bag of pretzels. What's wrong here?



Problem 1-g2.

1-j1 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?

▷ Solution, p. 231



Problem 1-j1.

1-j2 How many cm^2 is 1 mm^2?

▷ Solution, p. 231

1-j3 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope.

▷ Solution, p. 232

1-j4 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

▷ Solution, p. 232

1-j5 How many cubic inches are there in a cubic foot? The answer is not 12.

✓

1-j6 Assume a dog's brain is twice as great in diameter as a cat's, but each animal's brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

1-k1 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.

▷ Solution, p. 232

1-k2 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD's area is therefore lost?

√

1-k3 The Earth's surface is about 70% water. Mars's diameter is about half the Earth's, but it has no surface water. Compare the land areas of the two planets.

√

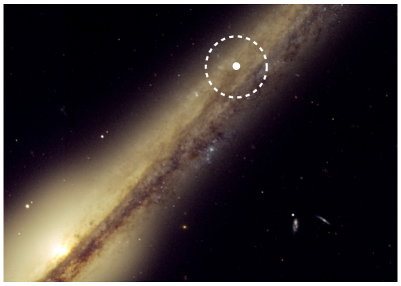
1-k4 At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let's call this fraction f. Mathematicians have proved that the best possible result is $f \approx 0.7405$, which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has $f \approx 0.64$. Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box?

√

Problem 1-k4.

1-k5 Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I've indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don't have similar photos of our own Milky Way galaxy, because we can't see it from the outside.) So far we haven't received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010?

√



Problem 1-k5.

1-k6 X-ray images aren't only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other.

✓

1-m1 A taxon (plural taxa) is a group of living things. For example, *Homo sapiens* and *Homo neanderthalensis* are both taxa — specifically, they are two different species within the genus *Homo*. Surveys by botanists show that the number of plant taxa native to a given contiguous land area A is usually approximately proportional to $A^{1/3}$. (a) There are 70 different species of lupine native to Southern California, which has an area of about $200,000$ km^2. The San Gabriel Mountains cover about $1,600$ km^2. Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with? ✓
(b) What is the interpretation of the fact that the exponent, 1/3, is less than one?

★

1-m2 The population density of Los Angeles is about 4000 people/km^2. That of San Francisco is about 6000 people/km^2. How many times farther away is the average person's nearest neighbor in LA than in San Francisco? The answer is not 1.5.

✓ ★

1-m3 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box?

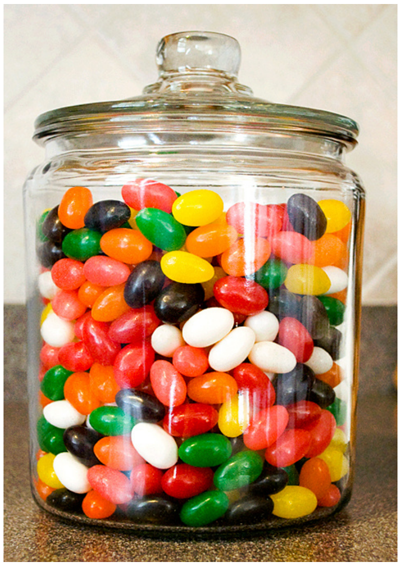
✓ ★

1-p1 Estimate the number of jellybeans in the figure.

▷ Solution, p. 233

1-p2 Suppose you took enough water out of the oceans to reduce sea level by 1 mm, and you took that water and used it to fill up water bottles. Make an order-of-magnitude estimate of how many water bottles could you fill.

1-p3 If you filled up a small classroom with pennies, about much money would be in the room?



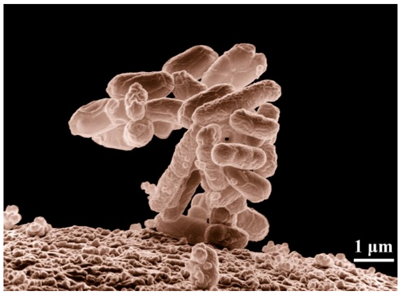
Problem 1-p1.

1-p4 Estimate the mass of one of the hairs in Albert Einstein's moustache, in units of kg.

1-p5 Estimate the number of blades of grass on a football field.

1-p6 Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world's population to live.

1-p7 (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 μm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of 10^{13}. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?



Problem 1-p7.

1-q1 Estimate the number of man-hours required for building the Great Wall of China.

▷ Solution, p. 233 ⋆

1-q2 Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world's waste stockpiles have about 10^3 metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than 2×10^{-13} g/cm^3 of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world's inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world's oceans, thereby becoming mixed uniformly into the world's water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.

⋆

2 Kinematics in one dimension

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

2.1 Velocity

The motion of a particle in one dimension can be described using the function $x(t)$ that gives its position at any time. Its *velocity* is defined by the derivative

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{2.1}$$

From the definition, we see that the SI units of velocity are meters per second, m/s. Positive and negative signs indicate the direction of motion, relative to the direction that is arbitrarily called positive when we pick our coordinate system. In the case of constant velocity, we have

$$v = \frac{\Delta x}{\Delta t}, \tag{2.2}$$

where the notation Δ (Greek uppercase "delta," like Latin "D") means "change in," or "final minus initial." When the velocity is not constant, this equation is false, although the quantity $\Delta x/\Delta t$ can be interpreted as a kind of average velocity.

Velocity can only be defined if we choose some arbitrary reference point that we consider to be at rest. Therefore velocity is relative, not absolute. A person aboard a cruising passenger jet might consider the cabin to be at rest, but someone on the ground might say that the plane was moving very fast — relative to the dirt.

To convert velocities from one *frame of reference* to another, we add a constant. If, for example, v_{AB} is the velocity of A relative to B, then

$$v_{AC} = v_{AB} + v_{BC}. \tag{2.3}$$

The *principle of inertia* states that if an object is not acted on by a force, its velocity remains constant. For example, if a rolling soccer ball slows down, the change in its velocity is not because the ball naturally "wants" to slow down but because of a frictional force that the grass exerts on it.

A frame of reference in which the principle of inertia holds is called an *inertial* frame of reference. The earth's surface defines a very nearly inertial frame of reference, but so does the cabin of a cruising passenger jet. Any frame of reference moving at constant velocity, in a straight line, relative to an inertial frame is also an inertial frame. An example of a noninertial frame of reference is a car in an amusement park ride that maneuvers violently.

2.2 Acceleration

The *acceleration* of a particle is defined as the time derivative of the velocity, or the second derivative of the position with respect to time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \tag{2.4}$$

It measures the rate at which the velocity is changing. Its units are m/s/s, more commonly written as m/s^2.

Unlike velocity, acceleration is not just a matter of opinion. Observers in different inertial frames of reference agree on accelerations. An acceleration is caused by the force that one object exerts on another.

In the case of constant acceleration, simple algebra and calculus give the following relations:

$$a = \frac{\Delta v}{\Delta t} \tag{2.5}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{2.6}$$

$$v_f^2 = v_0^2 + 2a\Delta x, \tag{2.7}$$

where the subscript 0 (read "nought") means initial, or $t = 0$, and f means final.

Free fall

Galileo showed by experiment that when the only force acting on an object is gravity, the object's acceleration has a value that is independent of the object's mass. This is because the greater force of gravity on a heavier object is exactly compensated for by the object's greater *inertia*, meaning its tendency to resist a change in its motion. For example, if you stand up now and drop a coin side by side with your shoe, you should see them hit the ground at almost the same time, despite the huge disparity in mass. The magnitude of the acceleration of falling objects is notated g, and near the earth's surface g is approximately 9.8 m/s^2. This number is a measure of the strength of the earth's gravitational field.

The inclined plane

If an object slides frictionlessly on a ramp that forms an angle θ with the horizontal, then its acceleration equals $g \sin\theta$. This can be shown based on a looser, generalized statement of the principle of inertia, which leads to the conclusion that the gain in speed on a slope depends only on the vertical drop.[1] For $\theta = 90°$, we recover the case of free fall.

Graphs of position, velocity, and acceleration

The motion of an object can be represented visually by a stack of graphs of x versus t, v versus t, and a versus t. Figure **??** shows two examples. The slope of the tangent line at a given point on one graph equals the value of the function at the same time in the graph below.

---

[1] For details of this argument, see Crowell, Mechanics, `lightandmatter.com`, section 3.6.

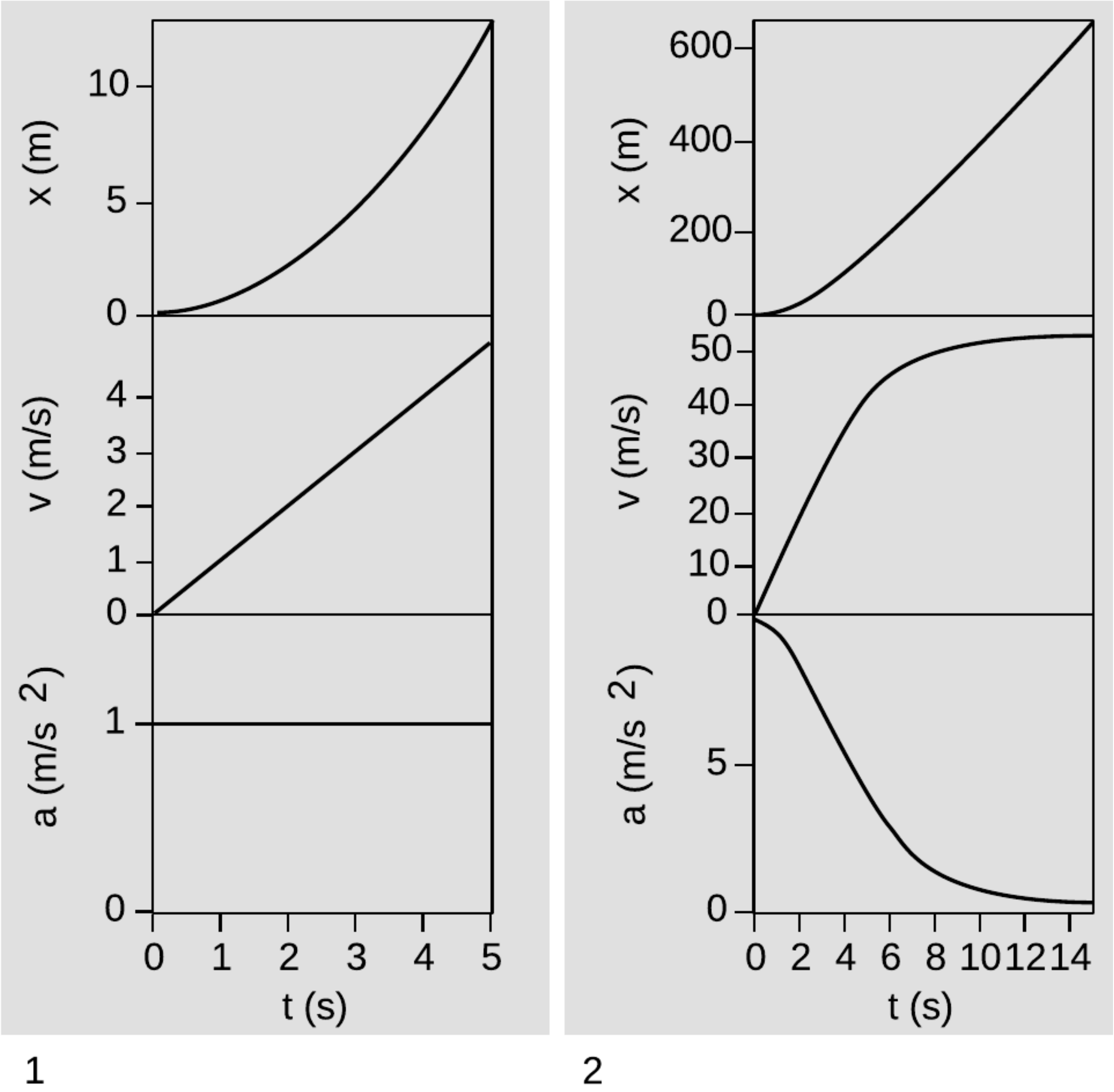


Figure 2.1: 1. Graphs representing the motion of an object moving with a constant acceleration of 1 m/s^2. 2. Graphs for a parachute jumper who initially accelerates at g, but later accelerates more slowly due to air resistance.

Problems

2-a1 You're standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train's velocity? Explain.

2-a2 Interpret the general rule $v_{AB} = -v_{BA}$ in words.

2-a3 Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols, using the notation with two subscripts introduced in section 2.1.

2-a4 Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe an inertial frame of reference in which your car was *speeding up* during that same period of time.

⋆

2-b1 (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.1. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

2-b2 An airport has a moving walkway to help people move across and/or between terminals quickly. Suppose that you're walking north on such a walkway, where the walkway has speed 3.0 m/s relative to the airport, and you walk at a speed of 2.0 m/s. You pass by your friend, who is off the walkway, traveling south at 1.5 m/s.
(a) What is the magnitude of your velocity with respect to the moving walkway? ✓
(b) What is the magnitude of your velocity with respect to the airport? ✓
(c) What is the magnitude of your velocity with respect to your friend? ✓
(d) If it takes you 45 seconds to get across the airport terminal, how long does it take your friend? ✓

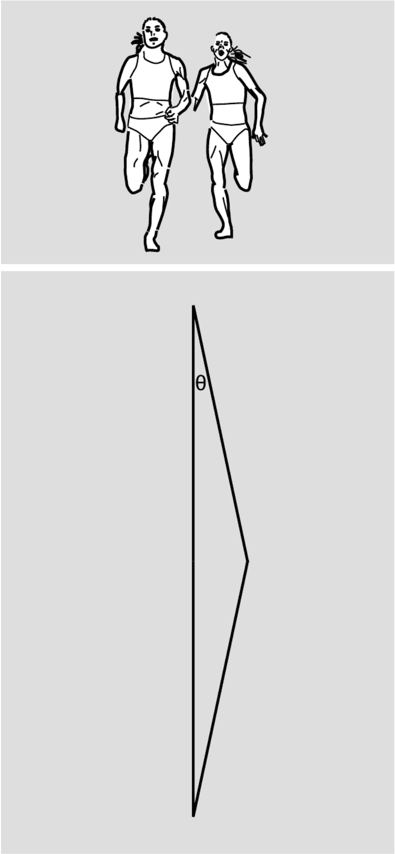
2-b3 On a 20 km bike ride, you ride the first 10 km at an average speed of 8 km/hour. What average speed must you have over the next 10 km if your average speed for the whole ride is to be 12 km/hour? ✓

2-b4 (a) In a race, you run the first half of the distance at speed u, and the second half at speed v. Find the over-all speed, i.e., the total distance divided by the total time. ✓
(b) Check the units of your equation .
(c) Check that your answer makes sense in the case where $u = v$.
(d) Show that the dependence of your result on u and v makes sense. That is, first check whether making u bigger makes the result bigger, or smaller. Then compare this with what you expect physically.

2-b5 An object starts moving at $t = 0$, and its position is given by $x = At^5 - Bt^2$, where t is in seconds and x is in meters. A is a non-zero constant. (a) Infer the units of A and B.
(b) Find the velocity as a function of t. ✓
(c) What is the average velocity from 0 to t as a function of time? ✓
(d) At what time t ($t > 0$) is the velocity at t equal to the average velocity from 0 to t? ✓

2-b6 (a) Let R be the radius of the Earth and T the time (one day) that it takes for one rotation. Find the speed at which a point on the equator moves due to the rotation of the earth. ✓
(b) Check the units of your equation .
(c) Check that your answer to part a makes sense in the case where the Earth stops rotating completely, so that T is infinitely long.
(d) Nairobi, Kenya, is very close to the equator. Plugging in numbers to your answer from part a, find Nairobi's speed in meters per second. See the table in the back of the book for the relevant data. For comparison, the speed of sound is about 340 m/s. ✓

2-c1 In running races at distances of 800 meters and longer, runners do not have their own lanes, so in order to pass, they have to go around their opponents. Suppose we adopt the simplified geometrical model suggested by the figure, in which the two runners take equal times to trace out the sides of an isoceles triangle, deviating from parallelism by the angle θ. The runner going straight runs at speed v, while the one who is passing must run at a greater speed. Let the difference in speeds be Δv.
(a) Find Δv in terms of v and θ. √
(b) Check the units of your equation .
(c) Check that your answer makes sense in the special case where $\theta = 0$, i.e., in the case where the runners are on an extremely long straightaway.
(d) Suppose that $\theta = 1.0$ degrees, which is about the smallest value that will allow a runner to pass in the distance available on the straightaway of a track, and let $v = 7.06$ m/s, which is the women's world record pace at 800 meters. Plug numbers into your equation from part a to determine Δv, and comment on the result. √



Problem 2-c1.

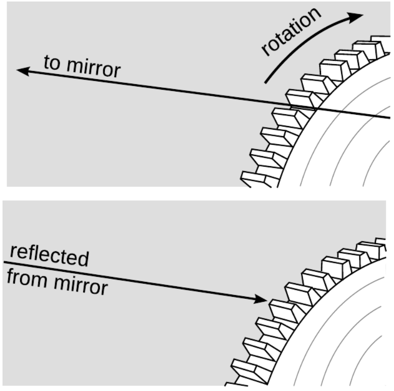
⋆

2-c2 In 1849, Fizeau carried out the first terrestrial measurement of the speed of light; previous measurements by Roemer and Bradley had involved astronomical observation. The figure shows a simplified conceptual representation of Fizeau's experiment. A ray of light from a bright source was directed through the teeth at the edge of a spinning cogwheel. After traveling a distance L, it was reflected from a mirror and returned along the same path. The figure shows the case in which the ray passes between two teeth, but when it returns, the wheel has rotated by half the spacing of the teeth, so that the ray is blocked. When this condition is achieved, the observer looking through the teeth toward the far-off mirror sees it go completely dark. Fizeau adjusted the speed of the wheel to achieve this condition and recorded the rate of rotation to be f rotations per second. Let the number of teeth on the wheel be n.

(a) Find the speed of light c in terms of L, n, and f. √
(b) Check the units of your equation . (Here f's units of rotations per second should be taken as inverse seconds, s^{-1}, since the number of rotations in a second is a unitless count.)
(c) Imagine that you are Fizeau trying to design this experiment. The speed of light is a huge number in ordinary units. Use your equation from part a to determine whether increasing c requires an increase in L, or a decrease. Do the same for n and f. Based on this, decide for each of these variables whether you want a value that is as big as possible, or as small as possible.
(d) Fizeau used $L = 8633$ m, $f = 12.6\ \text{s}^{-1}$, and $n = 720$. Plug in to your equation from part

a and extract the speed of light from his data. ✓

⋆



Problem 2-c2.

2-e1 What is the acceleration of a car that moves at a steady velocity of 100 km/h for 100 seconds? Explain your answer. [Based on a problem by Hewitt.]

2-e2 Alice drops a rock off a cliff. Bubba shoots a gun straight down from the edge of the same cliff. Compare the accelerations of the rock and the bullet while they are in the air on the way down. [Based on a problem by Serway and Faughn.]

2-e3 A toy car is released on one side of a piece of track that is bent into an upright U shape. The car goes back and forth. When the car reaches the limit of its motion on one side, its velocity is zero. Is its acceleration also zero? Explain using a $v-t$ graph. [Based on a problem by Serway and Faughn.]

2-e4 If an object starts accelerating from rest, we have $v^2 = 2a\Delta x$ for its speed after it has traveled a distance Δx. Explain in words why it makes sense that the equation has velocity squared, but distance only to the first power. Don't recapitulate the derivation in the book, or give a justification based on units. The point is to explain what this feature of the equation tells us about the way speed increases as more distance is covered.

2-f1 On New Year's Eve, a stupid person fires a pistol straight up. The bullet leaves the gun at a speed of 100 m/s. How long does it take before the bullet hits the ground?

2-f2 A physics homework question asks, "If you start from rest and accelerate at 1.54 m/s^2 for 3.29 s, how far do you travel by the end of that time?" A student answers as follows:

$$1.54 \times 3.29 = 5.07\ \text{m}$$

His Aunt Wanda is good with numbers, but has never taken physics. She doesn't know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?

2-f3 You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well? ✓

2-f4 Consider the following passage from Alice in Wonderland, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall *never* come to an end? "I wonder how many miles I've fallen by this time?" she said aloud. "I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think" (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a *very* good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)...

Alice doesn't know much physics, but let's try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of 10 m/s^2. This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, g is zero, because the earth is pulling you equally in every direction at once).

√

2-f5 You shove a box with initial velocity 2.0 m/s, and it stops after sliding 1.3 m. What is the magnitude of the deceleration, assuming it is constant?

√

2-f6 You're an astronaut, and you've arrived on planet X, which is airless. You drop a hammer from a height of 1.00 m and find that it takes 350 ms to fall to the ground. What is the acceleration due to gravity on planet X?

√

2-i1 Mr. Whiskers the cat can jump 2.0 meters vertically (undergoing free-fall while in the air). (a) What initial velocity must he have in order to jump this high? √
(b) How long does it take him to reach his maximum height from the moment he leaves the ground? √
(c) From the start of the jump to the time when he lands on the ground again, how long is he in the air?

√

2-i2 A baseball pitcher throws a fastball. Her hand accelerates the ball from rest to 45.0 m/s over a distance 1.5 meters. For the purposes of this problem, we will make the simplifying assumption that this acceleration is constant. (a) What is the ball's acceleration? √
(b) How much time does it take for the pitcher to accelerate the ball? √
(c) If home plate is 18.0 meters away from where the pitcher releases the baseball, how much total time does the baseball take to get there, assuming it moves with constant velocity as soon as it leaves the pitcher's hand? Include both the time required for acceleration and the time the ball spends on the fly.

√

2-i3 The photo shows Apollo 16 astronaut John Young jumping on the moon and saluting at the top of his jump. The video footage of the jump shows him staying aloft for 1.45 seconds. Gravity on the moon is 1/6 as strong as on the earth. Compute the height of the jump.

√

Problem 2-i3.

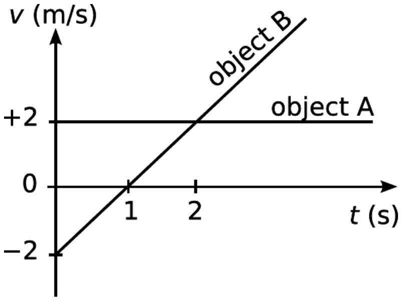
2-i4 Find the error in the following calculation. A student wants to find the distance traveled by a car that accelerates from rest for 5.0 s with an acceleration of 2.0 m/s^2. First he solves $a = \Delta v/\Delta t$ for $\Delta v = 10$ m/s. Then he multiplies to find (10 m/s)(5.0 s) = 50 m. Do not just recalculate the result by a different method; if that was all you did, you'd have no way of knowing which calculation was correct, yours or his.

2-i5 A naughty child drops a golf ball from the roof of your apartment building, and you see it drop past your window. It takes the ball time T to traverse the window's height H. Find the initial speed of the ball when it first came into view.

√

2-i6 Objects A and B, with $v(t)$ graphs shown in the figure, both leave the origin at time $t = 0$ s. When do they cross paths again?

√



Problem 2-i6.

2-i7 An elevator is moving upward at constant speed of 2.50 m/s. A bolt in the elevator's ceiling, 3.00 m above the floor, works loose and falls. (a) How long does it take for the bolt to fall to the floor? $\checkmark$
(b) What is the speed of the bolt just as it hits the floor, according to an observer in the elevator? $\checkmark$
(c) What is the speed of the bolt just as it hits the elevator's floor, according to an observer standing on one of the floor landings of the building? $\checkmark$

2-i8 You're in your Honda, cruising on the freeway at velocity u, when, up ahead at distance L, a Ford pickup truck cuts in front of you while moving at constant velocity $u/2$. Like half the speed that any reasonable person would go! Let positive x be the direction of motion, and let your position be $x = 0$ at $t = 0$. To avoid a collision, you immediately slam on the brakes and start decelerating with acceleration $-a$, where a is a positive constant.
(a) Write an equation for $x_F(t)$, the position of the Ford as a function of time, as they trundle onward obliviously. $\checkmark$
(b) Write an equation for $x_H(t)$, the position of your Honda as a function of time. $\checkmark$
(c) By subtracting one from the other, find an expression for the distance between the two vehicles as a function of time, $d(t)$. $\checkmark$
(d) Find the minimum value of a that avoids a collision. $\checkmark$
(e) Show that your answer to part e has units that make sense.
(f) Show that the dependence of your answer on the variables makes sense physically.

2-i9 You're in your Honda on the freeway traveling behind a Ford pickup truck. The truck is moving at a steady speed of 30.0 m/s, you're speeding at 40.0 m/s, and you're cruising 45 meters behind the Ford. At $t = 0$, the Ford slams on his/her brakes, and decelerates at a rate of 5.0 m/s^2. You don't notice this until $t = 1.0$ s, where you begin decelerating at 10.0 m/s^2. Let positive x be the direction of motion, and let your position be $x = 0$ at $t = 0$. The goal is to find the motion of each vehicle and determine whether there is a collision.
(a) Doing this entire calculation purely numerically would be very cumbersome, and it would be difficult to tell whether you had made mistakes. Translate the given information into algebra symbols, and find an equation for $x_F(t)$, the position of the Ford as a function of time. $\checkmark$
(b) Write a similar symbolic equation for $x_H(t)$ (for $t > 1$ s), the position of the Honda as a function of time. Why isn't this formula valid for $t < 1$ s? $\checkmark$
(c) By subtracting one from the other, find an expression for the distance between the two vehicles as a function of time, $d(t)$ (valid for $t > 1$ s until the truck stops). Does the equation $d(t) = 0$ have any solutions? What does this tell you? $\checkmark$
(d) Because this is a fairly complicated calculation, we will find the result in two different ways and check them against each other. Plug numbers back in to the results of parts a and b, replacing the symbols in the constant coefficients, and graph the two functions using a graphing calculator or an online utility such as desmos.com.
(e) As you should have discovered in parts c and d, the two vehicles do not collide. At what time does the minimum distance occur, and what is that distance?

$\checkmark$

2-i10 You're standing on the roof of your science building, which is 10.0 meters above the ground. You have a rock in your hand, which you can throw with a maximum speed of 10.0 m/s.
(a) How long would it take for the rock to hit the ground if you released the rock from rest? √
(b) How long would it take for the rock to hit the ground if you threw the rock straight downward? √
(c) How long would it take for the rock to hit the ground if you threw the rock straight upward? √
(d) If you threw the rock straight upward, how high would it get above the ground? √

2-i11 You're standing on the roof of your science building. You drop a rock from rest and notice that it takes an amount of time T to hit the ground. Express your answers to the following questions in terms of T and the acceleration due to gravity, g.
(a) How high is the building? √
(b) How fast must you throw the rock straight up if the rock is to take $2T$ to hit the ground? √
(c) For the situation described in part b, how long does it take from the time you let the rock go to when the rock reaches maximum height? √
(d) For the same situation, what is the maximum height that the rock gets to above the ground? √

2-i12 You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get halfway there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.
(a) Find a formula for the time, T, required for the round trip, in terms of d, the distance from our sun to the star, and a, the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.
(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of $d = 4 \times 10^{16}$ m. Suppose you use an acceleration of $a = 10\ \text{m/s}^2$, just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?
(c) Using the same numbers for d and a, find your maximum speed. Compare this to the speed of light, which is 3.0×10^8 m/s. (Later in this course, you will learn that there are some new things going on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you've learned so far.) √

2-k1 If the acceleration of gravity on Mars is 1/3 that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance.
▷ Solution, p. 239

2-k2 Starting from rest, a ball rolls down a ramp, traveling a distance L and picking up a final speed v. How much of the distance did the ball have to cover before achieving a speed of $v/2$? [Based on a problem by Arnold Arons.] ⋆

2-k3 Suppose you can hit a tennis ball vertically upward with a certain initial speed, independent of what planet you're on.
(a) If the height the ball reaches on Earth is H, what is the height the ball will reach on Pluto, where the acceleration due to gravity is about 1/15th the value on Earth? √
(b) If the amount of time the ball spends in the air on Earth is 3.0 seconds, how long would it spend in the air on Pluto? √

2-k4 You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)

2-k5 Most people don't know that *Spinosaurus aegyptiacus*, not *Tyrannosaurus rex*, was the biggest theropod dinosaur. We can't put a dinosaur on a track and time it

in the 100 meter dash, so we can only infer from physical models how fast it could have run. When an animal walks at a normal pace, typically its legs swing more or less like pendulums of the same length ℓ. As a further simplification of this model, let's imagine that the leg simply moves at a fixed acceleration as it falls to the ground. That is, we model the time for a quarter of a stride cycle as being the same as the time required for free fall from a height ℓ. *S. aegyptiacus* had legs about four times longer than those of a human. (a) Compare the time required for a human's stride cycle to that for *S. aegyptiacus.* ✓
(b) Compare their running speeds.
✓

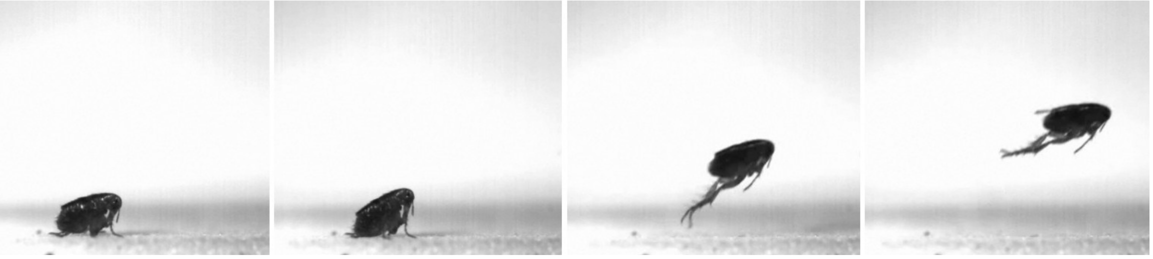
2-k6 Engineering professor Qingming Li used sensors and video cameras to study punches delivered in the lab by British former welterweight boxing champion Ricky "the Hitman" Hatton. For comparison, Li also let a TV sports reporter put on the gloves and throw punches. The time it took for Hatton's best punch to arrive, i.e., the time his opponent would have had to react, was about 0.47 of that for the reporter. Let's assume that the fist starts from rest and moves with constant acceleration all the way up until impact, at some fixed distance (arm's length). Compare Hatton's acceleration to the reporter's.
✓

2-k7 Aircraft carriers originated in World War I, and the first landing on a carrier was performed by E.H. Dunning in a Sopwith Pup biplane, landing on HMS Furious. (Dunning was killed the second time he attempted the feat.) In such a landing, the pilot slows down to just above the plane's stall speed, which is the minimum speed at which the plane can fly without stalling. The plane then lands and is caught by cables and decelerated as it travels the length of the flight deck. Comparing a modern US F-14 fighter jet landing on an Enterprise-class carrier to Dunning's original exploit, the stall speed is greater by a factor of 4.8, and to accomodate this, the length of the flight deck is greater by a factor of 1.9. Which deceleration is greater, and by what factor?
✓

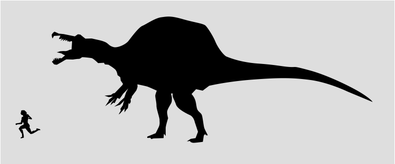
2-k8 In college-level women's softball in the U.S., typically a pitcher is expected to be at least 1.75 m tall, but Virginia Tech pitcher Jasmin Harrell is 1.62 m. Although a pitcher actually throws by stepping forward and swinging her arm in a circle, let's make a simplified physical model to estimate how much of a disadvantage Harrell has had to overcome due to her height. We'll pretend that the pitcher gives the ball a constant acceleration in a straight line, and that the length of this line is proportional to the pitcher's height. Compare the acceleration Harrell would have to supply with the acceleration that would suffice for a pitcher of the nominal minimum height, if both were to throw a pitch at the same speed.
✓

2-k9 When the police engage in a high-speed chase on city streets, it can be extremely dangerous both to the police and to other motorists and pedestrians. Suppose that the police car must travel at a speed that is limited by the need to be able to stop before hitting a baby carriage, and that the distance at which the driver first sees the baby carriage is fixed. Tests show that in a panic stop from high speed, a police car based on a Chevy Impala has a deceleration 9% greater than that of a Dodge Intrepid. Compare the maximum safe speeds for the two cars.
✓

2-n1 The figure shows a practical, simple experiment for determining g to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights Δx_1 and Δx_2. A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity T defined as $T = \Delta t_2 - \Delta t_1$, i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound



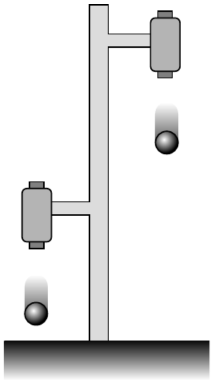
Problem 2-n2.



Problem 2-k5.

when they are released, we have no way of measuring the individual times Δt_2 and Δt_1.
(a) Find an equation for g in terms of the measured quantities T, Δx_1 and Δx_2. √
(b) Check the units of your equation.
(c) Check that your equation gives the correct result in the case where Δx_1 is very close to zero. However, is this case realistic?
(d) What happens when $\Delta x_1 = \Delta x_2$? Discuss this both mathematically and physically.

⋆



Problem 2-n1.

2-n2 Some fleas can jump as high as 30 cm. The flea only has a short time to build up speed — the time during which its center of mass is accelerating upward but its feet are still in contact with the ground. Make an order-of-magnitude estimate of the acceleration the flea needs to have while straightening its legs, and state your answer in units of g, i.e., how many "g's it pulls." (For comparison, fighter pilots black out or die if they exceed about 5 or 10 g's.)

⋆

2-n3 The speed required for a low-earth orbit is 7.9×10^3 m/s. When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to $3g$ to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn't necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth's surface.

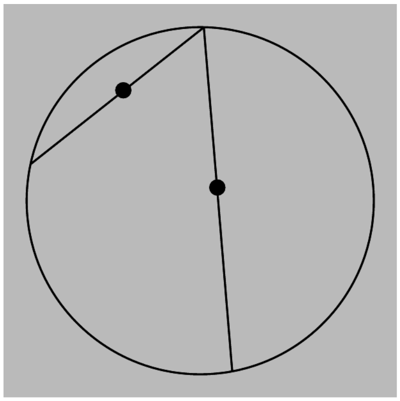
⋆

2-n4 When an object slides frictionlessly down a plane inclined at an angle θ, its acceleration equals $g\sin\theta$ (p. 16). Suppose that a person on a bike is to coast down a ramp, starting from rest, and then coast back up an identical ramp, tracing a "V." The horizontal distance is fixed to be $2w$, and we want to set the depth of the "V" so as to achieve the minimal possible value t^* for the total time.
(a) Based only on units, infer the form of the expression for t^* in terms of w, up to a unitless multiplicative constant.
(b) Find the angle that minimizes the time.
(c) Complete the determination of t^* by finding the unitless constant.

▷ Solution, p. 241 ⋆

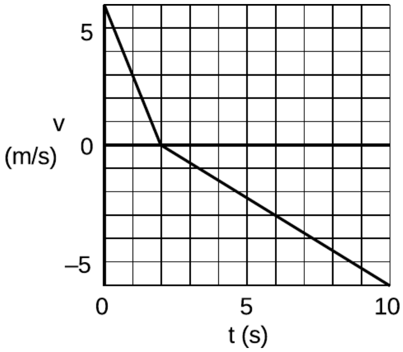
2-n5 The figure shows a circle in a vertical plane, with two wires positioned along chords of the circle. The top of each wire coincides with the top of the circle. Beads slide frictionlessly on the wires. If the beads are released simultaneously at the top, which one wins the race? You will need the fact that the acceleration equals $g\sin\theta$ (p. 16).

⋆



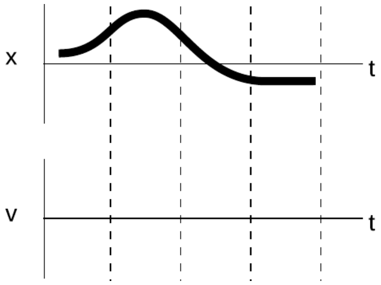
Problem 2-n5.

2-o1 The graph represents the motion of a ball that rolls up a hill and then back down. When does the ball return to the location it had at $t = 0$?



Problem 2-o1.

2-o2 The top part of the figure shows the position-versus-time graph for an object moving in one dimension. On the bottom part of the figure, sketch the corresponding v-versus-t graph.



Problem 2-o2.

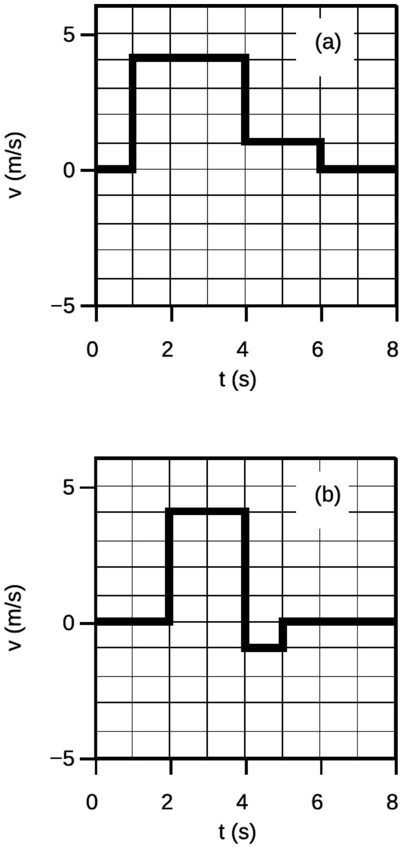
2-o3 For each of the two graphs, find the change in position Δx from beginning to end.

✓

2-o4 The graph represents the velocity of a bee along a straight line. At $t = 0$, the bee is at the hive. (a) When is the bee farthest from the hive? (b) How far is the bee at its farthest point from the hive? (c) At $t = 13$ s, how far is the bee from the hive?

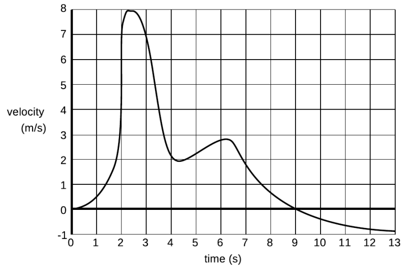
✓

2-o5 Decide whether the following statements about one-dimensional motion are true or false:
(a) The area under a $v(t)$ curve gives the acceleration of the object.



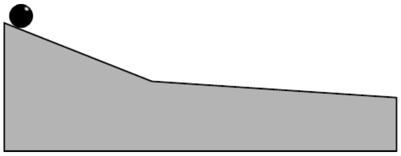
Problem 2-o3.

(b) The area under a $a(t)$ curve gives the change in velocity of the object.
(c) The slope of a $a(t)$ curve at time T is the value $v(T)$.
(d) The slope of the $v(t)$ curve at time T is the value $a(T)$.
(e) The displacement Δx from t_1 to t_2 is equal to the area under the $a(t)$ curve from t_1 to t_2.



Problem 2-o4.

2-o6 (a) The ball is released at the top of the ramp shown in the figure. Friction is negligible. Use physical reasoning to draw $v-t$ and $a-t$ graphs. Assume that the ball doesn't bounce at the point where the ramp changes slope. (b) Do the same for the case where the ball is rolled up the slope from the right side, but doesn't quite have enough speed to make it over the top.



Problem 2-o6.

2-o7 You throw a rubber ball up, and it falls and bounces several times. Draw graphs of position, velocity, and acceleration as functions of time.

2-p1 (a) Express the chain rule in Leibniz ("d") notation, and show that it always results in an answer whose units make sense.
(b) An object has a position as a function of time given by $x = A\cos(bt)$, where A and b are constants. Infer the units of A and b, and interpret their physical meanings.
(c) Find the velocity of this object, and check that the chain rule has indeed given an answer with the right units.

2-p2 In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car's top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with *constant* acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part b and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time?

▷ Solution, p. 243 ⋆

2-p3 A honeybee's position as a function of time is given by $x = 10t - t^3$, where t is in seconds and x in meters. What is its velocity at $t = 3.0$ s?

√

2-p4 Objects A and B move along the x axis. The acceleration of both objects as functions of time is given by $a(t) = (3.00\ \text{m/s}^3)t$. Object A starts (at $t = 0$) from rest at the origin, and object B starts at $x = 5.00$ m, initially moving in the negative x direction with speed 9.00 m/s.

(a) What is A's velocity at time $t = 2.00$ s? √

(b) What is A's position at the same time? √

(c) What is B's velocity at the same time? √

(d) What is B's position at the same time? √

(e) Consider a frame of reference in which A is at rest, such as the frame that would naturally be adopted by an observer moving along with A. Describe B's motion in this frame.

(f) After they start, is there any time at which A and B collide?

2-p5 The position of a particle moving on the x-axis is described by the equation $x(t) = t^3 - 4t^2$ (with x in meters and t in seconds). Consider the times $t = -1$, 0, 1, 2, and 3 seconds. For which of these times is the particle slowing down?

2-p6 Freddi Fish$^{(\text{TM})}$ has a position as a function of time given by $x = a/(b + t^2)$. (a) Infer the units of the constants a and b. (b) Find her maximum speed. (c) Check that your answer has the right units.

√

2-p7 Let t be the time that has elapsed since the Big Bang. In that time, one would imagine that light, traveling at speed c, has been able to travel a maximum distance ct. (In fact the distance is several times more than this, because according to Einstein's theory of general relativity, space itself has been expanding while the ray of light was in transit.) The portion of the universe that we can observe would then be a sphere of radius ct, with volume $v = (4/3)\pi r^3 = (4/3)\pi(ct)^3$. Compute the rate $\mathrm{d}v/\mathrm{d}t$ at which the volume of the observable universe is increasing, and check that your answer has the right units.

√

2-p8 Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion of the form $x = cte^{-bt}$, where x is the position of some point on the door, and c and b are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.) In this example, the door moves in the positive direction up until a certain time, then stops and settles back in the negative direction, eventually approaching $x = 0$. This would be the type of motion we would get if someone flung a door open and the door closer then brought it back closed again.

(a) Infer the units of the constants b and c.

(b) Find the door's maximum speed (i.e., the greatest absolute value of its velocity) as it comes back to the closed position. √

(c) Show that your answer has units that make sense.

⋆

2-p9 A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by an equation of the form

$$y = b - c\left(t + ke^{-t/k}\right),$$

where e is the base of natural logarithms, and b, c, and k are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.
(a) What units would b, c, and k have to have for the equation to make sense?
(b) Find the person's velocity, v, as a function of time. [You will need to use the chain rule, and the fact that $\mathrm{d}(e^x)/\,\mathrm{d}x = e^x$.] √
(c) Use your answer from part (b) to get an interpretation of the constant c. [Hint: e^{-x} approaches zero for large values of x.]
(d) Find the person's acceleration, a, as a function of time. √
(e) Use your answer from part (d) to show that if she waits long enough to open her chute, her acceleration will become very small.

⋆

3 Kinematics in three dimensions

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

3.1 Vectors

Most of the things we want to measure in physics fall into two categories, called *vectors* and *scalars*. A scalar is something that doesn't change when you turn it around, while a vector does change when you rotate it, and the way in which it changes is the same as the way in which a pointer such as a pencil or an arrow would change. For example, temperature is a scalar: a hot cup of coffee doesn't change its temperature when we turn it around. Force is a vector. When I play tug-of-war with my dog, her force and mine are the same in strength, but they're in opposite directions. If we swap positions, our forces reverse their directions, just as a pair of arrows would.

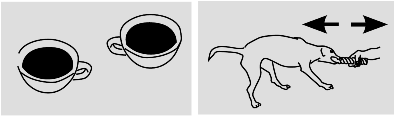


Figure 3.1: Temperature is a scalar. Force is a vector.

To distinguish vectors from scalars, we write them differently, e.g., p for a scalar and boldface $\mathbf{p}$ for a vector. In handwriting, a vector is written with an arrow over it, $\vec{p}$.

Not everything is a scalar or a vector. For example, playing cards are designed in a symmetric way, so that they look the same after a 180-degree rotation. The orientation of the card is not a scalar, because it changes under a rotation, but it's not a vector, because it doesn't behave the way an arrow would under a 180-degree rotation.

In kinematics, the simplest example of a vector is a motion from one place to another, called a displacement vector.

A vector $\mathbf{A}$ has a magnitude $|\mathbf{A}|$, which means its size, length, or amount. Rotating a vector can change the vector, but will never change its magnitude.

Scalars are just numbers, and we do arithmetic on them in the usual way. Vectors are different. Vectors can be added graphically by placing them tip to tail, and then drawing a vector from the tail of the first vector to the tip of the second vector. A vector can be multiplied by a scalar to give a new vector. For instance, if $\mathbf{A}$ is a vector, then $2\mathbf{A}$ is a vector that has the same direction but twice the magnitude. Multiplying by -1 is the same as flipping the vector, $-\mathbf{A} = (-1)\mathbf{A}$. Vector subtraction can be accomplished by flipping and adding.

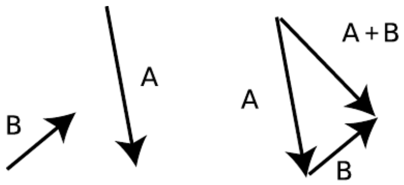


Figure 3.2: Graphical addition of vectors.

Suppose that a sailboat undergoes a displacement $\mathbf{h}$ while moving near a pier. We can define a number called the *component* of $\mathbf{h}$ parallel to the pier, which is the distance the boat has moved along the pier, ignoring an motion toward or away from the pier. If we arbitrarily define one direction along the pier as positive, then the component has a sign.

Often it is convenient to work with components of a vector along the coordinate axes. If we pick a Cartesian coordinate system with x, y, and z axes, then any vector can be specified according to its x, y, and z components. We have previously given a graphical definition for vector addition. This is equivalent to adding components.

Unit vector notation

Suppose we want to tell someone that a certain vector **A** in two dimensions has components $A_x = 3$ and $A_y = 7$. A more compact way of notating this is $\mathbf{A} = 3\hat{\mathbf{x}} + 7\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, read "x-hat" and "y-hat," are the vectors with magnitude one that point in the positive x and y directions. Some authors notate the unit vectors as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ rather than $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

Rotational invariance

Certain vector operations are useful and others are not. Consider the operation of multiplying two vectors component by component to produce a third vector:

$$\begin{aligned} R_x &= P_x Q_x \\ R_y &= P_y Q_y \\ R_z &= P_z Q_z . \end{aligned}$$

This operation will never be useful in physics because it can give different results depending on our choice of coordinates. That is, if we change our coordinate system by rotating the axes, then the resulting vector **R** will of course have different components, but these will not (except in exceptional cases) be the components of the same vector expressed in the new coordinates. We say that this operation is not *rotationally invariant.*

The universe doesn't come equipped with coordinates, so if any vector operation is to be useful in physics, it must be rotationally invariant. Vector addition, for example, is rotationally invariant, since we can define it using tip-to-tail graphical addition, and this definition doesn't even refer to any coordinate system. This rotational invariance would still have held, but might not have been so obvious, if we had defined addition in terms of addition of components.

Dot and cross product

The vector *dot product* $\mathbf{A} \cdot \mathbf{B}$ is defined as the (signed) component of **A** parallel to **B**. It is a scalar. If we know the magnitudes of the vectors and the angle θ_{AB} between them, we can compute the dot product as $|\mathbf{A}||\mathbf{B}| \cos \theta_{AB}$. If we know the components of the vectors in a particular coordinate system, we can express the dot product as $A_x B_x + A_y B_y + A_z B_z$.

The dot product is useful simply as a geometrical tool, but later in this course we will also see that it has physical applications that include mechanical work, as well as many examples in electricity and magnetism, such as electric flux.

There is also a way of multiplying two vectors to obtain a vector result. This is called the vector *cross product*, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. The magnitude of the cross product is the area of the parallelogram illustrated in figure 3.3. The direction of the cross product is perpendicular to the plane in which **A** and **B** lie. There are two such directions, and of these two, we choose the one defined by the right-hand rule illustrated in figure 3.4.

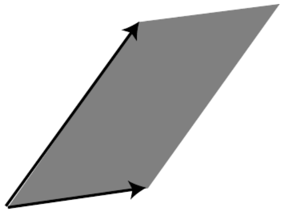


Figure 3.3: The magnitude of the cross product is the area of the shaded parallelogram.

Important physical applications of the cross product include torque, angular momentum, and magnetic forces.

Unlike ordinary multiplication of real numbers, the cross product is anticommutative, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The magnitude of the cross product can be expressed as $|\mathbf{A}||\mathbf{B}| \sin \theta_{AB}$. In terms

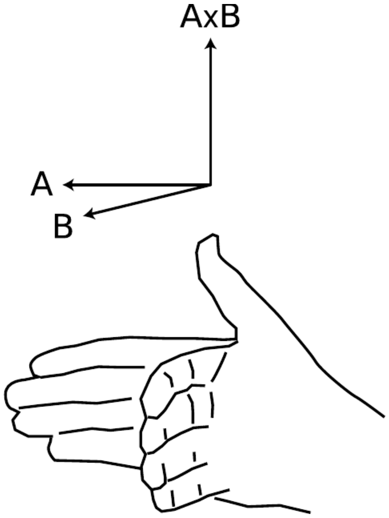


Figure 3.4: The right-hand rule for the direction of the vector cross product.

of the components, we have

$$\begin{aligned}\mathbf{C}_x &= A_yB_z - B_yA_z\\ \mathbf{C}_y &= A_zB_x - B_zA_x\\ \mathbf{C}_z &= A_xB_y - B_xA_y.\end{aligned}$$

3.2 Motion

Velocity and acceleration

If an object undergoes an infinitesimal displacement $\mathrm{d}\mathbf{r}$ in an infinitesimal time interval $\mathrm{d}t$, then its velocity vector is the derivative $\mathbf{v} = \mathrm{d}\mathbf{r}/\,\mathrm{d}t$. This type of derivative of a vector can be computed by differentiating each component separately. The acceleration is the second derivative $\mathrm{d}^2\,\mathbf{r}/\,\mathrm{d}t^2$.

The velocity vector has a magnitude that indicates the speed of motion, and a direction that gives the direction of the motion. We saw in section 2.1 that velocities add in relative motion. To generalize this to more than one dimension, we use vector addition.

The acceleration vector does *not* necessarily point in the direction of motion. It points in the direction that an accelerometer would point, as in figure 3.6.

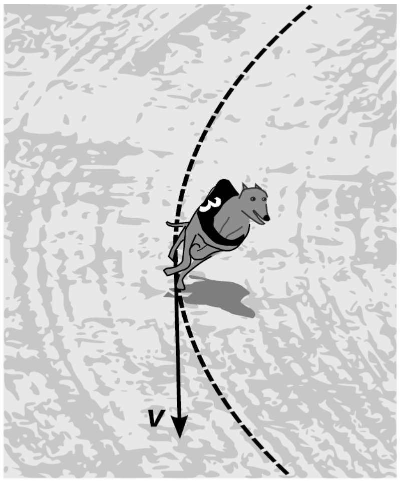


Figure 3.5: The racing greyhound's velocity vector is in the direction of its motion, i.e., tangent to its curved path.

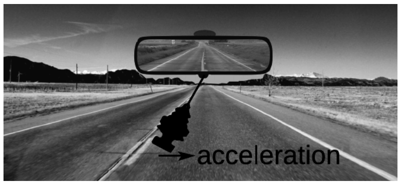


Figure 3.6: The car has just swerved to the right. The air freshener hanging from the rear-view mirror acts as an accelerometer, showing that the acceleration vector is to the right.

Projectiles and the inclined plane

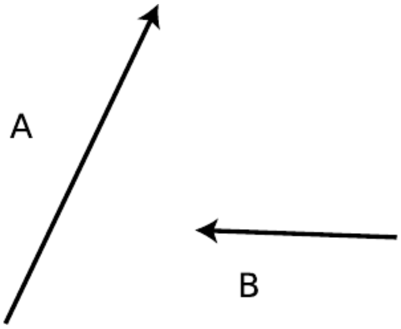
Forces cause accelerations, not velocities. In particular, the downward force of gravity causes a downward acceleration vector. After a projectile is launched, the only force on it is gravity, so its acceleration vector points straight down. Therefore the horizontal part of its motion has constant velocity. The vertical and horizontal motions of a projectile are independent. Neither affects the other.

Problems

3-a1 The figure shows vectors **A** and **B**. Graphically calculate the following.

$\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{A}$, $-2\mathbf{B}$, $\mathbf{A}-2\mathbf{B}$

No numbers are involved.



Problem 3-a1.

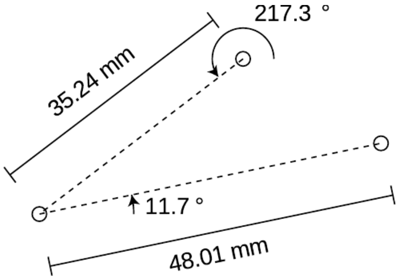
3-a2 Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.

(a) Choose a coordinate system, and translate these data into Δx and Δy values with the proper plus and minus signs.

(b) Find the components of the $\Delta\mathbf{r}$ vector pointing from Bangkok to Hanoi. √

3-a3 If you walk 35 km at an angle 25° counterclockwise from east, and then 22 km at 230° counterclockwise from east, find the distance and direction from your starting point to your destination. √

3-a4 A machinist is drilling holes in a piece of aluminum according to the plan shown in the figure. She starts with the top hole, then moves to the one on the left, and then to the one on the right. Since this is a high-precision job, she finishes by moving in the direction and at the angle that should take her back to the top hole, and checks that she ends up in the same place. What are the distance and direction from the right-hand hole to the top one? √



Problem 3-a4.

3-a5 Suppose someone proposes a new operation in which a vector **A** and a scalar B are added together to make a new vector **C** like this:

$$\begin{aligned} C_x &= A_x + B \\ C_y &= A_y + B \\ C_z &= A_z + B \end{aligned}$$

Prove that this operation won't be useful in physics, because it's not rotationally invariant.

3-d1 Find the angle between the following two vectors:

$$\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

√

3-d2 Let a and b be any two numbers (not both zero), and let $\mathbf{u} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$. Suppose we want to find a (nonzero) second vector $\mathbf{v}$ in the x-y plane that is perpendicular to $\mathbf{u}$. Use the vector dot product to write down a condition for $\mathbf{v}$ to satisfy, find a suitable $\mathbf{v}$, and check using the dot product that it is indeed a solution.

3-g1 Find a vector that is perpendicular to both of the following two vectors:

$$\hat{\mathbf{x}}+2\hat{\mathbf{y}}+3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}}+5\hat{\mathbf{y}}+6\hat{\mathbf{z}}$$

√

3-g2 Which of the following expressions make sense, and which are nonsense? For those that make sense, indicate whether the result is a vector or a scalar.
(a) $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$
(b) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$
(c) $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$

3-g3 Vector $\mathbf{A}=(3.0\hat{\mathbf{x}}-4.0\hat{\mathbf{y}})$ meters, and vector $\mathbf{B}=(5.0\hat{\mathbf{x}}+12.0\hat{\mathbf{y}})$ meters. Find the following: (a) The magnitude of vector $\mathbf{A}-2\mathbf{B}$. √
(b) The dot product $\mathbf{A}\cdot\mathbf{B}$. √
(c) The cross product $\mathbf{A}\times\mathbf{B}$ (expressing the result in terms of its components). √
(d) The value of $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$. √
(e) The angle between the two vectors. √

3-g4 Prove the anticommutative property of the vector cross product, $\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$, using the expressions for the components of the cross product. Note that giving an example does not constitute a proof of a general rule.

3-g5 Label the following statements about vectors as true or false.
(a) The angle between $a\hat{\mathbf{x}}+b\hat{\mathbf{y}}$ and $b\hat{\mathbf{x}}+a\hat{\mathbf{y}}$ is zero.
(b) The three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$ form a triangle.
(c) The three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}-\mathbf{B}$ form a triangle.
(d) The cross product between two vectors is always perpendicular to each of the two vectors.
(e) If the angle between two vectors is greater than 90°, then the dot product between the two vectors is negative.
(f) A unit vector has magnitude 1 (and no units).

3-g6 Find three vectors with which you can demonstrate that the vector cross product need not be associative, i.e., that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ need not be the same as $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

3-g7 Can the vector cross product be generalized to four dimensions? The generalization should, like the three-dimensional version, take two vectors as inputs, give a vector as an output, and be rotationally invariant. (This is of real-world interest because Einstein's theory of relativity can be interpreted as describing time as a kind of fourth dimension.)

⋆

3-g8 A certain function f takes two vectors as inputs and gives an output that is also a vector. The function can be defined in such a way that it is rotationally invariant, and it is also well defined regardless of the units of the vectors. It takes on the following values for the following inputs:

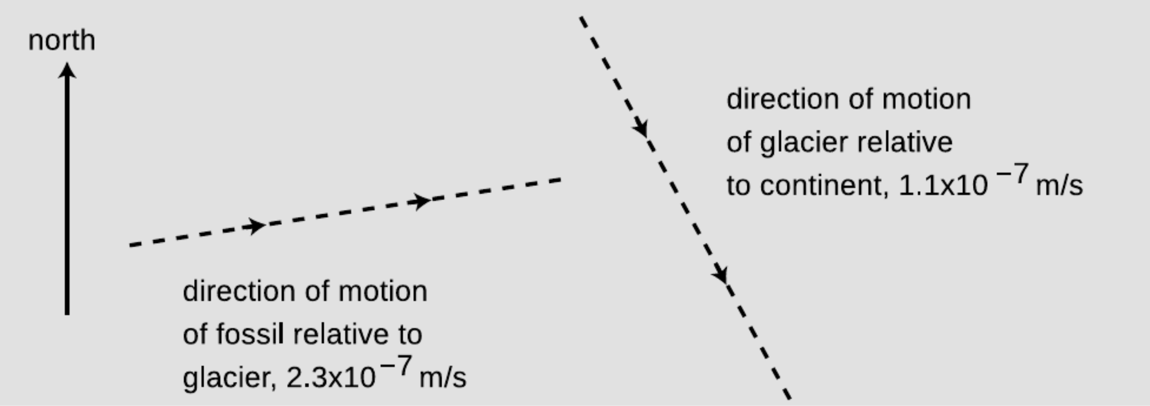
$$f(\hat{\mathbf{x}},\hat{\mathbf{y}})=-\hat{\mathbf{z}}$$
$$f(2\hat{\mathbf{x}},\hat{\mathbf{y}})=-8\hat{\mathbf{z}}$$
$$f(\hat{\mathbf{x}},2\hat{\mathbf{y}})=-2\hat{\mathbf{z}}$$

Prove that the given information uniquely determines f, and give an explicit expression for it.

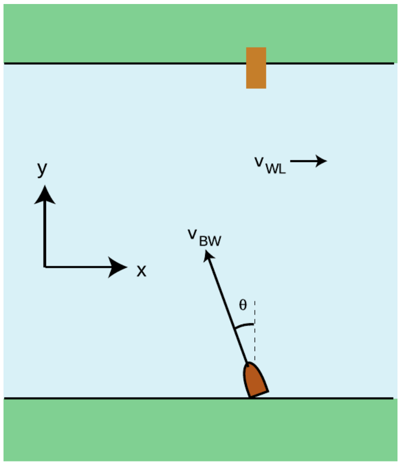
⋆

3-j1 Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet.

3-j2 As shown in the figure, you wish to cross a river and arrive at a dock that is directly across from you, but the river's current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle θ, given the magnitude, $|\mathbf{v}_{WL}|$, of the water's velocity relative to the land, and the maximum speed, $|\mathbf{v}_{BW}|$, of which the boat is capable relative to the water.


north
direction of motion of fossil relative to glacier, 2.3x10^{-7} m/s
direction of motion of glacier relative to continent, 1.1x10^{-7} m/s

Problem 3-j7.



Problem 3-j2.

▷ Solution, p. 249

3-j3 It's a calm day in Los Angeles with no wind. You're in your car on the Ventura Freeway going 105 km/hour when it starts to rain. You notice out the driver's side window that the raindrops make an angle of 70° with respect to the vertical.

(a) What is the speed of the raindrops as measured by someone at rest relative to the freeway? √

(b) What is the speed of the raindrops as measured by you? √

3-j4 A border collie and a rottweiler, both initially at the same location, start chasing two different objects. The border collie starts chasing a stick thrown in the $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ direction, and the rottweiler starts chasing your neighbor in the $-\hat{\mathbf{y}}$ direction. Both dogs move at speed v. For both parts of this problem, give your results as exact expressions, not decimal approximations.

(a) What is the velocity of the rottweiler as measured by the border collie? √

(b) How long does it take for the distance between the two dogs to be D? √

3-j5 A plane can fly at $u = 150$ m/s with respect to the air, and the wind is from the southwest at $v = 50$ m/s.

(a) In what direction should the plane head in order to fly directly north (with respect to the ground)? Give the angle in degrees west of north. √

(b) What is the plane's speed as measured by an observer on the ground? √

3-j6 A plane flies toward a city directly north and a distance D away. The wind speed is u, and the plane's speed with respect to the wind is v.

(a) If the wind is blowing from the west (towards the east), what direction should the plane head

(what angle west of north)? $\checkmark$
(b) How long does it take the plane to get to the city? $\checkmark$
(c) Check that your answer to part b has units that make sense.
(d) Comment on the behavior of your answer in the case where $u = v$.

3-j7 As shown in the diagram, a dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil's velocity relative to the continent. You will need a ruler and protractor. $\checkmark$

3-j8 Andrés and Brenda are going to race to see who can first get to a town across a river of width 20.0 m. The water in the river is moving at a constant 0.60 m/s, each person can swim with speed 1.00 m/s with respect to the water, and each person can run 4.00 m/s on land.

Andrés is going to row in such a way that he moves straight towards the town across the river. Brenda, however, decides to get to the other side of the river as quickly as she can, and run.
(a) How long does it take Andrés to swim to the other side of the river? Call this T_A $\checkmark$
(b) How long does it take Brenda to get to the other side of the river (not at the town, since the river carries her downstream)? Call this t_1. $\checkmark$
(c) How long does it take Brenda to run to the town on the other side of the river? Call this t_2. $\checkmark$
(d) How long ($T_B = t_1 + t_2$) does the total trip take Brenda? Who wins the race? $\checkmark$

3-j9 César is on one bank of a river in which the water flows at speed w. He can swim at speed u and run at speed v. On the other side, directly across from him, is a town that he wants to reach in the minimum possible time. Depending on the direction in which he chooses to swim, he may need to run some distance along the far bank in order to get to the town. Show that, surprisingly, the optimal angle depends on the variables v and w only through their sum $v + w$. $\star$

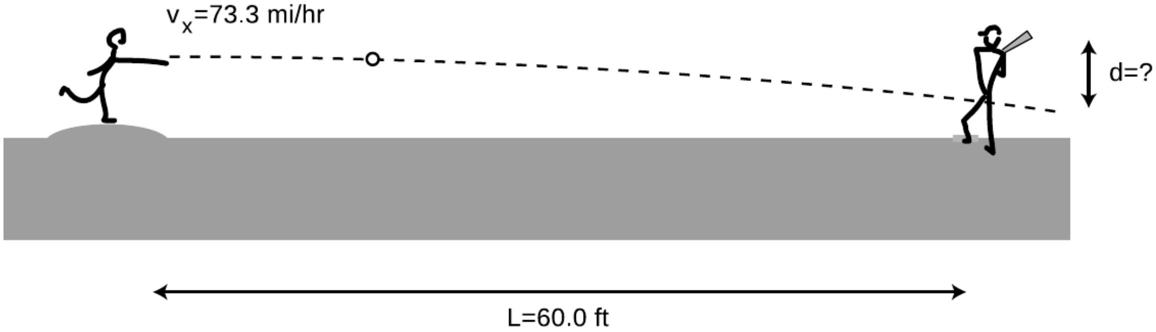
3-m1 Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

3-m2 The figure shows the path followed by Hurricane Irene in 2005 as it moved north. The dots show the location of the center of the storm at six-hour intervals, with lighter dots at the time when the storm reached its greatest intensity. Find the time when the storm's center had a velocity vector to the northeast and an acceleration vector to the southeast. Explain.

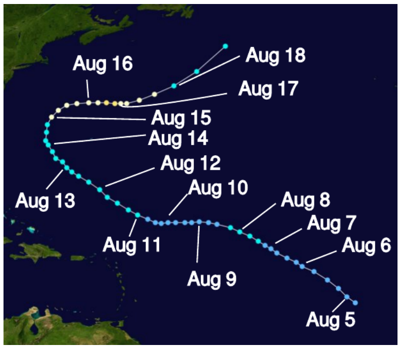
3-m3 A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and 7° north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.) $\checkmark$

3-m4 Two cars go over the same speed bump in a parking lot, Maria's Maserati at 25 miles per hour and Park's Porsche at 37. How many times greater is the vertical acceleration of the Porsche? Hint: Remember that acceleration depends both on how much the velocity changes and on how much time it takes to change. $\checkmark$

3-p1 Two daredevils, Wendy and Bill, go over Niagara Falls. Wendy sits in an inner tube, and lets the 30 km/hr velocity of the river throw her out horizontally over the falls. Bill paddles a kayak, adding an extra 10 km/hr to his velocity. They go over the edge of the falls at the same moment, side by side. Ignore air friction. Explain your reasoning.
(a) Who hits the bottom first?
(b) What is the horizontal component of Wendy's velocity on impact?
(c) What is the horizontal component of Bill's



Problem 3-s2.



Problem 3-m2.

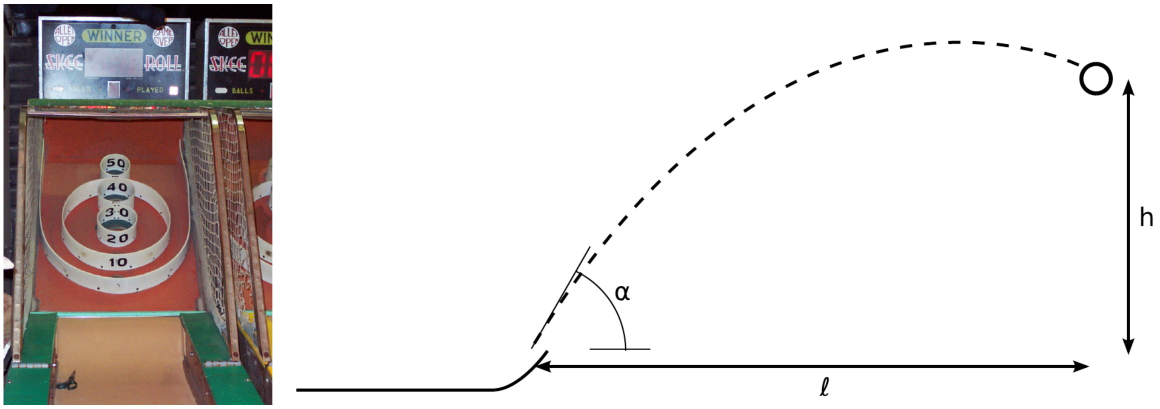
velocity on impact?
(d) Who is going faster on impact?

3-p2 At the 2010 Salinas Lettuce Festival Parade, the Lettuce Queen drops her bouquet while riding on a float moving toward the right. Sketch the shape of its trajectory in her frame of reference, and compare with the shape seen by one of her admirers standing on the sidewalk.

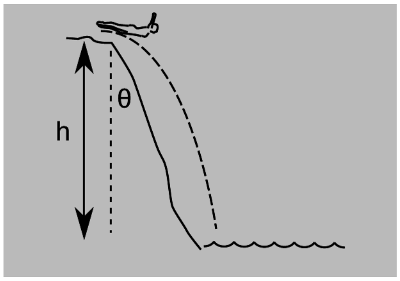
3-s1 A gun is aimed horizontally to the west. The gun is fired, and the bullet leaves the muzzle at $t = 0$. The bullet's position vector as a function of time is $\mathbf{r} = b\hat{\mathbf{x}} + ct\hat{\mathbf{y}} + dt^2\hat{\mathbf{z}}$, where b, c, and d are positive constants.
(a) What units would b, c, and d need to have for the equation to make sense?
(b) Find the bullet's velocity and acceleration as functions of time.
(c) Give physical interpretations of b, c, d, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

3-s2 A baseball pitcher throws a pitch clocked at $v_x = 73.3$ miles/hour. He throws horizontally. By what amount, d, does the ball drop by the time it reaches home plate, $L = 60.0$ feet away?
(a) First find a symbolic answer in terms of L, v_x, and g. ✓
(b) Plug in and find a numerical answer. Express your answer in units of ft. (Note: 1 foot=12 inches, 1 mile=5280 feet, and 1 inch=2.54 cm) ✓

3-s3 You're running off a cliff into a pond. The cliff is $h = 5.0$ m above the water, but the cliff is not strictly vertical; it slopes down to the pond at an angle of $\theta = 20°$ with respect to the vertical. You want to find the minimum speed you need to jump off the cliff in order to land in the water.
(a) Find a symbolic answer in terms of h, θ, and g. ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence on the variables g, h, and θ makes sense, and check the special cases $\theta = 0$ and $\theta = 90°$.
(d) Plug in numbers to find the numerical result. ✓



Problem 3-s12.



Problem 3-s3.

3-s4 A batter hits a baseball at speed v, at an angle θ above horizontal.
(a) Find an equation for the range (horizontal distance to where the ball falls), R, in terms of the relevant variables. Neglect air friction and the height of the ball above the ground when it is hit.
(b) Interpret your equation in the cases of θ=0 and $\theta = 90°$.
(c) Find the angle that gives the maximum range.

3-s5 A tennis ball is thrown from the ground with speed 15 m/s at an angle of 45° above the horizontal.
(a) How long is the ball in the air (from the throw to when it lands on the ground)? √
(b) What is the maximum height that the ball reaches? √
(c) What is the range of the ball (the horizontal distance the ball has traveled by the time it lands)?
√

3-s6 (a) A ball is thrown straight up with velocity v. Find an equation for the height to which it rises. √
(b) Generalize your equation for a ball thrown at an angle θ above horizontal, in which case its initial velocity components are $v_x = v\cos\theta$ and $v_y = v\sin\theta$. √

3-s7 The first time he played golf, now-deceased North Korean leader Kim Jong-Il is said to have gotten 11 holes in one. Suppose that his son, Kim Jong-Un, wants to top his father's feat by hitting a golf ball all the way from Pyongyang to Seoul, a distance of 195 km. Ignoring air resistance and the curvature of the earth, how fast does he need to hit the ball? Note that the maximum range (assuming no air resistance) is achieved for a launch angle of 45°.

3-s8 Two footballs, one white and one green, are on the ground and kicked by two different footballers. The white ball, which is kicked straight upward with initial speed v_0, rises to height H. The green ball is hit with twice the

initial speed but reaches the same height.
(a) What is the y-component of the green ball's initial velocity vector? Give your answer in terms of v_0 alone. ✓
(b) Which ball is in the air for a longer amount of time?
(c) What is the range of the green ball? Your answer should only depend on H. ✓

3-s9 You throw a rock horizontally from the edge of the roof of a building of height 10.0 m. The rock hits the ground at exactly twice its initial speed. How fast was the rock thrown off the roof? Express your answer to three significant figures. ✓

3-s10 You throw a rock horizontally from the edge of the roof of a building of height h with speed v_0. What is the (positive) angle between the final velocity vector and the horizontal when the rock hits the ground? ✓

3-s11 Standing on the edge of the roof of a building of height h, you throw a rock with speed v_0 at 30° above the horizontal.
(a) How high above the ground does the rock get? ✓
(b) How far away from the building does the rock land? ✓

3-s12 The figure shows an arcade game called skee ball that is similar to bowling. The player rolls the ball down a horizontal alley. The ball then rides up a curved lip and is launched at an initial speed u, at an angle α above horizontal. Suppose we want the ball to go into a hole that is at horizontal distance ℓ and height h, as shown in the figure.
(a) Find the initial speed u that is required, in terms of the other variables and g. ✓
(b) Check that your answer to part a has units that make sense.
(c) Check that your answer to part a depends on g in a way that makes sense. This means that you should first determine on physical grounds whether increasing g should increase u, or decrease it. Then see whether your answer to part a has this mathematical behavior.
(d) Do the same for the dependence on h.
(e) Interpret your equation in the case where $\alpha = 90°$.
(f) Interpret your equation in the case where $\tan\alpha = h/\ell$.
(g) Find u numerically if $h = 70$ cm, $\ell = 60$ cm, and $\alpha = 65°$. ✓

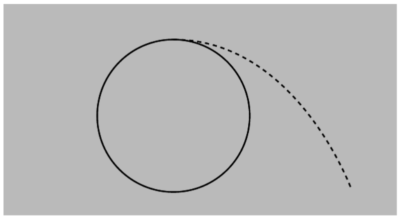
3-s13 A particle leaves point P at time $t = 0$ s with initial velocity $(-2.0\hat{\mathbf{x}} + 4.0\hat{\mathbf{y}})$ m/s. Point P is located on the x axis at position $(x, y) = (10.0\text{ m}, 0)$. If the particle experiences constant acceleration $\mathbf{a} = (-5.0\hat{\mathbf{y}})$ m/s^2, then which axis does it cross first, x or y, and at what location?

3-s14 A Hot Wheels car is rolling along a horizontal track at speed $v_0 = 6.0$ m/s. It then comes to a ramp inclined at an angle $\theta = 30°$ above the horizontal, and the car undergoes a deceleration of $g\sin\theta = 4.9$ m/s^2 when moving along the ramp. The track ends at the top of the ramp, so the car is launched into the air. By the time the car reaches the top of the ramp, its speed has gone down to 3.0 m/s.
(a) How high is the top of the ramp (vertical height, not distance along the ramp)? ✓
(b) After the car is achieves lift-off, how long does it spend in the air before hitting the ground? ✓

3-s15 A Hot Wheels car is rolling along a horizontal track at speed v_0. It then comes to a ramp inclined at an angle θ above the horizontal. The car undergoes a deceleration of $g\sin\theta$ while rolling up the ramp. The track ends after a distance L, so the car is launched into the air.
(a) What is the speed of the car when it leaves the ramp? ✓
(b) How high does the car get above the ground? ✓

3-s16 The figure shows a vertical cross-section of a cylinder. A gun at the top shoots a bullet horizontally. What is the minimum speed at which the bullet must be shot in order to completely clear the cylinder?

⋆



Problem 3-s16.

3-s17 A battleship has a gun that fires shells with muzzle velocity u, and the gunner wants to hit an enemy ship at range L. The gun is to be fired at some angle θ above the horizontal. Find the two possible values of θ that are possible under normal conditions. Under what conditions are there one solution or no solutions at all? To what extent is the form of the solution determined by units?

4 Newton's laws, part 1

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

4.1 Newton's first law

Isaac Newton (1643-1727) originated the idea of explaining all events, both on earth and in the heavens, using a set of simple and universal mathematical laws. His three laws talk about *forces*, so what is a force?

In figure 4.1, the legendary Baron von Munchausen lifts himself and his horse out of a swamp by pulling up on his own pigtail. This is not actually possible, because an object can't accelerate by exerting a force on itself. A force is always an interaction between *two* objects.

Figure 4.1: Escaping from a swamp.

The left side of figure 4.2 shows a hand making a force on a rope. Two objects: hand and rope.

A force refers to a direct cause, not an indirect one. A pool player makes a force on the cue stick, but not on the cue ball.

To finish defining what we mean by a force, we need to say how we would measure a force numerically. In the right-hand side, the stretching of a spring is a measure of the hand's force. The SI unit of force is the newton (N), which we will see later is actually defined in a convenient way in terms of the base units of the SI. Force is a vector.

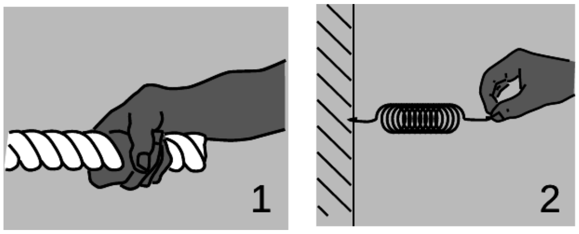


Figure 4.2: Forces.

Suppose that we can prevent any forces at all from acting on an object, perhaps by moving it far away from all other objects, or surrounding it with shielding. (For example, there is a nickel-iron alloy marketed as "mu-metal" which blocks magnetic forces very effectively.) *Newton's first law* states that in this situation, the object has a zero acceleration vector, i.e., its velocity vector is constant. If the object is already at rest, it remains at rest. If it is already in motion, it remains in motion at constant speed in the same direction.

Newton's first law is a more detailed and quantitative statement of the law of inertia. The first law holds in an inertial frame of reference; in fact, this is just a restatement of what we mean by an inertial frame.

The first law may not seem very useful for applications near the earth's surface, since an object there will always be subject at least to the

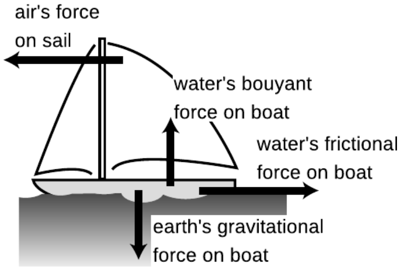


Figure 4.3: The four forces on the sailboat cancel out.

force of gravity. But the first law can also be extended to apply to cases in which forces do act on an object, but they cancel out. An example is the sailboat in figure 4.3.

An object can rotate or change its shape. A cat does both of these things when it falls and brings its feet under itself before it hits the ground. In such a situation, it is not immediately obvious what is meant by "the" velocity of the object. We will see later that Newton's first law can still be made to hold in such cases if we measure its motion by using a special point called its center of mass, which is the point on which it would balance. In the example of Baron von Munchausen, it is certainly possible for one part of his body to accelerate another part of his body by making a force on it; however, this will have no effect on the motion of his center of mass.

4.2 Newton's second law

Newton's second law tells us what happens when the forces acting on an object do *not* cancel out. The object's acceleration is then given by

$$\mathbf{a} = \frac{\mathbf{F}_{\text{total}}}{m}, \tag{4.1}$$

where F_{total} is the vector sum of all the forces, and m is the object's mass. Mass is a permanent property of an object that measures its inertia, i.e., how much it resists a change in its motion. Since the SI unit of mass is the kilogram, it follows from Newton's second law that the newton is related to the base units of the SI as $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$.

The force that the earth's gravity exerts on an object is called its *weight*, which is not the same thing as its mass. An object of mass m has weight mg (problem 4-a1, p. 46).

4.3 Newton's third law

We have seen that a force is always an interaction between two objects. *Newton's third law* states that these forces come in pairs. If object A exerts a force on object B, then B also exerts a force on A. The two forces have equal magnitudes but are in opposite directions. In symbols,

$$\mathbf{F}_{\text{A on B}} = -\mathbf{F}_{\text{B on A}}. \tag{4.2}$$

Newton's third law holds regardless of whether everything is in a state of equilibrium. It might seem as though the two forces would cancel out, but they can't cancel out because it doesn't even make sense to add them in the first place. They act on different objects, and it only makes sense to add forces if they act on the same object.

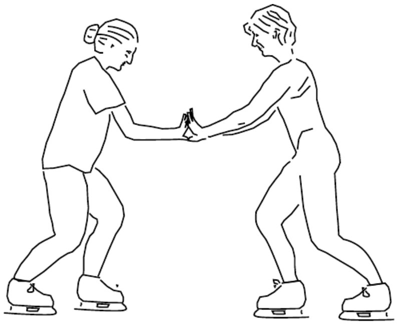


Figure 4.4: Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.

The pair of forces related by Newton's third law are always of the same type. For example, the hand in the left side of figure 4.2 makes a frictional force to the right on the rope. Newton's third law tells us that the rope exerts a force on the hand that is to the left and of the same strength. Since one of these forces is frictional, the other is as well.

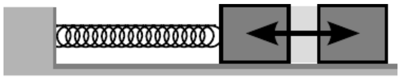
Problems

4-a1 (a) Why would it not make sense to say that the force of gravity acting on an object equals g?
(b) Why would it not make sense to say that the force of gravity acting on an object of mass m equals m?
(c) Use Newton's second law to prove that the magnitude of the gravitational force acting on an object of mass m equals mg.

▷ Solution, p. 255

4-a2 You are given a large sealed box, and are not allowed to open it. Which of the following experiments measure its mass, and which measure its weight? [Hint: Which experiments would give different results on the moon?]
(a) Put it on a frozen lake, throw a rock at it, and see how fast it scoots away after being hit.
(b) Drop it from a third-floor balcony, and measure how loud the sound is when it hits the ground.
(c) As shown in the figure, connect it with a spring to the wall, and watch it vibrate.

▷ Solution, p. 256



Problem 4-a2.

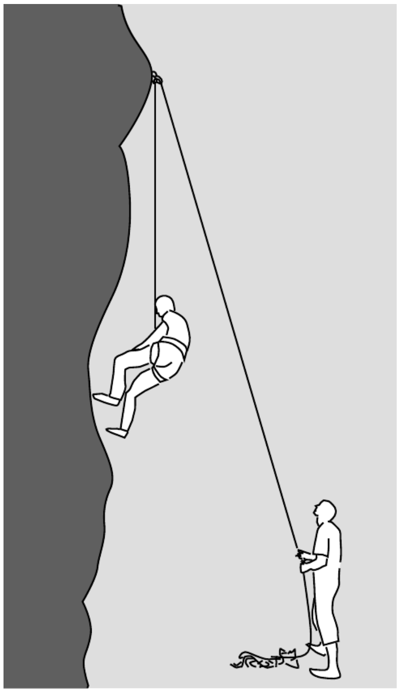
4-a3 (a) Compare the mass of a one-liter water bottle on earth, on the moon, and in interstellar space.
(b) Do the same for its weight.

4-a4 In the figure, the rock climber has finished the climb, and his partner is lowering him back down to the ground at approximately constant speed. The following is a student's analysis of the forces acting *on the climber*. The arrows give the directions of the forces.

force of the earth's gravity, ↓
force from the partner's hands, ↑
force from the rope, ↑

The student says that since the climber is moving down, the sum of the two upward forces must be slightly less than the downward force of gravity.

Correct all mistakes in the above analysis.



Problem 4-a4.

4-a5 A car is accelerating forward along a straight road. If the force of the road on the car's wheels, pushing it forward, is a constant 3.0 kN, and the car's mass is 1000 kg, then how long will the car take to go from 20 m/s to 50 m/s?

▷ Solution, p. 256

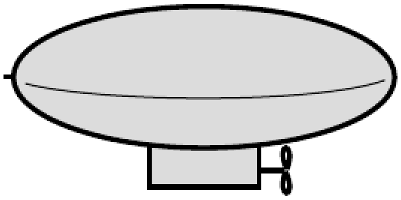
4-a6 An object is observed to be moving at constant speed in a certain direction. Can you conclude that no forces are acting on it? Explain. [Based on a problem by Serway and Faughn.]

4-a7 A book is pushed along a frictionless table with a constant horizontal force. The book starts from rest and travels 2.0 m in 1.0 s. If the same force continues, how far will the book travel in the next 1.0 s?

√

4-d1 A blimp is initially at rest, hovering, when at $t = 0$ the pilot turns on the engine driving the propeller. The engine cannot instantly get the propeller going, but the propeller speeds up steadily. The steadily increasing force between the air and the propeller is given by the equation $F = kt$, where k is a constant. If the mass of the blimp is m, find its position as a function of time. (Assume that during the period of time you're dealing with, the blimp is not yet moving fast enough to cause a significant backward force due to air resistance.)

√



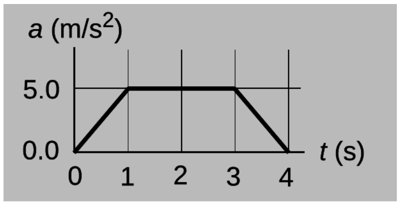
Problem 4-d1.

4-g1 The acceleration of a 1.0 kg object is given in the graph.

(a) What is the *maximum* force that acts on the object over the time interval $[0.0, 4.0]$ s? √

(b) What is the *average* force over the same interval?

√



Problem 4-g1.

4-g2 Flaca has brought a bathroom scale with her in an elevator and is standing on it. Just before the elevator arrives at the top floor, as the car is slowing down, she notices that according to the scale, her weight appears to be off by 3% from its normal value W.

(a) Does the scale read $0.97W$, or $1.03W$? √

(b) What is the magnitude of the acceleration of the elevator?

√

4-g3 A person who normally weighs 890 N is standing on a scale inside an elevator. The elevator is moving upward with a speed of 10 m/s, and then begins to decelerate at a rate of 5.0 m/s^2.

(a) Before the elevator begins to decelerate, what is the reading on the scale? √

(b) What about while the elevator is slowing down?

√

4-g4 A bullet of mass m is fired from a pistol, accelerating from rest to a speed v in the barrel's length L.

(a) What is the force on the bullet? (Assume this force is constant.) √

(b) Check that the units of your answer to part a make sense.

(c) Check that the dependence of your answer on each of the three variables makes sense.

4-g5 You push a cup of mass M across a table, using a force of magnitude F. Because of a second, frictional force, the cup's acceleration only has magnitude $F/3M$. What is the magnitude of this frictional force?

√

4-g6 In an experiment, a force is applied to two different unknown masses. This force causes the first object, with mass m_1, to have acceleration a_1, and gives an object of mass m_2 an acceleration a_2, where $a_1 > a_2$.

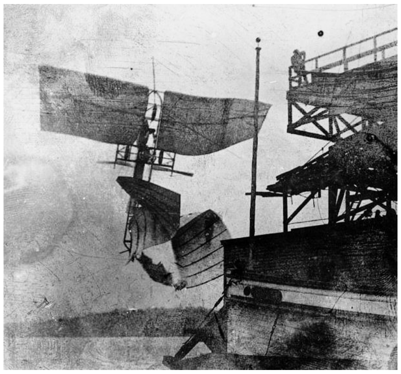
(a) Which mass is heavier: m_1 or m_2?

(b) Based on the experimental data (a_1 and a_2), what acceleration would the force give to an object of mass $m_1 + m_2$?

√

4-j1 At low speeds, every car's acceleration is limited by traction, not by the engine's power. Suppose that at low speeds, a certain car is normally capable of an acceleration of 3 m/s^2. If it is towing a trailer with half as much mass as the car itself, what acceleration can it achieve? [Based on a problem from PSSC Physics.]

4-j2 The tires used in Formula 1 race cars can generate traction (i.e., force from the road) that is as much as 1.9 times greater than with the tires typically used in a passenger car. Suppose that we're trying to see how fast a car can cover a fixed distance starting from rest, and traction is the limiting factor. By what factor is this time reduced when switching from ordinary tires to Formula 1 tires? √



Problem 4-j3.

4-j3 At the turn of the 20th century, Samuel Langley engaged in a bitter rivalry with the Wright brothers to develop human flight. Langley's design used a catapult for launching. For safety, the catapult was built on the roof of a houseboat, so that any crash would be into the water. This design required reaching cruising speed within a fixed, short distance, so large accelerations were required, and the forces frequently damaged the craft, causing dangerous and embarrassing accidents. Langley achieved several uncrewed, unguided flights, but never succeeded with a human pilot. If the force of the catapult is fixed by the structural strength of the plane, and the distance for acceleration by the size of the houseboat, by what factor is the launch velocity reduced when the plane's 340 kg is augmented by the 60 kg mass of a small man? √

4-j4 In the 1964 Olympics in Tokyo, the best men's high jump was 2.18 m. Four years later in Mexico City, the gold medal in the same event was for a jump of 2.24 m. Because of Mexico City's altitude (2400 m), the acceleration of gravity there is lower than that in Tokyo by about 0.01 m/s^2. Suppose a high-jumper has a mass of 72 kg.

(a) Compare his mass and weight in the two locations.

(b) Assume that he is able to jump with the same initial vertical velocity in both locations, and that all other conditions are the same except for gravity. How much higher should he be able to jump in Mexico City? √

(Actually, the reason for the big change between '64 and '68 was the introduction of the "Fosbury flop.")

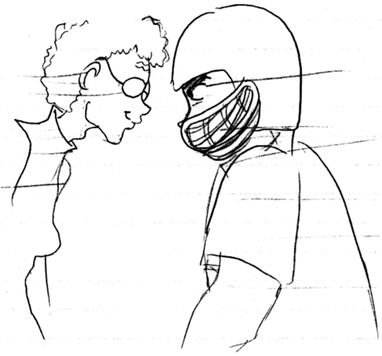
4-j5 Your friend, who's kind of an idiot, jumps out of a third-story window. After falling 7.0 m, he lands on his stomach so that as his body compresses on impact, his center of mass only moves 0.020 m. What is the average force of the ground on your friend as he smacks the floor? Express your answer in terms of his weight W. √

4-j6 A book is pushed along a frictionless table with a constant horizontal force. The book starts from rest and travels 2.0 m in 1.0 s. If the same experiment is carried out again, but with a book having twice the mass, how far will this heavier book travel? √

4-m1 In each case, identify the force that causes the acceleration, and give its Newton's-third-law partner. Describe the effect of the partner force. (a) A swimmer speeds up. (b) A golfer hits the ball off of the tee. (c) An archer fires an arrow. (d) A locomotive slows down.

▷ Solution, p. 257

4-m2 A little old lady and a pro football player collide head-on. Compare their forces on each other, and compare their accelerations. Explain.



Problem 4-m2.

4-m3 The earth is attracted to an object with a force equal and opposite to the force of the earth on the object. If this is true, why is it that when you drop an object, the earth does not have an acceleration equal and opposite to that of the object?

4-m4 When you stand still, there are two forces acting on you, the force of gravity (your weight) and the normal force of the floor pushing up on your feet. Are these forces equal and opposite? Does Newton's third law relate them to each other? Explain.

4-m5 Some garden shears are like a pair of scissors: one sharp blade slices past another. In the "anvil" type, however, a sharp blade presses against a flat one rather than going past it. A gardening book says that for people who are not very physically strong, the anvil type can make it easier to cut tough branches, because it concentrates the force on one side. Evaluate this claim based on Newton's laws. [Hint: Consider the forces acting on the branch, and the motion of the branch.]

$\star$

4-m6 Pick up a heavy object such as a backpack or a chair, and stand on a bathroom scale. Shake the object up and down. What do you observe? Interpret your observations in terms of Newton's third law.

$\star$

4-p1 (a) Let T be the maximum tension that an elevator's cable can withstand without breaking, i.e., the maximum force it can exert. If the motor is programmed to give the car an acceleration a ($a > 0$ is upward), what is the maximum mass that the car can have, including passengers, if the cable is not to break? $\checkmark$
(b) Interpret the equation you derived in the special cases of $a = 0$ and of a downward acceleration of magnitude g.

4-p2 While escaping from the palace of the evil Martian emperor, Sally Spacehound jumps from a tower of height h down to the ground. Ordinarily the fall would be fatal, but she fires her blaster rifle straight down, producing an upward force of magnitude F_B. This force is insufficient to levitate her, but it does cancel out some of the force of gravity. During the time t that she is falling, Sally is unfortunately exposed to fire from the emperor's minions, and can't dodge their shots. Let m be her mass, and g the strength of gravity on Mars.
(a) Find the time t in terms of the other variables.
(b) Check the units of your answer to part a.
(c) For sufficiently large values of F_B, your answer to part a becomes nonsense — explain what's going on. $\checkmark$

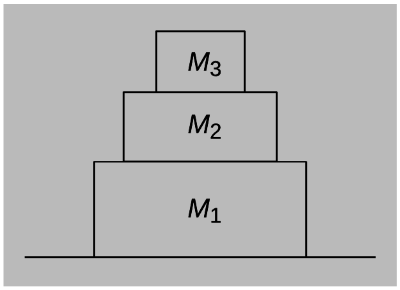
4-p3 A helicopter of mass m is taking off vertically. The only forces acting on it are the earth's gravitational force and the force, F_{air}, of the air pushing up on the propeller blades.
(a) If the helicopter lifts off at $t = 0$, what is its vertical speed at time t?
(b) Check that the units of your answer to part a make sense.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what

would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically.*
(d) Plug numbers into your equation from part a, using $m = 2300$ kg, $F_{air} = 27000$ N, and $t = 4.0$ s.
✓

4-p4 A uranium atom deep in the earth spits out an alpha particle. An alpha particle is a fragment of an atom. This alpha particle has initial speed v, and travels a distance d before stopping in the earth.
(a) Find the force, F, from the dirt that stopped the particle, in terms of v, d, and its mass, m. Don't plug in any numbers yet. Assume that the force was constant. ✓
(b) Show that your answer has the right units.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically.*
(d) Evaluate your result for $m = 6.7 \times 10^{-27}$ kg, $v = 2.0 \times 10^{4}$ km/s, and $d = 0.71$ mm. ✓

4-p5 A car is pushing a truck from behind. The car has mass M, and the truck has mass $3M$. If the maximum force that the ground can provide to the cars' tires is F, what is the maximum force between the two vehicles? Assume no other horizontal forces act on the truck.
✓

4-p6 Blocks of mass M_1, M_2, and M_3 are stacked on a table as shown in the figure. Let the upward direction be positive.
(a) What is the force on block 2 from block 3? ✓
(b) What is the force on block 2 from block 1?
✓



Problem 4-p6.

4-s1 When I cook rice, some of the dry grains always stick to the measuring cup. To get them out, I turn the measuring cup upside-down and hit the "roof" with my hand so that the grains come off of the "ceiling." (a) Explain why static friction is irrelevant here. (b) Explain why gravity is negligible. (c) Explain why hitting the cup works, and why its success depends on hitting the cup hard enough.
⋆

4-s2 The following reasoning leads to an apparent paradox; explain what's wrong with the logic. A baseball player hits a ball. The ball and the bat spend a fraction of a second in contact. During that time they're moving together, so their accelerations must be equal. Newton's third law says that their forces on each other are also equal. But $a = F/m$, so how can this be, since their masses are unequal? (Note that the paradox isn't resolved by considering the force of the batter's hands on the bat. Not only is this force very small compared to the ball-bat force, but the batter could have just thrown the bat at the ball.)
⋆

5 Newton's laws, part 2

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

5.1 Classification of forces

A fundamental and still unsolved problem in physics is the classification of the forces of nature. Ordinary experience suggests to us that forces come in different types, which behave differently. Frictional forces seem clearly different from magnetic forces.

But some forces that appear distinct are actually the same. For instance, the friction that holds a nail into the wall seems different from the kind of friction that we observe when fluids are involved — you can't drive a nail into a waterfall and make it stick. But at the atomic level, both of these types of friction arise from atoms bumping into each other. The force that holds a magnet on your fridge also seems different from the force that makes your socks cling together when they come out of the dryer, but it was gradually realized, starting around 1800 and culminating with Einstein's theory of relativity in 1905, that electricity and magnetism are actually closely related things, and observers in different states of motion do not even agree on what is an electric force and what is a magnetic one. The tendency has been for more and more superficially disparate forces to become unified in this way.

Today we have whittled the list down to only three types of interactions at the subatomic level (called the gravitational, electroweak, and strong forces). It is possible that some future theory of physics will reduce the list to only one — which Star Wars fans would then probably want to call "The Force."

Nevertheless, there is a practical classification of forces that works pretty well for objects on the human scale, and that is usually more convenient. Figure **??** on p. **??** shows this scheme in the form of a tree.

5.2 Friction

If you push a refrigerator across a kitchen floor, you will find that as you make more and more force, at first the fridge doesn't move, but that eventually when you push hard enough, it unsticks and starts to slide. At the moment of unsticking, static friction turns into kinetic friction. Experiments support the following approximate model of friction when the objects are solid, dry, and rigid. We have two unitless coefficients μ_s and μ_k, which depend only on the types of surfaces. The maximum force of static friction is limited to

$$F_s \leq \mu_s F_N, \tag{5.1}$$

where F_N is the normal force between the surfaces, i.e., the amount of force with which they are being pressed together. Kinetic friction is given by

$$F_k = \mu_k F_N. \tag{5.2}$$

5.3 Elasticity

When a force is applied to a solid object, it will change its shape, undergoing some type of deformation such as flexing, compression, or expansion. If the force is small enough, then this change is proportional to the force, and when the force is removed the object will resume its original shape. A simple example is a spring being stretched or compressed. If the spring's relaxed length is x_0, then its length x is related to the

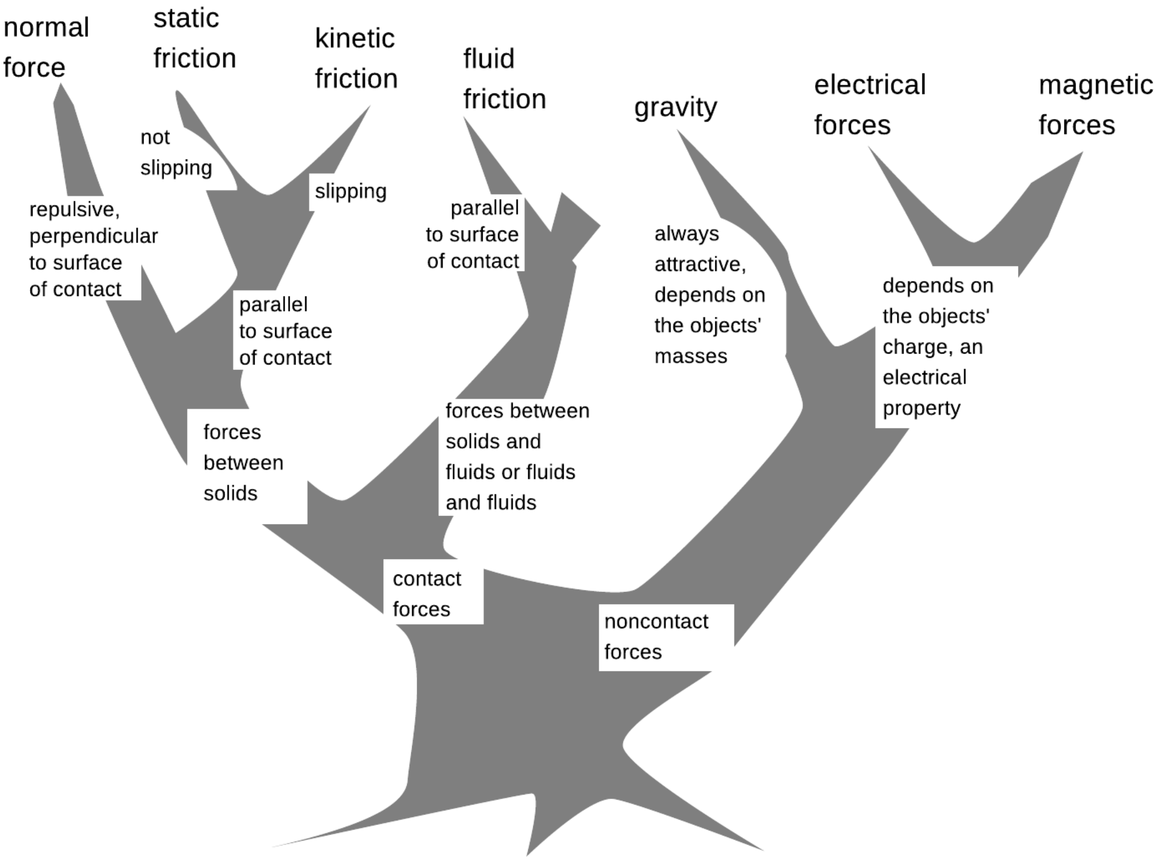

normal force
static friction
kinetic friction
fluid friction
gravity
electrical forces
magnetic forces
not slipping
slipping
repulsive, perpendicular to surface of contact
parallel to surface of contact
parallel to surface of contact
always attractive, depends on the objects' masses
depends on the objects' charge, an electrical property
forces between solids
forces between solids and fluids or fluids and fluids
contact forces
noncontact forces

Figure 5.1: A practical classification scheme for forces.

force applied to it by *Hooke's law*,

$$F \approx k(x - x_0). \qquad (5.3)$$

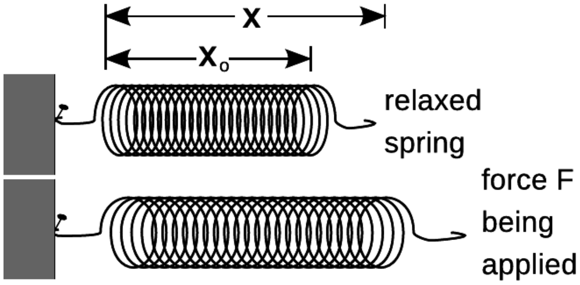


Figure 5.2: Hooke's law.

5.4 Ropes, pulleys, tension, and simple machines

If you look carefully at a piece of rope or yarn while tightening it, you will see a physical change in the fibers. This is a manifestation of the fact that there is tension in the rope. Tension is a scalar with units of newtons. For a rope of negligible mass, the tension is constant throughout the rope, and it equals the magnitudes of the forces at its ends. This is still true if the rope goes around a frictionless post, or a massless pulley with a frictionless axle. You can't push with a rope, you can only pull. A rigid object such as a pencil can, however, sustain compression, which is equivalent to negative tension.

A pulley is an example of a *simple machine*, which is a device that can amplify a force by some factor, while reducing the amount of motion by the inverse of that factor. Another example of a simple machine is the gear system on a bicycle.

We can put more than one simple machine together in order to give greater amplification of forces or to redirect forces in different directions. For an idealized system,[1] the fundamental principles are:

1. The total force acting on any pulley is zero.[2]
2. The tension in any given piece of rope is constant throughout its length.
3. The length of every piece of rope remains the same.

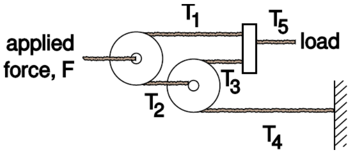


Figure 5.3: A complicated pulley system. The bar is massless.

As an example, let us find the mechanical advantage T_5/F of the pulley system shown in figure 5.3. By rule 2, $T_1 = T_2$, and by rule 1, $F = T_1 + T_2$, so $T_1 = T_2 = F/2$. Similarly, $T_3 = T_4 = F/4$. Since the bar is massless, the same reasoning that led to rule 1 applies to the bar as well, and $T_5 = T_1 + T_3$. The mechanical advantage is $T_5/F = 3/4$, i.e., this pulley system *reduces* the input force.

5.5 Analysis of forces

Newton's second law refers to the total force acting on a particular object. Therefore whenever we want to apply the second law, a necessary preliminary step is to pick an object and list all the forces acting on it. In addition, it may be helpful to determine the types and directions of the forces and also to identify the Newton's-third-law partners of those forces, i.e., all the forces that our object exerts back on other things.

---

[1] In such a system: (1) The ropes and pulleys have negligible mass. (2) Friction in the pulleys' bearings is negligible. (3) The ropes don't stretch.

[2] $F = ma$, and $m = 0$ since the pulley's mass is assumed to be negligible.

force acting on Fifi	*force related to it by Newton's third law*
planet earth's gravitational force $F_W = mg$ on Fifi, ↓	Fifi's gravitational force on earth, ↑
belt's kinetic frictional force F_k on Fifi, →	Fifi's kinetic frictional force on belt, ←
belt's normal force F_N on Fifi, ↑	Fifi's normal force on belt, ↓

Table 5.1: Analysis of the forces on the dog.

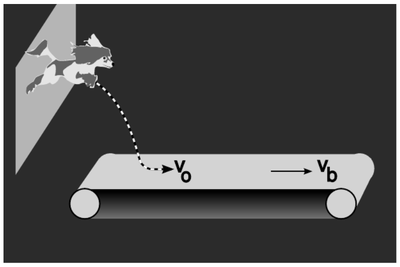


Figure 5.4: The spy dog lands on the moving conveyor belt.

As an example, consider figure 5.4. Fifi is an industrial espionage dog who loves doing her job and looks great doing it. She leaps through a window and lands at initial horizontal speed v_o on a conveyor belt which is itself moving at the greater speed v_b. Unfortunately the coefficient of kinetic friction μ_k between her foot-pads and the belt is fairly low, so she skids, and the effect on her coiffure is *un désastre*. Table 5.1 shows the resulting analysis of the forces in which she participates.

Problems

In problems 5-a1-5-a5, analyze the forces using a table in the format shown in section 5.5 on p. 53. Analyze the forces in which the italicized object participates.

5-a1 Some people put a spare car key in a little magnetic *box* that they stick under the chassis of their car. Let's say that the box is stuck directly underneath a horizontal surface, and the car is parked. (See instructions above.)

5-a2 Analyze two examples of *objects* at rest relative to the earth that are being kept from falling by forces other than the normal force. Do not use objects in outer space, and do not duplicate problem 5-a1 or 5-a5. (See instructions above.)

5-a3 A *person* is rowing a boat, with her feet braced. She is doing the part of the stroke that propels the boat, with the ends of the oars in the water (not the part where the oars are out of the water). (See instructions above.)

5-a4 A *farmer* is in a stall with a cow when the cow decides to press him against the wall, pinning him with his feet off the ground. Analyze the forces in which the farmer participates. (See instructions above.)

5-a5 A propeller *plane* is cruising east at constant speed and altitude. (See instructions above.)

5-a6 Someone tells you she knows of a certain type of Central American earthworm whose skin, when rubbed on polished diamond, has $\mu_k > \mu_s$. Why is this not just empirically unlikely but logically suspect?

⋆

5-d1 The figure shows a boy hanging in three positions: (1) with his arms straight up, (2) with his arms at 45 degrees, and (3) with his arms at 60 degrees with respect to the vertical. Compare the tension in his arms in the three cases.

5-d2 For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a symmetric "Y" at an angle θ. The metal piece at the center is called a carabiner. The usual advice is to make $\theta < 90°$; for large values of θ, the stress placed on the anchors can be many times greater than the actual load L, so that two anchors are actually *less* safe than one.

(a) Find the force S at each anchor in terms of L and θ. ✓

(b) Verify that your answer makes sense in the case of $\theta = 0$.

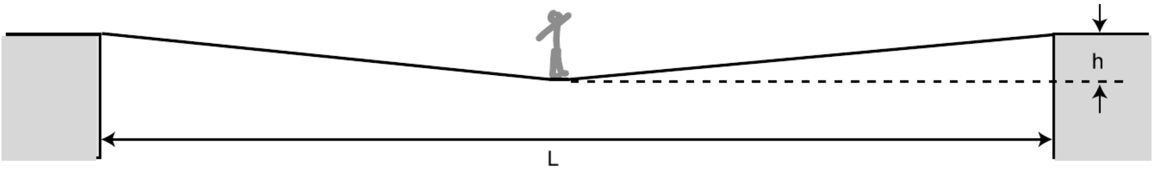
(c) Interpret your answer in the case of $\theta = 180°$.

(d) What is the smallest value of θ for which S equals or exceeds L, so that for larger angles a failure of at least one anchor is *more* likely than it would have been with a single anchor? ✓

5-d3 Problem 5-d2 discussed a possible correct way of setting up a redundant anchor for mountaineering. The figure for this problem shows an incorrect way of doing it, by arranging the rope in a triangle (which we'll take to be isoceles). One of the bad things about the triangular arrangement is that it requires more force from the anchors, making them more likely to fail. (a) Using the same notation as in problem 5-d2, find S in terms of L and θ. ✓

(b) Verify that your answer makes sense in the case of $\theta = 0$, and compare with the correct setup.

5-d4 A person of mass M stands in the middle of a tightrope, which is fixed at the ends to two buildings separated by a horizontal distance L. The rope sags in the middle, stretching and



Problem 5-d4.

Problem 5-a3.

Problem 5-a4.

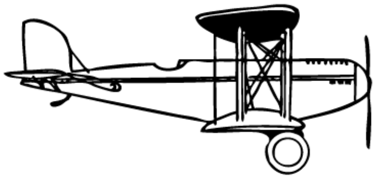
lengthening the rope slightly.

(a) If the tightrope walker wants the rope to sag vertically by no more than a height h, find the minimum tension, T, that the rope must be able to withstand without breaking, in terms of h, g, M, and L. ✓

(b) Based on your equation, explain why it is not possible to get $h = 0$, and give a physical interpretation.

5-d5 The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose.

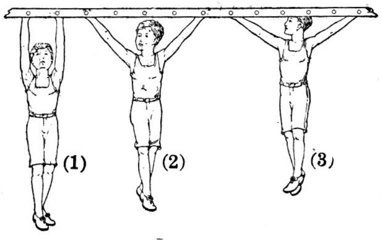
(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.



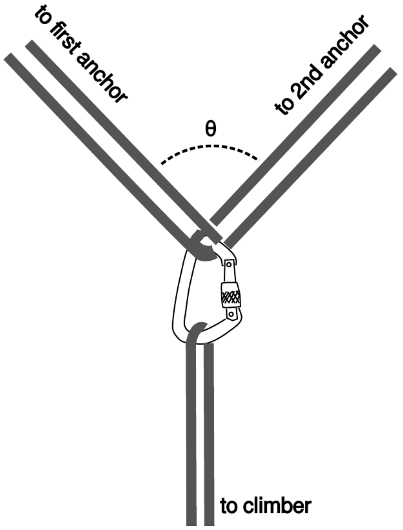
Problem 5-a5.

(b) On an asteroid, where g can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?

5-d6 Your hand presses a block of mass m against a wall with a force $\mathbf{F}_H$ acting at an angle θ, as shown in the figure. Find the minimum and maximum possible values of $|\mathbf{F}_H|$ that can keep the block stationary, in terms of m, g, θ, and μ_s, the coefficient of static friction between the block and the wall. Check both your answers in



Problem 5-d1.



Problem 5-d2.

the case of $\theta = 90°$, and interpret the case where the maximum force is infinite.

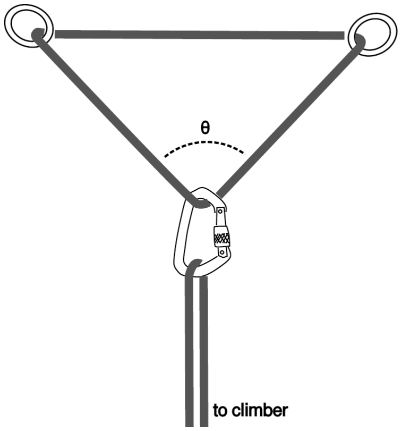
√ ⋆

5-d7 A telephone wire of mass m is strung between two poles, making an angle θ with the horizontal at each end. (a) Find the tension at the center. √
(b) Which is greater, the tension at the center or at the ends?

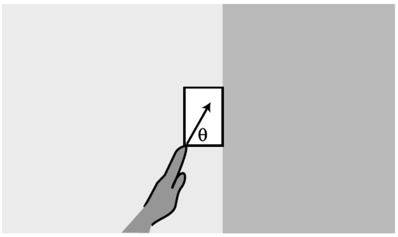
⋆

5-d8 The photo shows a coil of rope wound around a smooth metal post. A large amount of tension is applied at the bottom of the coil, but only a tiny force, supplied by a piece of sticky tape, is needed at the top to keep the rope from slipping. Show that the ratio of these two forces increases exponentially with the number of turns of rope, and find an expression for that ratio.

√ ⋆⋆

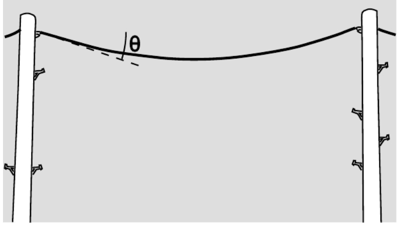


Problem 5-d3.

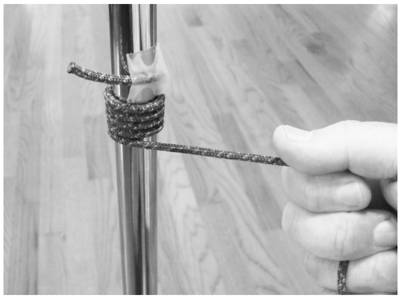


Problem 5-d6.

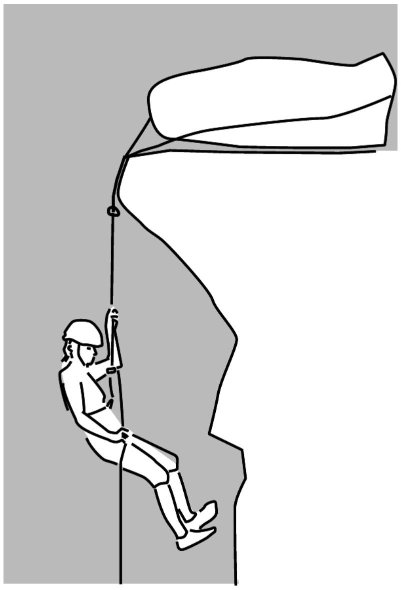
5-d9 The figure shows a mountaineer doing a vertical rappel. Her anchor is a big boulder. The American Mountain Guides Association suggests as a rule of thumb that in this situation, the boulder should be at least as big as a refrigerator, and should be sitting on a surface that is horizontal rather than sloping. The goal of this problem is to estimate what coefficient of static friction μ_s between the boulder and the ledge is required if this setup is to hold the person's body weight. For comparison, reference books meant for civil engineers building walls out of granite blocks state that granite on granite typically has a $\mu_s \approx 0.6$. We expect the result of our calculation to be much less than this, both because a large margin of safety



Problem 5-d7.



Problem 5-d8.



Problem 5-d9.

is desired and because the coefficient could be much lower if, for example, the surface was sandy rather than clean. We will assume that there is no friction where the rope goes over the lip of the cliff, although in reality this friction significantly reduces the load on the boulder.

(a) Let m be the mass of the climber, V the volume of the boulder, ρ its density, and g the strength of the gravitational field. Find the minimum value of μ_s. ✓

(b) Show that the units of your answer make sense.

(c) Check that its dependence on the variables makes sense.

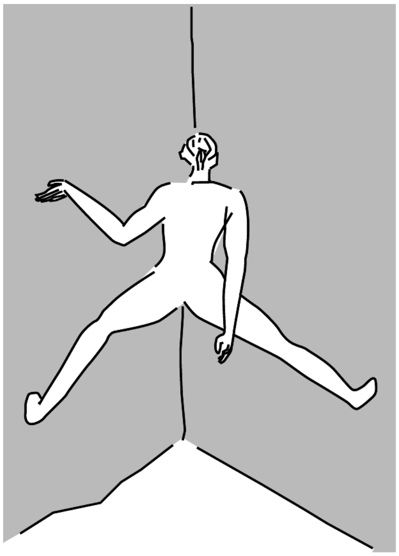
(d) Evaluate your result numerically. The volume of my refrigerator is about 0.7 m^3, the density of granite is about 2.7 g/cm^3, and standards bodies use a body mass of 80 kg for testing climbing equipment. ✓

5-d10 The figure shows a rock climber wedged into a dihedral or "open book" consisting of two vertical walls of rock at an angle θ relative to one another. This position can be maintained without any ledges or holds, simply by pressing the feet against the walls. The left hand is being used just for a little bit of balance. (a) Find the minimum coefficient of friction between the rubber climbing shoes and the rock. (b) Interpret the behavior of your expression at extreme values of θ. (c) Steven Won has done tabletop experiments using climbing shoes on the rough back side of a granite slab from a kitchen countertop, and has estimated $\mu_s = 1.17$. Find the corresponding maximum value of θ.

▷ Solution, p. 264

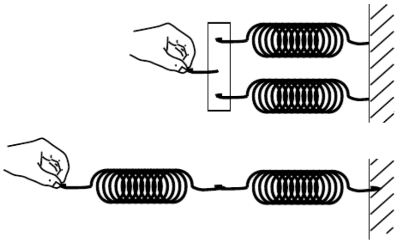
5-g1 The figure shows two different ways of combining a pair of identical springs, each with spring constant k. We refer to the top setup as parallel, and the bottom one as a series arrangement.

(a) For the parallel arrangement, analyze the forces acting on the connector piece on the left, and then use this analysis to determine the equivalent spring constant of the whole setup. Explain whether the combined spring constant should be interpreted as being stiffer or less stiff.



Problem 5-d10.

(b) For the series arrangement, analyze the forces acting on each spring and figure out the same things.



Problem 5-g1.

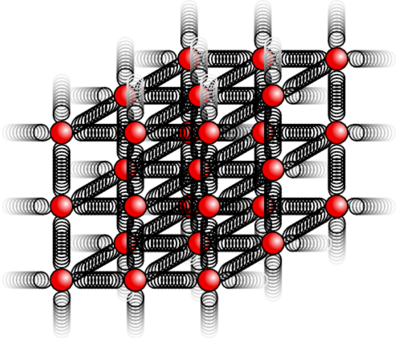
5-g2 Generalize the results of problem 5-g1 to the case where the two spring constants are unequal.

⋆

5-g3 (a) Using the solution of problem 5-g1, which is given in the back of the book, predict how the spring constant of a fiber will depend on its length and cross-sectional area.
(b) The constant of proportionality is called the Young's modulus, E, and typical values of the Young's modulus are about 10^{10} to 10^{11}. What units would the Young's modulus have in the SI (meter-kilogram-second) system?

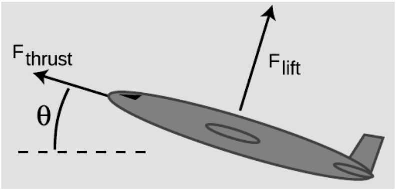
5-g4 This problem depends on the results of problems 5-g1 and 5-g3, whose solutions are in the back of the book. When atoms form chemical bonds, it makes sense to talk about the spring constant of the bond as a measure of how "stiff" it is. Of course, there aren't really little springs — this is just a mechanical model. The purpose of this problem is to estimate the spring constant, k, for a single bond in a typical piece of solid matter. Suppose we have a fiber, like a hair or a piece of fishing line, and imagine for simplicity that it is made of atoms of a single element stacked in a cubical manner, as shown in the figure, with a center-to-center spacing b. A typical value for b would be about 10^{-10} m.
(a) Find an equation for k in terms of b, and in terms of the Young's modulus, E, defined in problem 16 and its solution.
(b) Estimate k using the numerical data given in problem 5-g3.
(c) Suppose you could grab one of the atoms in a diatomic molecule like H_2 or O_2, and let the other atom hang vertically below it. Does the bond stretch by any appreciable fraction due to gravity?

⋆



Problem 5-g4.

5-j1 A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of $\theta = 17°$ with respect to horizontal. Its engines supply a thrust of $F_{\text{thrust}} = 200$ kN, and the lift from its wings is $F_{\text{lift}} = 654$ kN. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg?



Problem 5-j1.

5-j2 A toy manufacturer is playtesting teflon booties that slip on over your shoes. In the parking lot, giggling engineers find that when they start with an initial speed of 1.2 m/s, they glide for 2.0 m before coming to a stop. What is the coefficient of friction between the asphalt and the booties?

✓

5-j3 A small piece of styrofoam packing material is dropped from rest at a height 2.00 m above the ground at time $t = 0$. The magnitude of its acceleration is given by $a = g - bv$, where v is the speed of the styrofoam, $g = 9.8$ m/s^2, and b is a positive constant. After falling 0.500 m, the styrofoam effectively reaches terminal speed and then takes 5.00 s more to reach the ground.

(a) What is the acceleration (magnitude and direction) when $t = 0$? What about when the styrofoam is halfway to the ground?

(b) Find the terminal speed of the styrofoam. ✓

(c) What is the value of the constant b, with units? ✓

(d) What is the acceleration when the speed is 0.150 m/s? ✓

(e) Write $a = \mathrm{d}v/\,\mathrm{d}t$ and solve the differential equation for $v(t)$ (without plugging in numbers), and plot the result.

✓

5-m1 Ice skaters with masses m_1 and m_2 push off from each other with a constant force F, which lasts until they lose contact. The distance between their centers of mass is ℓ_0 initially and ℓ_f when they lose contact.

(a) Find the amount of time T for which they remain in contact.

(b) Show that your answer in part a has units that make sense.

(c) Show that your answer has the right dependence on F.

(d) Interpret the case where one of the masses is very small.

▹ Solution, p. 266

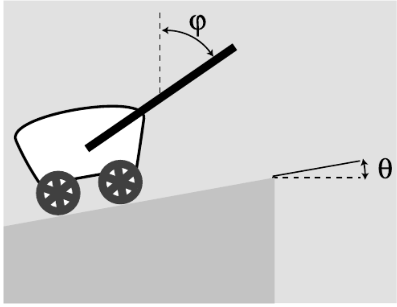
5-m2 A wagon is being pulled at constant speed up a slope θ by a rope that makes an angle ϕ with the vertical.

(a) Assuming negligible friction, show that the tension in the rope is given by the equation

$$F_T = \frac{\sin\theta}{\sin(\theta+\phi)} F_W,$$

where F_W is the weight force acting on the wagon.

(b) Interpret this equation in the special cases of $\phi = 0$ and $\phi = 180° - \theta$.

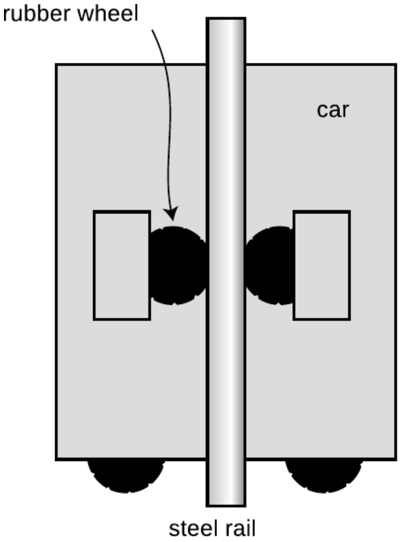


Problem 5-m2.

5-m3 Today's tallest buildings are really not that much taller than the tallest buildings of the 1940's. One big problem with making an even taller skyscraper is that every elevator needs its own shaft running the whole height of the building. So many elevators are needed to serve the building's thousands of occupants that the elevator shafts start taking up too much of the space within the building. An alternative is to have elevators that can move both horizontally and vertically: with such a design, many elevator cars can share a few shafts, and they don't get in each other's way too much because they can detour around each other. In this design, it becomes impossible to hang the cars from cables, so they would instead have to ride on rails which they grab onto with wheels. Friction would keep them from slipping. The figure shows such a frictional elevator in its vertical travel mode. (The wheels on the bottom are for when it needs to switch to horizontal motion.)

(a) If the coefficient of static friction between rubber and steel is μ_s, and the maximum mass of the car plus its passengers is M, how much force must there be pressing each wheel against the rail in order to keep the car from slipping? (Assume the car is not accelerating.) √

(b) Show that your result has physically reasonable behavior with respect to μ_s. In other words, if there was less friction, would the wheels need to be pressed more firmly or less firmly? Does your equation behave that way?



Problem 5-m3.

5-m4 A skier of mass m is coasting down a slope inclined at an angle θ compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is μ_k, and in addition the skier experiences an air friction force of magnitude bv^2, where b is a constant.

(a) Find the maximum speed that the skier will attain, in terms of the variables m, g, θ, μ_k, and b. √

(b) For angles below a certain minimum angle θ_{min}, the equation gives a result that is not mathematically meaningful. Find an equation for θ_{min}, and give a physical explanation of what is happening for $\theta < \theta_{min}$. √

5-m5 Driving down a hill inclined at an angle θ with respect to horizontal, you slam on the brakes to keep from hitting a deer. Your antilock brakes kick in, and you don't skid.

(a) Analyze the forces. (Ignore rolling resistance and air friction.)

(b) Find the car's maximum possible deceleration, a (expressed as a positive number), in terms of g, θ, and the relevant coefficient of friction. √

(c) Explain physically why the car's mass has no effect on your answer.

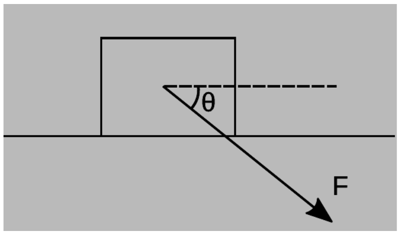
(d) Discuss the mathematical behavior and physical interpretation of your result for negative values of θ.

(e) Do the same for very large positive values of θ.

5-m6 An ice skater builds up some speed, and then coasts across the ice passively in a straight line. (a) Analyze the forces, using a table in the format shown in section 5.5.
(b) If his initial speed is v, and the coefficient of kinetic friction is μ_k, find the maximum theoretical distance he can glide before coming to a stop. Ignore air resistance. $\surd$
(c) Show that your answer to part b has the right units.
(d) Show that your answer to part b depends on the variables in a way that makes sense physically.
(e) Evaluate your answer numerically for $\mu_k = 0.0046$, and a world-record speed of 14.58 m/s. (The coefficient of friction was measured by De Koning et al., using special skates worn by real speed skaters.) $\surd$
(f) Comment on whether your answer in part e seems realistic. If it doesn't, suggest possible reasons why.

5-m7 A cop investigating the scene of an accident measures the length L of a car's skid marks in order to find out its speed v at the beginning of the skid. Express v in terms of L and any other relevant variables. $\surd$

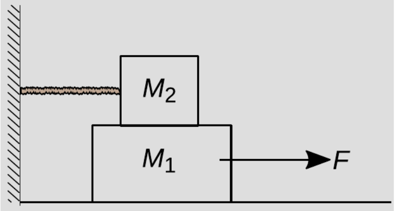
5-m8 A force F is applied to a box of mass M at an angle θ below the horizontal (see figure). The coefficient of static friction between the box and the floor is μ_s, and the coefficient of kinetic friction between the two surfaces is μ_k.
(a) What is the magnitude of the normal force on the box from the floor? $\surd$
(b) What is the minimum value of F to get the box to start moving from rest? $\surd$
(c) What is the value of F so that the box will move with constant velocity (assuming it is already moving)? $\surd$
(d) If θ is greater than some critical angle θ_crit, it is impossible to have the scenario described in part c. What is θ_crit? $\surd$



Problem 5-m8.

5-m9 A ramp of length L is inclined at angle θ to the horizontal. You would like to push your backpack, which has mass m, to the top of the ramp. The coefficient of kinetic friction between the backpack and the ramp is μ_k, and you will push the backpack with a force of magnitude F.
(a) How long will it take the backpack to reach the top of the ramp if you apply your push parallel to the ramp's surface? $\surd$
(b) How long will it take the backpack to reach the top of the ramp if you apply your push parallel to the the ground? $\surd$

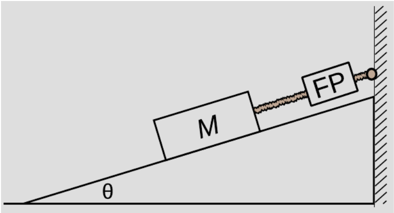
5-m10 Blocks M_1 and M_2 are stacked as shown, with M_2 on top. M_2 is connected by a string to the wall, and M_1 is pulled to the right with a force F big enough to get M_1 to move. The coefficient of kinetic friction has the same value μ_k among all surfaces (i.e., the block-block and ground-block interfaces).
(a) Analyze the forces in which each block participates, as in section 5.5.
(b) Determine the tension in the string. $\surd$
(c) Find the acceleration of the block of mass M_1. $\surd$



Problem 5-m10.

5-m11 (a) A mass M is at rest on a fixed, frictionless ramp inclined at angle θ with respect to the horizontal. The mass is connected to the force probe, as shown. What is the reading on the force probe? ✓

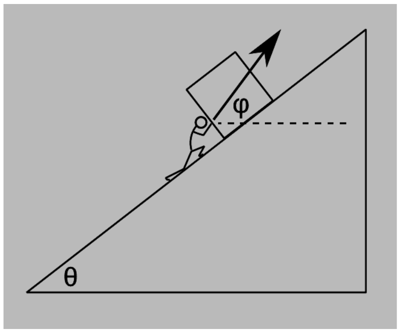
(b) Check that your answer to part a makes sense in the special cases $\theta = 0$ and $\theta = 90°$.



Problem 5-m11.

5-m12 You are pushing a box up a ramp that is at an angle θ with respect to the horizontal. Friction acts between the box and the ramp, with coefficient μ. Suppose that your force is fixed in magnitude, but can be applied at any desired angle φ above the horizontal. Find the optimal value of φ.

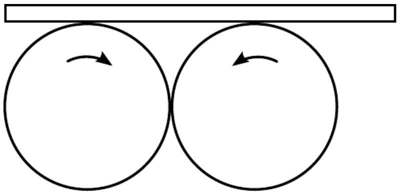
✓ ★



Problem 5-m12.

5-m13 Two wheels of radius r rotate in the same vertical plane with angular velocities $+\Omega$ and $-\Omega$ (rates of rotation in radians per second) about axes that are parallel and at the same height. The wheels touch one another at a point on their circumferences, so that their rotations mesh like gears in a gear train. A board is laid on top of the wheels, so that two friction forces act upon it, one from each wheel. Characterize the three qualitatively different types of motion that the board can exhibit, depending on the initial conditions.

★★



Problem 5-m13.

5-p1 A tugboat of mass m pulls a ship of mass M, accelerating it. The speeds are low enough that you can ignore fluid friction acting on their hulls, although there will of course need to be fluid friction acting on the tug's propellers.

(a) Analyze the forces in which the tugboat participates, using a table in the format shown in section 5.5. Don't worry about vertical forces.

(b) Do the same for the ship.

(c) If the force acting on the tug's propeller is F, what is the tension, T, in the cable connecting the two ships? [Hint: Write down two equations, one for Newton's second law applied to each object. Solve these for the two unknowns T and a.] ✓

(d) Interpret your answer in the special cases of $M = 0$ and $M = \infty$.

5-p2 Unequal masses M and m are suspended from a pulley as shown in the figure.

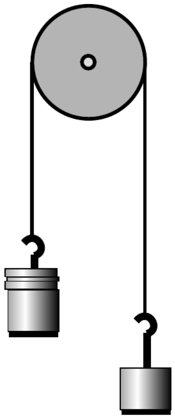
(a) Analyze the forces in which mass m participates, using a table in the format shown in section 5.5. [The forces in which the other mass participates will of course be similar, but not numerically the same.]

(b) Find the magnitude of the accelerations of the two masses. [Hints: (1) Pick a coordinate system, and use positive and negative signs consistently to indicate the directions of the forces and accelerations. (2) The two accelerations of the two masses have to be equal in magnitude but of opposite signs, since one side eats up rope at the same rate at which the other side pays it out. (3) You need to apply Newton's second law twice, once to each mass, and then solve the two equations for the unknowns: the acceleration, a, and the tension in the rope, T.] ✓

(c) Many people expect that in the special case of $M = m$, the two masses will naturally settle down to an equilibrium position side by side. Based on your answer from part b, is this correct?

(d) Find the tension in the rope, T. ✓

(e) Interpret your equation from part d in the special case where one of the masses is zero. Here "interpret" means to figure out what happens mathematically, figure out what should happen physically, and connect the two.



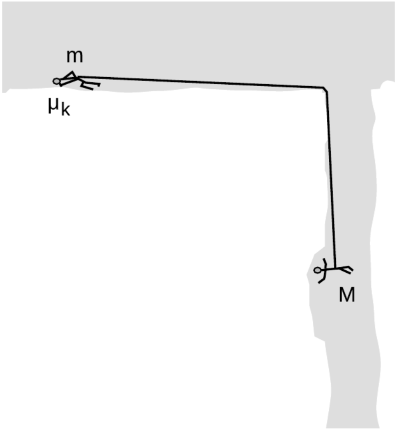
Problem 5-p2.

5-p3 Mountain climbers with masses m and M are roped together while crossing a horizontal glacier when a vertical crevasse opens up under the climber with mass M. The climber with mass m drops down on the snow and tries to stop by digging into the snow with the pick of an ice ax. Alas, this story does not have a happy ending, because this doesn't provide enough friction to stop. Both m and M continue accelerating, with M dropping down into the crevasse and m being dragged across the snow, slowed only by the kinetic friction with coefficient μ_k acting between the ax and the snow. There is no significant friction between the rope and the lip of the crevasse.

(a) Find the acceleration a. ✓

(b) Check the units of your result.

(c) Check the dependence of your equation on the variables. That means that for each variable, you should determine what its effect on a should be physically, and then what your answer from part a says its effect would be mathematically.



Problem 5-p3.

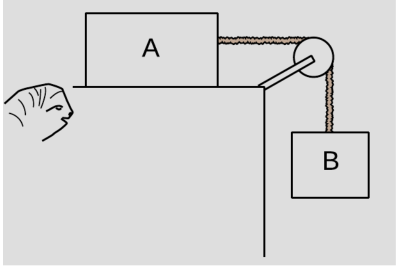
5-p4 Consider the system shown in the figure. Block A has mass M_A and block B has mass M_B. There is friction between the table and block A. Once block B is set into downward motion, it descends at a constant speed.

(a) Analyze the forces in which each block participates as described in section 5.5

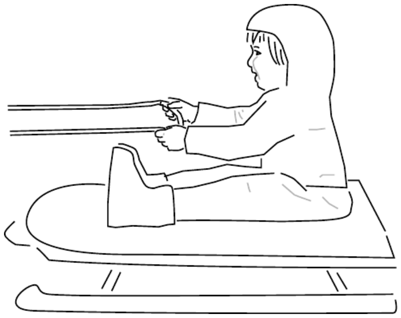
(b) Calculate the coefficient of kinetic friction between block A and the tabletop. ✓

(c) A sloth, also of mass M_A, falls asleep on top of block A. If block B is now set into downward motion, what is its acceleration (magnitude and direction)?

✓



Problem 5-p4.

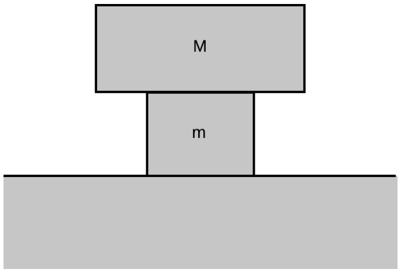


Problem 5-p5.

5-p5 Ginny has a plan. She is going to ride her sled while her dog Foo pulls her, and she holds on to his leash. However, Ginny hasn't taken physics, so there may be a problem: she may slide right off the sled when Foo starts pulling.
(a) Analyze all the forces in which Ginny participates, making a table as in section 5.5.
(b) Analyze all the forces in which the sled participates.
(c) The sled has mass m, and Ginny has mass M. The coefficient of static friction between the sled and the snow is μ_1, and μ_2 is the corresponding quantity for static friction between the sled and her snow pants. Ginny must have a certain minimum mass so that she will not slip off the sled. Find this in terms of the other three variables. ✓
(d) Interpreting your equation from part c, under what conditions will there be no physically realistic solution for M? Discuss what this means physically.

5-p6 The figure shows a stack of two blocks, sitting on top of a table that is bolted to the floor. All three objects are made from identical wood, with their surfaces finished identically using the same sandpaper. We tap the middle block, giving it an initial velocity v to the right. The tap is executed so rapidly that almost no initial velocity is imparted to the top block.
(a) Find the time that will elapse until the slipping between the top and middle blocks stops. Express your answer in terms of v, m, M, g, and the relevant coefficient of friction. ✓
(b) Show that your answer makes sense in terms of units.
(c) Check that your result has the correct behavior when you make m bigger or smaller. Explain. This means that you should discuss the mathematical behavior of the result, and then explain how this corresponds to what would really happen physically.
(d) Similarly, discuss what happens when you make M bigger or smaller.
(e) Similarly, discuss what happens when you make g bigger or smaller.

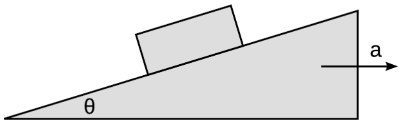
⋆



Problem 5-p6.

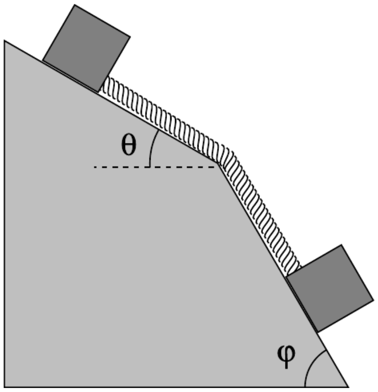
5-p7 (a) A block is sitting on a wedge inclined at an angle θ with respect to horizontal. Someone grabs the wedge and moves it horizontally with acceleration a. The motion is in the direction shown by the arrow in the figure. Find the maximum acceleration that can be applied

without causing the block to slide downhill. √
(b) Show that your answer to part a has the right units.
(c) Show that it also has the right dependence on θ, by comparing its mathematical behavior to its physically expected behavior.
⋆



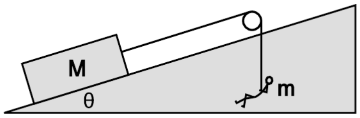
Problem 5-p7.

5-p8 The two blocks shown in the figure have equal mass, m, and the surface is frictionless. (a) What is the tension in the massless rope? √
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $\phi \le \theta$ and $\theta = 0$, $\phi = 90°$.
⋆



Problem 5-p8.

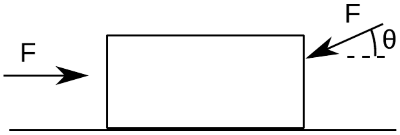
5-p9 (a) The person with mass m hangs from the rope, hauling the box of mass M up a slope inclined at an angle θ. There is friction between the box and the slope, described by the usual coefficients of friction. The pulley, however, is frictionless. Find the magnitude of the box's acceleration. √
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $M = 0$ and $\theta = -90°$.



Problems 5-p9 and 5-p10.

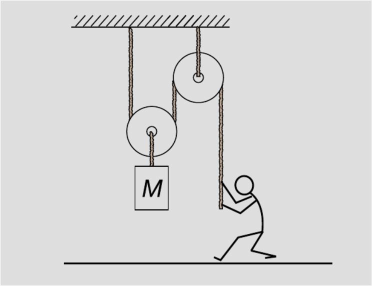
5-p10 The physical situation is the same as in problem 5-p9, except that we make different assumptions about friction. (a) Suppose that there is no friction between the block and the ramp. Find the value of m/M so that the system is in equilibrium. √
(b) If there is instead a coefficient of static friction μ_s between the block and the ramp, find the minimum and maximum values that m/M can have so that the blocks remain at rest.
√

5-p11 The figure shows a block acted on by two external forces, each of magnitude F. One of the forces is horizontal, but the other is applied at a downward angle θ. Gravity is negligible compared to these forces. The block rests on a surface with friction described by a coefficient of friction μ_s. (a) Find the minimum value of μ_s that is required if the block is to remain at rest. √
(b) Show that this expression has the correct limit as θ approaches zero.



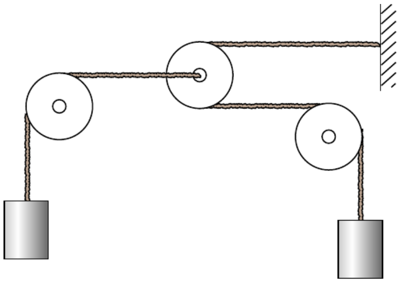
Problem 5-p11.

5-s1 A person can pull with a maximum force F. What is the maximum mass that the person can lift with the pulley setup shown in the figure?
√



Problem 5-s1.

5-s2 In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two masses are identical. Find the upward acceleration of the mass on the left, in terms of g only. Assume all the ropes and pulleys are massless and frictionless.

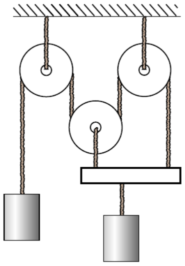


Problem 5-s2.

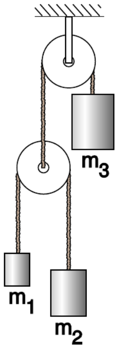
5-s3 The two masses are identical. Find the upward acceleration of the mass on the right, in terms of g only. Assume all the ropes and pulleys, as well as the cross-bar, are massless, and the pulleys are frictionless. The right-hand mass has been positioned away from the bar's center, so that the bar will not twist.

5-s4 Find the upward acceleration of mass m_1 in the figure.

√



Problem 5-s3.



Problem 5-s4.

6 Circular motion

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

6.1 Uniform circular motion

Figure 6.1 shows an overhead view of a person swinging a rock on a rope. A force from the string is required to make the rock's velocity vector keep changing direction. If the string breaks, the rock will follow Newton's first law and go straight instead of continuing around the circle. Circular motion requires a force with a component toward the center of the circle.

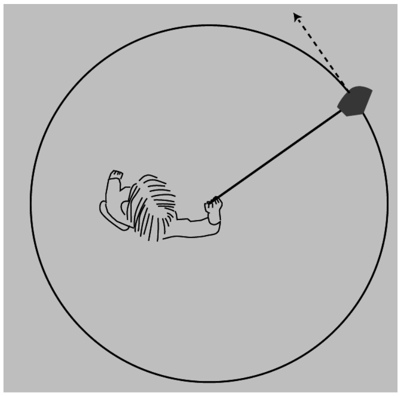


Figure 6.1: Overhead view of a person swinging a rock on a rope.

Uniform circular motion is the special case in which the speed is constant. In uniform circular motion, the acceleration vector is toward the center, and therefore total force acting on the object must point *directly* toward the center. We can define the *angular velocity* ω, which is the number of radians per second by which the object's angle changes, $\omega = \mathrm{d}\theta/\mathrm{d}t$. For uniform circular motion, ω is constant, and the magnitude of the acceleration is

$$a = \omega^2 r = \frac{v^2}{r}, \tag{6.1}$$

where r is the radius of the circle (see problem 6-d1, p. 72). These expressions can also be related to the *period* of the rotation T, which is the time for one revolution. We have $\omega = 2\pi/T$.

6.2 Rotating frames

When you're in the back seat of a car going around a curve, not looking out the window, there is a strong tendency to adopt a frame of reference in which the car is at rest. This is a noninertial frame of reference, because the car is accelerating. In a noninertial frame, Newton's laws are violated. For example, the air freshener hanging from the mirror in figure 3.6 on p. 33 will swing as the car enters the curve, but this motion is not caused by a force made by any identifiable object. In the inertial frame of someone standing by the side of the road, the air freshener simply continued straight while the *car* accelerated.

6.3 Nonuniform motion

In nonuniform circular motion, we have not just the acceleration $a_r = \omega^2 r$ in the radial direction (toward the center of the circle) but also a component $a_t = \mathrm{d}v/\mathrm{d}t$ in the tangential direction.

6.4 Rotational kinematics

Angular velocity and acceleration

If a rigid body such as a top rotates about a fixed axis, then every particle in that body per-

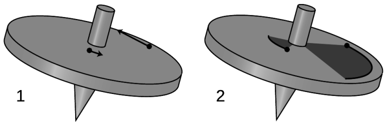


Figure 6.2: Different atoms in the top have different velocity vectors, 1, but sweep out the same angle in a given time, 2.

forms circular motion about a point on that axis. Every atom has a different velocity vector, figure 6.2. Since all the velocities are different, we can't measure the top's speed of rotation of the top by giving a single velocity. But every particle covers the same angle in the same amount of time, so we can specify the speed of rotation consistently in terms of angle per unit time. Let the position of some reference point on the top be denoted by its angle θ, measured in a circle around the axis. We measure all our angles in radians. We define the angular velocity as

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

The relationship between ω and t is exactly analogous to that between x and t for the motion of a particle through space. The angular velocity has units of radians per second, s^{-1}. We also define an angular acceleration,

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}.$$

with units s^{-2}.

The mathematical relationship between ω and θ is the same as the one between v and x, and similarly for α and a. We can thus make a system of analogies, and recycle all the familiar kinematic equations for constant-acceleration motion.

$$x \longleftrightarrow \theta$$

$$v \longleftrightarrow \omega$$

$$a \longleftrightarrow \alpha$$

Figure 6.3: Analogies between rotational and linear quantities.

Angular and linear quantities related

We often want to relate the angular quantities to the motion of a particular point on the rotating object. The velocity vector has tangential and radial components

$$v_t = \omega r$$

and

$$v_r = 0.$$

For the acceleration vector,

$$a_t = \alpha r$$

and

$$a_r = \omega^2 r.$$

Problems

6-a1 Show that the expression $|\mathbf{v}|^2/r$ has the units of acceleration.

6-a2 A plane is flown in a loop-the-loop of radius 1.00 km. The plane starts out flying upside-down, straight and level, then begins curving up along the circular loop, and is right-side up when it reaches the top. (The plane may slow down somewhat on the way up.) How fast must the plane be going at the top if the pilot is to experience no force from the seat or the seatbelt while at the top of the loop? ✓

6-a3 The bright star Sirius has a mass of 4.02×10^{30} kg and lies at a distance of 8.1×10^{16} m from our solar system. Suppose you're standing on a merry-go-round carousel rotating with a period of 10 seconds, and Sirius is on the horizon. You adopt a rotating, noninertial frame of reference, in which the carousel is at rest, and the universe is spinning around it. If you drop a corndog, you see it accelerate horizontally away from the axis, and you interpret this as the result of some horizontal force. This force does not actually exist; it only seems to exist because you're insisting on using a noninertial frame. Similarly, calculate the force that seems to act on Sirius in this frame of reference. Comment on the physical plausibility of this force, and on what object could be exerting it. ✓

6-a4 Lionel brand toy trains come with sections of track in standard lengths and shapes. For circular arcs, the most commonly used sections have diameters of 662 and 1067 mm at the inside of the outer rail. The maximum speed at which a train can take the broader curve without flying off the tracks is 0.95 m/s. At what speed must the train be operated to avoid derailing on the tighter curve? ✓

6-a5 Debbie is in Los Angeles, at a distance $r = 5280$ km from the Earth's axis of rotation.
(a) What is Debbie's speed (as measured by an observer not moving relative to the center of the Earth)? ✓
(b) What is Debbie's acceleration? Your answer indicates to what extent the apparent acceleration due to gravity is modified when not at one of the poles. For reference, $g = 9.81$ m/s^2 at the poles, but appears slightly less near the equator. ✓

6-a6 Some kids are playing around on a merry-go-round at the park. You notice that little Timmy can't seem to hold on to the edge of the merry-go-round (of radius 1.5 m) when the period of the merry-go-round (the time for one full revolution) is less than 2.0 s.
(a) What is the maximum acceleration that Timmy can handle? ✓
(b) If Timmy were on a merry-go-round of radius 6.0 m, what wold be the minimum period of revolution that would allow him to stay on? ✓

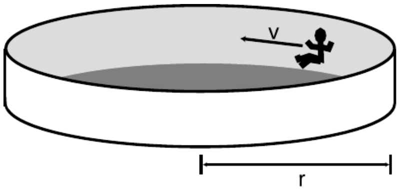
6-a7 A particle is undergoing uniform circular motion in the xy-plane such that its distance to the origin does not change. At one instant, the particle is moving with velocity $\mathbf{v} = (10.0\text{ m/s})\hat{\mathbf{x}}$ and with acceleration $\mathbf{a} = (2.0\text{ m/s}^2)\hat{\mathbf{y}}$.
(a) What is the radius of the circle? ✓
(b) How long does it take for the particle to travel once around the circle (i.e., what is the period of motion)? ✓
(c) In what direction will the particle be moving after a quarter of a period? Give your answer as an angle θ ($0 \leq \theta < 360°$) that the velocity vector makes with respect to the $+\hat{\mathbf{x}}$ direction, measured counterclockwise from the $+\hat{\mathbf{x}}$ direction. ✓

6-a8 A car is approaching the top of a hill of radius of curvature R.
(a) If the normal force that the driver feels at the top of the hill is 1/3 of their weight, how fast is the car going? ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense.

6-a9 An airplane is in a nosedive. In order not to crash, the pilot pulls up as much as she can. At the lowest point of the plane's trajectory, the plane is in a circular arc of radius 300 m, and the pilot experiences an acceleration of $5.0g$ (so that her weight feels like six times normal). What is the plane's speed at this point? ✓

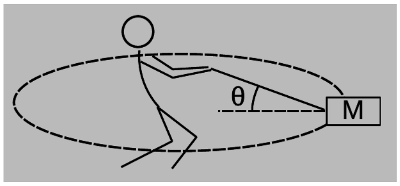
6-d1 In this problem, you'll derive the equation $|\mathbf{a}| = |\mathbf{v}|^2/r$ using calculus. Instead of comparing velocities at two points in the particle's motion and then taking a limit where the points are close together, you'll just take derivatives. The particle's position vector is $\mathbf{r} = (r\cos\theta)\hat{\mathbf{x}} + (r\sin\theta)\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the x and y axes. By the definition of radians, the distance traveled since $t = 0$ is $r\theta$, so if the particle is traveling at constant speed $v = |\mathbf{v}|$, we have $v = r\theta/\mathrm{t}$.
(a) Eliminate θ to get the particle's position vector as a function of time.
(b) Find the particle's acceleration vector.
(c) Show that the magnitude of the acceleration vector equals v^2/r.

6-g1 The amusement park ride shown in the figure consists of a cylindrical room that rotates about its vertical axis. When the rotation is fast enough, a person against the wall can pick his or her feet up off the floor and remain "stuck" to the wall without falling.
(a) Suppose the rotation results in the person having a speed v. The radius of the cylinder is r, the person's mass is m, the downward acceleration of gravity is g, and the coefficient of static friction between the person and the wall is μ_s. Find an equation for the speed, v, required, in terms of the other variables. (You will find that one of the variables cancels out.)
(b) Now suppose two people are riding the ride. Huy is wearing denim, and Gina is wearing polyester, so Huy's coefficient of static friction is three times greater. The ride starts from rest, and as it begins rotating faster and faster, Gina must wait longer before being able to lift her feet without sliding to the floor. Based on your equation from part a, how many times greater must the speed be before Gina can lift her feet without sliding down?



Problem 6-g1.

6-g2 Tommy the playground bully is whirling a brick tied to the end of a rope. The rope makes an angle θ with respect to the horizontal, and the brick undergoes circular motion with radius R.
(a) What is the speed of the brick? ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense, and comment on the limit $\theta \rightarrow 0$.

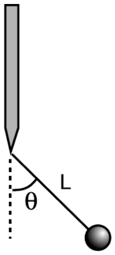


Problem 6-g2.

6-g3 The figure shows a ball on the end of a string of length L attached to a vertical rod which is spun about its vertical axis by a motor. The period (time for one rotation) is P.
(a) Analyze the forces in which the ball participates.
(b) Find how the angle θ depends on P, g, and L. [Hints: (1) Write down Newton's second law for the vertical and horizontal components of

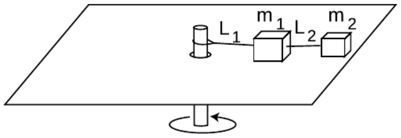
force and acceleration. This gives two equations, which can be solved for the two unknowns, θ and the tension in the string. (2) If you introduce variables like v and r, relate them to the variables your solution is supposed to contain, and eliminate them.] √

(c) What happens mathematically to your solution if the motor is run very slowly (very large values of P)? Physically, what do you think would actually happen in this case?



Problem 6-g3.

6-g4 The figure shows two blocks of masses m_1 and m_2 sliding in circles on a frictionless table. Find the tension in the strings if the period of rotation (time required for one rotation) is P. √



Problem 6-g4.

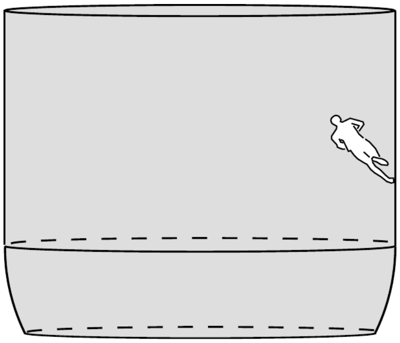
6-g5 In a well known stunt from circuses and carnivals, a motorcyclist rides around inside a big bowl, gradually speeding up and rising higher. Eventually the cyclist can get up to where the walls of the bowl are vertical. Let's estimate the conditions under which a running human could do the same thing.

(a) If the runner can run at speed v, and her shoes have a coefficient of static friction μ_s, what is the maximum radius of the circle? √

(b) Show that the units of your answer make sense.

(c) Check that its dependence on the variables makes sense.

(d) Evaluate your result numerically for $v = 10$ m/s (the speed of an olympic sprinter) and $\mu_s = 5$. (This is roughly the highest coefficient of static friction ever achieved for surfaces that are not sticky. The surface has an array of microscopic fibers like a hair brush, and is inspired by the hairs on the feet of a gecko. These assumptions are not necessarily realistic, since the person would have to run at an angle, which would be physically awkward.) √

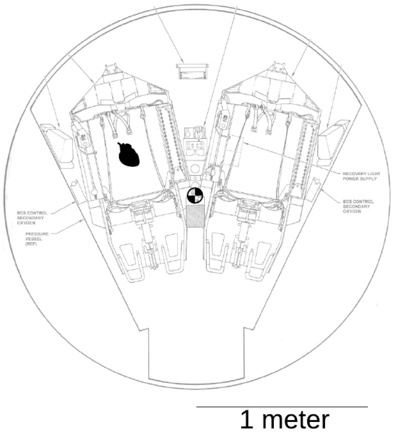


Problem 6-g5.

6-g6 A child places a toy on the outer rim of a merry-go-round that has radius R and period T. What is the lowest value of the coefficient of static friction between the toy and the merry-go-round that allows the toy to stay on the carousel? √

6-g7 The 1961-66 US Gemini program launched pairs of astronauts into earth orbit in tiny capsules, on missions lasting up to 14 days. The figure shows the two seats, in a cross-sectional view from the front, as if looking into a car through the windshield. During the Gemini 8 mission, a malfunctioning thruster in the Orbit Attitude and Maneuvering System (OAMS) caused the capsule to roll, i.e., to rotate in the plane of the page. The rate of rotation got faster and faster, reaching 296 degrees per second before pilot Neil Armstrong shut down the OAMS system by hand and succeeded in canceling the rotation using a separate set of re-entry thrusters. At the peak rate of rotation, the astronauts were approaching the physiological limits under which their hearts would no longer be able to circulate blood, potentially causing them to black out or go blind. Superimposing the approximate location of a human heart on the original NASA diagram, it looks like Armstrong's heart was about 45 cm away from the axis of rotation. Find the acceleration experienced by his heart, in units of g.

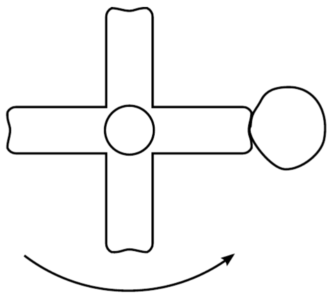
√


RECOVERY LIGHT POWER SUPPLY
ECS CONTROL SECONDARY OXYGEN
ECS CONTROL SECONDARY OXYGEN
PRESSURE VESSEL (REF)
1 meter

Problem 6-g7.

6-j1 When you're done using an electric mixer, you can get most of the batter off of the beaters by lifting them out of the batter with the motor running at a high enough speed. Let's imagine, to make things easier to visualize, that we instead have a piece of tape stuck to one of the beaters.
(a) Explain why static friction has no effect on whether or not the tape flies off.
(b) Analyze the forces in which the tape participates, using a table in the format shown in section 5.5.
(c) Suppose you find that the tape doesn't fly off when the motor is on a low speed, but at a greater speed, the tape won't stay on. Why would the greater speed change things? [Hint: If you don't invoke any law of physics, you haven't explained it.]

⋆



Problem 6-j1.

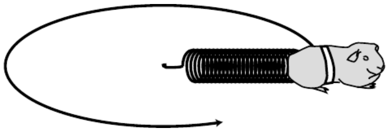
6-j2 The acceleration of an object in uniform circular motion can be given either by $|\mathbf{a}| = |\mathbf{v}|^2/r$ or, equivalently, by $|\mathbf{a}| = 4\pi^2 r/T^2$, where T is the time required for one cycle. Person A says based on the first equation that the acceleration in circular motion is greater when the circle is smaller. Person B, arguing from the second equation, says that the acceleration is smaller when the circle is smaller. Rewrite the two statements so that they are less misleading, eliminating the supposed paradox. [Based on a problem by Arnold Arons.]

⋆

6-j3 Psychology professor R.O. Dent requests funding for an experiment on compulsive thrill-seeking behavior in guinea pigs, in which the subject is to be attached to the end of a spring and whirled around in a horizontal circle. The spring has relaxed length b, and obeys Hooke's law with spring constant k. It is stiff enough to keep from bending significantly under the guinea pig's weight.
(a) Calculate the length of the spring when it is undergoing steady circular motion in which one rotation takes a time T. Express your result in terms of k, b, T, and the guinea pig's mass m. √
(b) The ethics committee somehow fails to veto

the experiment, but the safety committee expresses concern. Why? Does your equation do anything unusual, or even spectacular, for any particular value of T? What do you think is the physical significance of this mathematical behavior?

⋆



Problem 6-j3.

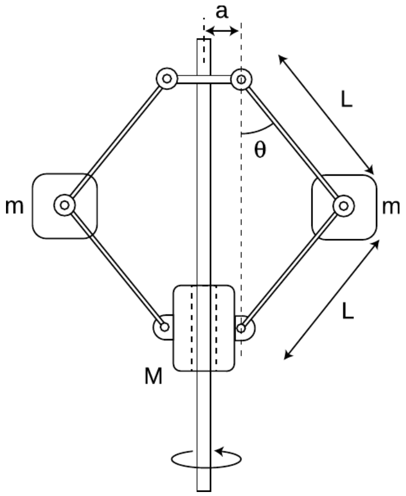
6-j4 The figure shows an old-fashioned device called a flyball governor, used for keeping an engine running at the correct speed. The whole thing rotates about the vertical shaft, and the mass M is free to slide up and down. This mass would have a connection (not shown) to a valve that controlled the engine. If, for instance, the engine ran too fast, the mass would rise, causing the engine to slow back down.

(a) Show that in the special case of $a = 0$, the angle θ is given by

$$\theta = \cos^{-1}\left(\frac{g(m+M)P^2}{4\pi^2 mL}\right),$$

where P is the period of rotation (time required for one complete rotation).

(b) There is no closed-form solution for θ in the general case where a is not zero. However, explain how the undesirable low-speed behavior of the $a = 0$ device would be improved by making a nonzero.



Problem 6-j4.

⋆

6-j5 A car exits the freeway via a circular off-ramp. The road is level (i.e., the curve is not banked at all), and the radius of curvature of the circular ramp is 1.00×10^2 m. The coefficient of friction between the tires and the road is $\mu = 0.8$.

(a) What is the maximum possible speed the car can travel on the offramp without slipping? √

For parts (b) through (d), the car is traveling at 20.0 m/s and decelerating at a rate of 3.0 m/s^2. The car is turning to the right.

(b) What is the absolute value of the radial component of the acceleration? What is the magnitude of the acceleration vector? √

(c) There is a pine-scented tree hanging from the rear view mirror. During the curve, the tree does not hang vertically. Describe in words which way the tree swings. Does it appear to lean left, or right? Forward, or backward? An analysis of the forces might help in your explanation.

(d) What angle does the tree make with respect to the vertical?

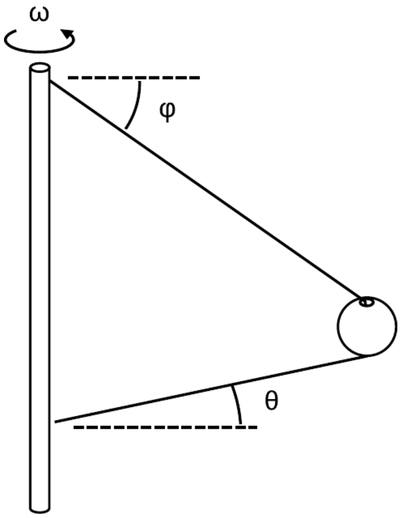
√ ⋆

6-j6 The vertical post rotates at frequency ω. The bead slides freely along the string, reaching an equilibrium in which its distance from the axis is r and the angles θ and ϕ have some particular values. Find ϕ in terms of θ, g, ω, and r.

√ ⋆

6-j7 A bead slides down along a piece of wire that is in the shape of a helix. The helix lies on the surface of a vertical cylinder of radius r, and the vertical distance between turns is d.

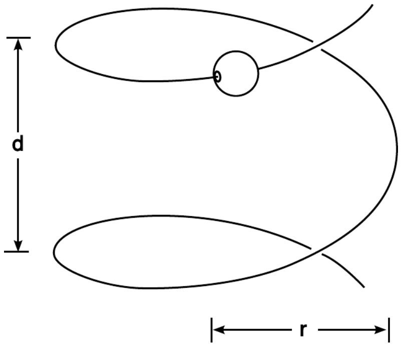
(a) Ordinarily when an object slides downhill under the influence of kinetic friction, the velocity-independence of kinetic friction implies that the



Problem 6-j6.

acceleration is constant, and therefore there is no limit to the object's velocity. Explain the physical reason why this argument fails here, so that the bead will in fact have some limiting velocity.
(b) Find the limiting velocity.
(c) Show that your result has the correct behavior in the limit of $r \to \infty$. [Problem by B. Korsunsky.]

✓ ⋆⋆



Problem 6-j7.

6-m1 A disk, initially rotating at 120 radians per second, is slowed down with a constant angular acceleration of magnitude 4.0 s^{-2}. How many revolutions does the disk make before it comes to rest?

✓

6-m2 A bell rings at the Tilden Park merry go round in Berkeley, California, and the carousel begins to move with an angular acceleration of $1.0 \times 10^{-2}\ \mathrm{s}^{-2}$. How much time does it take to perform its first revolution?

✓

6-m3 Neutron stars are the collapsed remnants of dead stars. They rotate quickly, and their rotation can be measured extremely accurately by radio astronomers. Some of them rotate at such a predictable rate that they can be used to count time about as accurately as the best atomic clocks. They do decelerate slowly, but this deceleration can be taken into account. One of the best-studied stars of this type[1] was observed continuously over a 10-year period. As of the benchmark date April 5, 2001, it was found to have

$$\omega = 1.091313551502333 \times 10^{3}\ \mathrm{s}^{-1}$$

and

$$\alpha = -1.085991 \times 10^{-14}\ \mathrm{s}^{-2},$$

where the error bars in the final digit of each number are about ±1. Astronomers often use the Julian year as their unit of time, where one Julian year is defined to be exactly 3.15576×10^{7} s. Find the number of revolutions that this pulsar made over a period of 10 Julian years, starting from the benchmark date.

✓

6-m4 A gasoline-powered car has a heavy wheel called a flywheel, whose main function is to add inertia to the motion of the engine so that it keeps spinning smoothly between power strokes of the cylinders. Suppose that a certain car's flywheel is spinning with angular velocity ω_0, but the car is then turned off, so that the engine and flywheel start to slow down as a result of friction. Assume that the angular acceleration is constant. After the flywheel has made N revolutions, it comes to rest. What is the magnitude of the angular acceleration? √

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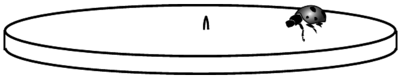
[1] Verbiest *et al.*, Astrophysical Journal 679 (675) 2008

6-m5 A rigid body rotates about a line according to $\theta = At^3 - Bt$ (valid for both negative and positive t).
(a) What is the angular velocity as a function of time? √
(b) What is the angular acceleration as a function of time? √
(c) There are two times when the angular velocity is zero. What is the positive time for which this is true? Call this t_+. √
(d) What is the average angular velocity over the time interval from 0 to t_+? √

6-m6 The angular acceleration of a wheel is $\alpha = 12t - 24t^3$, where α is in s^{-2} and t is the time in seconds. The wheel starts from rest at $t = 0$. How many revolutions has it turned between $t = 0$ and when it is again at rest? √

6-p1 (a) Find the angular velocities of the earth's rotation and of the earth's motion around the sun. √
(b) Which motion involves the greater acceleration?

6-p2 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. The turntable spins at constant angular frequency ω.
(a) Is the bug more likely to slip at small values of r, or large values?
(b) If the bug walks along a radius, what is the value of r at which it looses its footing? √



Problems 6-p2, 6-p3, and 6-p4.

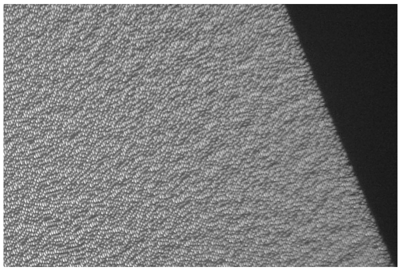
6-p3 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. Starting from rest, the turntable begins rotating with angular acceleration α. What is the magnitude of the angular frequency at which the bug starts to slide? √

6-p4 A 20.0 g cockroach is lounging at a distance $r = 5.00$ cm from the axis of the carousel of a microwave oven. Except for the species, the situation is similar to the one shown in the figure. The cockroach's angular coordinate is $\theta(t) = 12t^2 - 4.0t^3$, where t is in seconds and θ is in radians.
(a) Find the angular velocity and acceleration. √
(b) At what time $t_2 > 0$ is the cockroach at rest? At what time t_1, where $0 < t_1 < t_2$, is the cockroach moving the fastest? √
(c) Find the tangential and radial components of the *linear* acceleration of the cockroach at t_1. What is the direction of the acceleration vector at this time? √
(d) How many revolutions does the cockroach make from $t = 0$ to $t = t_2$? √
(e) Suppose that the carousel, instead of decelerating, had kept spinning at constant speed after t_1. Find the period and the frequency (in revolutions per minute). √

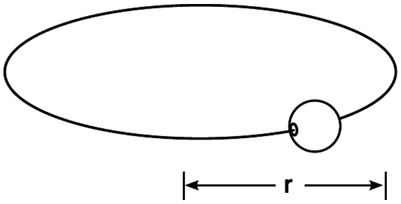
6-p5 A CD is initially moving counterclockwise with angular speed ω_0 and then starts decelerating with an angular acceleration of magnitude α.
(a) How long does it take for the CD to come to rest? √
(b) Suppose that, after a time equal to half that found in part a, point on the CD satisfies $|a_r| = |a_t|$. If the initial angular velocity of the CD was 40 s^{-1}, what is the total time it takes for the CD to come to rest (i.e., the numerical value of your answer in part a)? √

6-s1 The figure shows a microscopic view of the innermost tracks of a music CD. The pits

represent the pattern of ones and zeroes that encode the musical waveform. Because the laser that reads the data has to sweep over a fixed amount of data per unit time, the disc spins at a decreasing angular velocity as the music is played from the inside out. The linear velocity v, not the angular velocity, is constant. Each track is separated from its neighbors on either side by a fixed distance p, called the pitch. Although the tracks are actually concentric circles, we will idealize them in this problem as a type of spiral, called an Archimedean spiral, whose turns have constant spacing, p, along any radial line. Our goal is to find the angular acceleration of this idealized CD, in terms of the constants v and p, and the radius r at which the laser is positioned.
(a) Use geometrical reasoning to constrain the dependence of the result on p.
(b) Use units to further constrain the result up to a unitless multiplicative constant.
(c) Find the full result. [Hint: Find a differential equation involving r and its time derivative, and then solve this equation by separating variables.] √
(d) Consider the signs of the variables in your answer to part c, and show that your equation still makes sense when the direction of rotation is reversed.
(e) Similarly, check that your result makes sense regardless of whether we view the CD player from the front or the back. (Clockwise seen from one side is counterclockwise from the other.)
⋆


Problem 6-s1.

6-s2 Find the motion of a bead that slides with coefficient of kinetic friction μ on a circular wire of radius r. Neglect gravity. [This requires a couple of standard techniques for solving a differential equation, but not obscure or tricky ones.]
⋆



Problem 6-s2.

7 Conservation of energy

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

7.1 Conservation laws

Newton presented his laws of motion as universal ones that would apply to all phenomena. We now know that this is not true. For example, a ray of light has zero mass, so $a = F/m$ gives nonsense when applied to light. Today, physicists formulate the most fundamental laws of physics as *conservation laws*, which arise from *symmetry* principles.

An object has a symmetry if it remains unchanged under some sort of transformation such as a reflection, rotation, rotation, or translation in time or space. A sphere is symmetric under rotation. An object that doesn't change over time has symmetry with respect to time-translation.

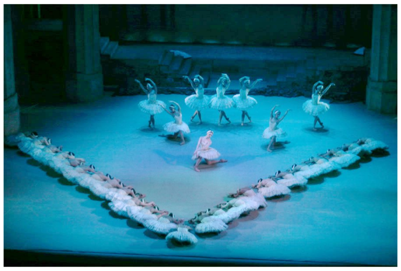


Figure 7.1: In this scene from Swan Lake, the choreography has a symmetry with respect to left and right.

The fundamentally important symmetries in physics are not symmetries of objects but symmetries of the laws of physics themselves. One such symmetry is that laws of physics do not seem to change over time. That is, they have time-translation symmetry. The gravitational forces that you see near the surface of the earth are determined by Newton's law of gravity, which we will state later in quantitative detail.

Suppose that Newton's law of gravity *did* change over time. (Such a change would have to be small, because precise experiments haven't shown objects to get heavier or lighter from one time to another.) If you knew of such a change, then you could exploit it to make money. On a day when gravity was weak, you could pay the electric company what it cost you to run an electric motor, and lift a giant weight to the top of a tower. Then, on a high-gravity day, you could lower the weight back down and use it to crank a generator, selling electric power back on the open market. You would have a kind of perpetual motion machine.

What you are buying from the electric company is a thing called energy, a term that has a specific technical meaning in physics. The fact that the law of gravity does does *not* seem to change over time tells us that we can't use a scheme like the one described above as a way to create energy out of nothing. In fact, experiments seem to show that no physical process can create or destroy energy, they can only transfer or transform it from one form into another. In other words, the total amount of energy in the universe can never change. A statement of this form is called a conservation law. Today, Newton's laws have been replaced by a set of conservation laws, including conservation of energy.

Writing conservation of energy symbolically, we have $E_1 + E_2 + \ldots = E_1' + E_2' + \ldots$, where the sum is over all the types of energy that are present, and the primed and unprimed letters distinguish the energies at some initial and final times. For more compact writing, we can use the

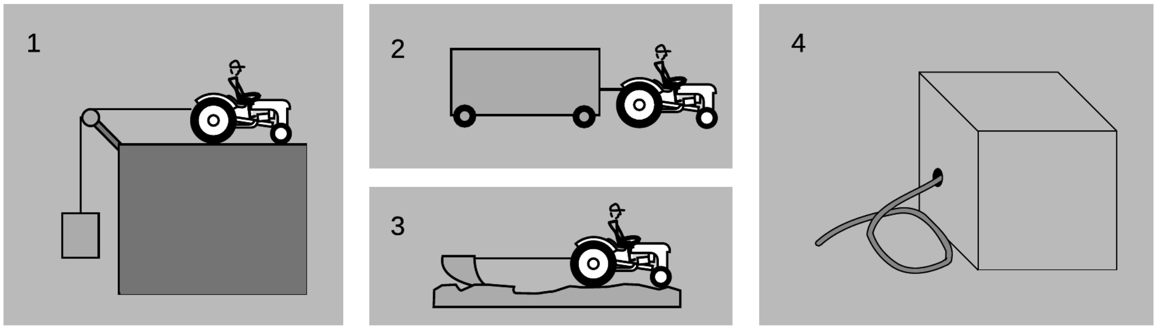


Figure 7.2: Work.

notation

$$\sum_k E_k = \sum_k E'_k, \tag{7.1}$$

where Σ, Greek uppercase sigma, stands for "sum," and k is an index taking on the values 1, 2, ...

Another conservation law is conservation of mass. Experiments by Lavoisier (1743-1794) showed that, for example, when wood was burned, the total mass of the smoke, hot gases, and charred wood was the same as the mass of the original wood. This view was modified in 1905 by Einstein's famous $E = mc^2$, which says that we can actually convert energy to mass and mass to energy. Therefore the separate laws of conservation of energy and mass are only approximations to a deeper, underlying conservation law that includes both quantities.

7.2 Work

Energy exists in various forms, such a the energy of sunlight, gravitational energy, and the energy of a moving object, called kinetic energy. Because it exists in so many forms, it is a tricky concept to define. By analogy, an amount of money can be expressed in terms of dollars, euros, or various other currencies, but modern governments no longer even attempt to define the value of their currencies in absolute terms such as ounces of gold. If we make radio contact with aliens someday, they will presumably not agree with us on how many units of energy there are in a liter of gasoline. We could, however, pick something arbitrary like a liter of gas as a standard of comparison.

Figure **??** shows a train of thought leading to a standard that turns out to be more convenient. In panel 1, the tractor raises the weight over the pulley. Gravitational energy is stored in the weight, and this energy could be released later by dropping or lowering the weight. In 2, the tractor accelerates the trailer, increasing its kinetic energy. In 3, the tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow. In all three examples, the energy of the gas in the tractor's tank is converted into some other form, and in all three examples there is a force F involved, and the tractor travels some distance d as it applies the force.

Now imagine a black box, panel 4, containing a gasoline-powered engine, which is designed to reel in a steel cable, exerting a force F. The box only communicates with the outside world via the hole through which its cable passes, and therefore the amount of energy transferred out through the cable can only depend on F and d. Since force and energy are both additive, this energy must be proportional to F, and since the energy transfer is additive as we reel in one section of cable and then a further section, the energy must also be proportional to d. As an arbitrary standard, we pick the constant of proportionality to be 1, so that the energy transferred, notated W for *work*, is given by

$$W = Fd. \tag{7.2}$$

This equation implicitly defines the SI unit of energy to be $\text{kg}\cdot\text{m}^2/\text{s}^2$, and we abbreviate this as one joule, $1\ \text{J} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$.

In general, we define *work* as the transfer of energy by a macroscopic force, with a plus sign if energy is flowing from the object exerting the force to the object on which the force is exerted. (In examples such as heat conduction, the forces are forces that occur in the collisions between atoms, which are not measurable by macroscopic devices such as spring scales and force probes.) Equation (7.2) is a correct rule for computing work in the special case when the force is exerted at a single well-defined point of contact, that point moves along a line, the force and the motion are parallel, and the force is constant. The distance d is a signed quantity.

When the force and the motion are not parallel, we have the generalization

$$W = \mathbf{F} \cdot \Delta\mathbf{x}, \tag{7.3}$$

in which $\cdot$ is the vector dot product. When force and motion are along the same line, but the force is not constant,

$$W = \int F\,\mathrm{d}x. \tag{7.4}$$

Applying both of these generalizations at once gives

$$W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{x}, \tag{7.5}$$

which is an example of the line integral from vector calculus.

The rate at which energy is transferred or transformed is the *power*,

$$P = \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{7.6}$$

The units of power can be abbreviated as watts, $1\ \text{W} = 1\ \text{J/s}$. For the conditions under which $W = \int F\,\mathrm{d}x$ is valid, we can use the fundamental theorem of calculus to find $F = \mathrm{d}W/\,\mathrm{d}x$, and since $\mathrm{d}W/\,\mathrm{d}t = (\mathrm{d}K/\,\mathrm{d}x)(\mathrm{d}x/\,\mathrm{d}t)$, the power transmitted by the force is

$$P = Fv. \tag{7.7}$$

7.3 Kinetic energy

Having chosen mechanical work as an arbitrary standard for defining transfers of energy, we are led by Newton's laws to an expression for the energy that an object has because of its motion, called *kinetic energy*, K. One form of this work-kinetic energy theorem is as follows. Let a force act on a particle of mass m in one dimension. By the chain rule, we have $\mathrm{d}K/\,\mathrm{d}x = (\mathrm{d}K/\,\mathrm{d}v)(\mathrm{d}v/\,\mathrm{d}t)(\mathrm{d}t/\,\mathrm{d}x) = (\mathrm{d}K/\,\mathrm{d}v)a/v$. Applying $a = F/m$ and $\mathrm{d}K/\,\mathrm{d}x = F$ (work) gives $\mathrm{d}K/\,\mathrm{d}v = mv$. Integration of both sides with respect to v results in

$$K = \frac{1}{2}mv^2, \tag{7.8}$$

where the constant of integration can be taken to be zero. The factor of 1/2 is ultimately a matter of convention; if we had wanted to avoid the 1/2 in this equation, we could have, but we would have had to define work as $2Fd$.

When we heat an object, we are increasing the kinetic energy of the random motion of its molecules. The amount of energy required to heat one kilogram a substance by one degree is called its specific heat capacity. A useful figure is that the specific heat of water is 4.2×10^3 J/kg°C.

7.4 Potential energy

Figure 7.3 shows someone lifting a heavy textbook at constant speed while a bug hitches a ride on top. The person's body has burned some calories, and although some of that energy went into body heat, an amount equal to Fd must

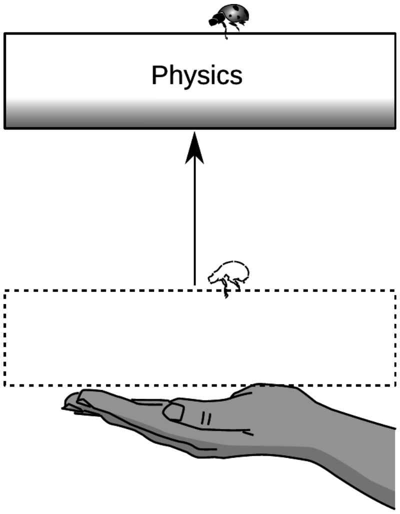


Figure 7.3: The book gains potential energy as it is raised.

have flowed into the book. But the bug may be excused for being skeptical about this. No measurement that the bug can do within its own immediate environment shows any changes in the properties of the book: there is no change in temperature, no vibration, nothing. Even if the bug looks at the walls of the room and is able to tell that it is rising, it finds that it is rising at constant speed, so there is no change in the book's kinetic energy. But if the person then takes her hand away and lets the book drop, the bug will have to admit that there is a spectacular and scary release of kinetic energy, which will later be transformed into sound and vibration when the book hits the floor.

If we are to salvage the law of conservation of energy, we are forced to invent a new type of energy, which depends on the height of the book in the earth's gravitational field. This is an energy of position, which is usually notated as PE or U. Any time two objects interact through a force exerted at a distance (gravity, magnetism, etc.), there is a corresponding position-energy, which is referred to as *potential energy.* The hand was giving gravitational potential energy to the book. Since the work done by the hand equals Fd, it follows that the potential energy must be given by

$$PE_{\text{grav}} = mgy, \tag{7.9}$$

where an arbitary additive constant is implied because we have to choose a reference level at which to define $y = 0$. In the more general case where the external force such as gravity is not constant and can point in any direction, we have

$$\Delta PE = -\int_1^2 \mathbf{F}\cdot \mathrm{d}\mathbf{x}, \tag{7.10}$$

where 1 and 2 stand for the inital and final positions. This is a line integral, which is general may depend on the path the object takes. But for a certain class of forces, which includes, to a good approximation, the earth's gravitational force on an object, the result is *not* dependent on the path, and therefore the potential energy is well defined.

A common example is an elastic restoring force $F = -kx$ (Hooke's law), such as the force of a spring. Calculating $\Delta PE = -\int F\,\mathrm{d}x$, we find

$$PE = \frac{1}{2}kx^2, \tag{7.11}$$

where the constant of integration is arbitrarily chosen to be zero. This is in fact a type of electrical potential energy, which varies as the lattice of atoms within the spring is distorted.

Problems

7-a1 "Big wall" climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial "portaledge." In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as "vertical ditch digging." (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case? ✓

7-a2 An airplane flies in the positive direction along the x axis, through crosswinds that exert a force $\mathbf{F} = (a + bx)\hat{\mathbf{x}} + (c + dx)\hat{\mathbf{y}}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position x. ✓

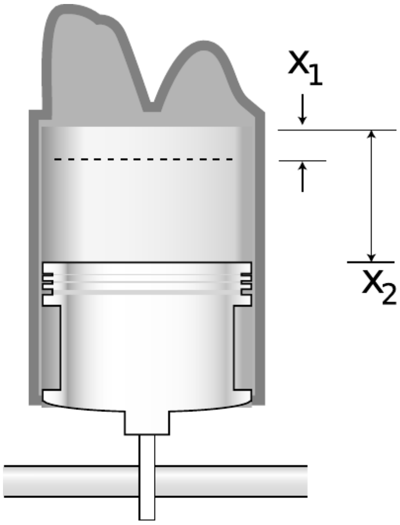
7-a3 In the power stroke of a car's gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture's force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by $F = a/x$, where x is the distance from the cylinder to the piston head, and a is a constant with units of $\text{N}\cdot\text{m}$. (Actually $a/x^{1.4}$ would be more accurate, but the problem works out more nicely with a/x!) The piston begins its stroke at $x = x_1$, and ends at $x = x_2$. The 1965 Rambler had six cylinders, each with $a = 220\ \text{N}\cdot\text{m}$, $x_1 = 1.2$ cm, and $x_2 = 10.2$ cm.

(a) Draw a neat, accurate graph of F vs x, on graph paper.

(b) From the area under the curve, derive the amount of work done in one stroke by one cylinder. ✓

(c) Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine's power, in units of horsepower (1 hp=746 W). ✓

(d) The compression ratio of an engine is defined as x_2/x_1. Explain in words why the car's power would be exactly the same if x_1 and x_2 were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would *not* quite be true with the more realistic force equation $F = a/x^{1.4}$.



Problem 7-a3.

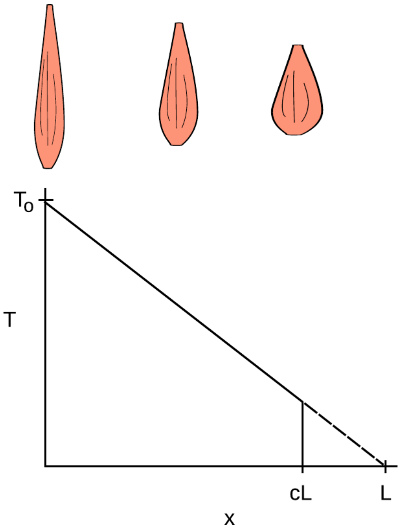
7-a4 (a) The crew of an 18th century warship is raising the anchor. The anchor has a mass of 5000 kg. The water is 30 m deep. The chain to which the anchor is attached has a mass per unit length of 150 kg/m. Before they start raising the anchor, what is the total weight of the anchor plus the portion of the chain hanging out of the ship? (Assume that the buoyancy of the anchor is negligible.)

(b) After they have raised the anchor by 1 m, what is the weight they are raising?

(c) Define $y = 0$ when the anchor is resting on the bottom, and $y = +30$ m when it has been raised up to the ship. Draw a graph of the force the crew has to exert to raise the anchor and chain, as a function of y. (Assume that they are raising it slowly, so water resistance is negligible.) It will not be a constant! Now find the area under the graph, and determine the work done by the crew in raising the anchor, in joules. (d) Convert your answer from (c) into units of kcal.

√

7-a5 The figure, redrawn from *Gray's Anatomy*, shows the tension of which a muscle is capable. The variable x is defined as the contraction of the muscle from its maximum length L, so that at $x = 0$ the muscle has length L, and at $x = L$ the muscle would theoretically have zero length. In reality, the muscle can only contract to $x = cL$, where c is less than 1. When the muscle is extended to its maximum length, at $x = 0$, it is capable of the greatest tension, T_o. As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at $x = L$. (a) Find the maximum work the muscle can do in one contraction, in terms of c, L, and T_o. √
(b) Show that your answer to part a has the right units.
(c) Show that your answer to part a has the right behavior when $c = 0$ and when $c = 1$.
(d) Gray also states that the absolute maximum tension T_o has been found to be approximately proportional to the muscle's cross-sectional area A (which is presumably measured at $x = 0$), with proportionality constant k. Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume, V, eliminating L and A. √
(e) Evaluate your result numerically for a biceps muscle with a volume of 200 cm^3, with $c = 0.8$ and $k = 100\ \text{N/cm}^2$ as estimated by Gray. √



Problem 7-a5.

7-d1 Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

7-d2 A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy?

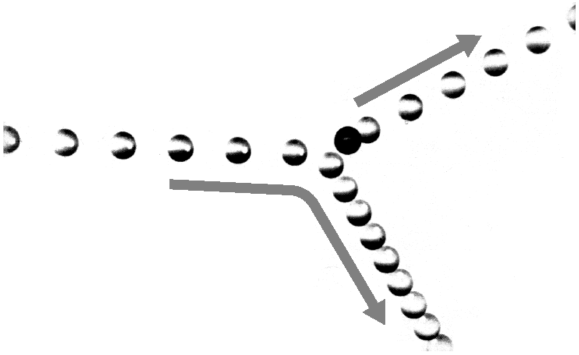
7-d3 A 7.00 kg bowling ball moves at 3.00 m/s. How fast must a 2.45 g ping-pong ball move so that the two balls have the same kinetic energy?

√

7-d4 You throw a ball straight up, and it lands with 2/3 the speed with which you threw it. What fraction of the initial kinetic energy was lost during the time the ball was in the air?

√

7-d5 The multiflash photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making *measurements* on the figure, determine *numerically* whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]



Problem 7-d5.

hw-colliding-balls

7-d6 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds? √

7-d7 Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B's kinetic energy? [Based on a problem by Arnold Arons.]

7-d8 One theory about the destruction of the space shuttle Columbia in 2003 is that one of its wings had been damaged on liftoff by a chunk of foam insulation that fell off of one of its external fuel tanks. The New York Times reported on June 5, 2003, that NASA engineers had recreated the impact to see if it would damage a mock-up of the shuttle's wing. "Before last week's test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball." In fact, the 0.80 kg piece of foam, moving at 240 m/s, did serious damage. A member of the board investigating the disaster said this demonstrated that "people's intuitive sense of physics is sometimes way off." (a) Compute the kinetic energy of the foam, and (b) compare with the energy of an 80 kg boulder moving at 2.4 m/s (the speed it would have if you dropped it from about knee-level). √
(c) The boulder is a hundred times more massive, but its speed is a hundred times smaller, so what's counterintuitive about your results?

7-d9 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes. √

7-d10 Experiments show that the power consumed by a boat's engine is approximately proportional to the third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is crusing at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat's crusing speed increased? [Based on a problem by Arnold Arons.]

7-g1 Estimate the kinetic energy of an Olympic sprinter.

7-g2 Estimate the kinetic energy of a buzzing fly's wing. (You may wish to review section 1.4 on order-of-magnitude estimates.)

7-g3 A blade of grass moves upward as it grows. Estimate its kinetic energy. (You may wish to review section 1.4 on order-of-magnitude estimates.)

7-g4 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world's coastal cities. The total sunlight that falls on Antarctica amounts to about 1×10^{16} watts. In the absence of natural or human-caused climate change, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires 3×10^3 J.

7-j1 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

7-j2 Can gravitational potential energy ever be negative? Note that the question refers to PE, not ΔPE, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

7-j3 In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck's bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

7-j4 (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work as a transfer of mechanical energy. (b) Now answer the question based on the $W = Fd$ rule.

7-j5 Does it make sense to say that work is conserved?

7-j6 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic energy increase, or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic energy increase, or decrease? Explain.

7-j7 Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE.

7-j8 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

7-j9 Decide whether the following statements regarding work and energy are true or false.
(a) The work done by a frictional force depends only on the initial and final points of the path of a particle.
(b) If a force is perpendicular to the direction of motion of an object, the force is not changing the kinetic energy of the object.
(c) The work done by a conservative force is zero.
(d) Doubling the amount of time a force is applied will double the work done on an object by the force.
(e) Since KE is always positive, the net work on a particle must be positive.

7-j10 When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

7-j11 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don't you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

7-j12 A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the solid doesn't heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids.

Don't repeat the following common mistakes:
In a solid, the atoms are packed more tightly and have less space between them. Not true. Ice floats because it's *less* dense than water.
In a liquid, the atoms are moving much faster. No, the difference in average speed between ice at −1°C and water at 1°C is only 0.4%.

7-j13 The figure above is from a classic 1920 physics textbook by Millikan and Gale. It represents a method for raising the water from the pond up to the water tower, at a higher level, without using a pump. Water is allowed into the drive pipe, and once it is flowing fast enough, it forces the valve at the bottom closed. Explain how this works in terms of conservation of mass and energy.

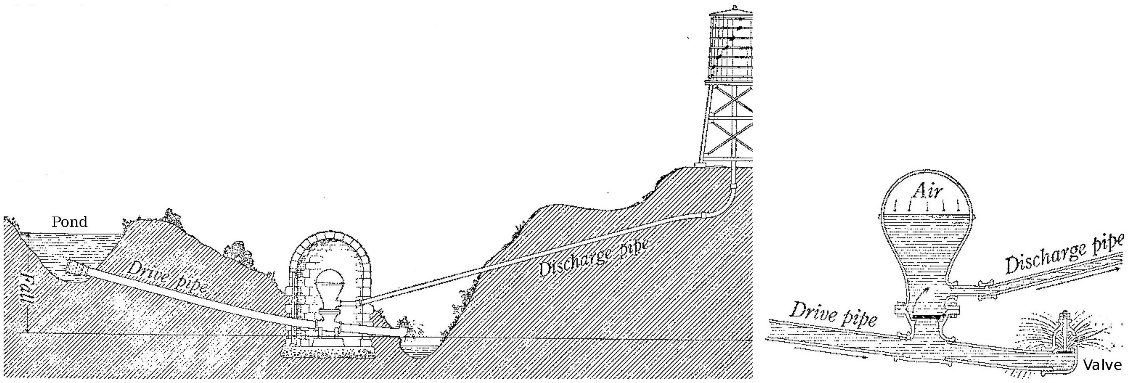
7-m1 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?

▷ Solution, p. 291

7-m2 Suppose that the cost of energy in your city is 15 cents per kilowatt-hour. A cost-efficient light bulb uses energy at a rate of 25 W. How much does it cost to leave the light on for the entire month of January?

✓

7-m3 Lisa times herself running up the stairs of her science building and finds that it takes her 23 s to reach the top floor. Her mass is 44 kg. If the vertical height reached is 11.0 m, what is minimum average power she would have to have produced during the climb (i.e., only taking into



Problem 7-j13.

account the energy required to overcome gravity)?

√

7-m4 How long will it take a 3.92 kW motor, operating at full power, to lift a 1150 kg car to a height of 25.0 m? Assume frictional forces are negligible. (To make this more vivid for people in the US, 3.92 kW is 5.26 horsepower.)

√

7-m5 A roller coaster starts from rest and descends 35 meters in its initial drop and then rises 23 meters before going over a hill. A passenger at the top of the hill feels an apparent weight which is 2/3 of her normal weight. By using the fact that the energy loss due to friction must be greater than zero, find a bound on the radius of curvature of the first hill. Is this an upper bound, or a lower bound?

√

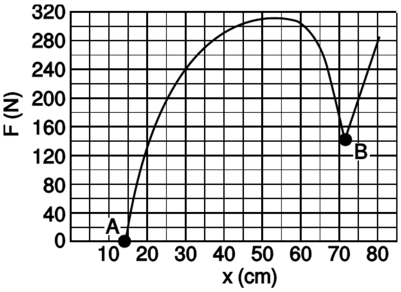
7-m6 A piece of paper of mass 4.5 g is dropped from a height 1.0 m above the ground. The paper dissipates 37 mJ of energy through frictional heating on its way down.
(a) How much kinetic energy does the paper have when it reaches the ground? √
(b) What is the speed of the paper when it hits the ground?

√

7-m7 Let E_b be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to 100°C. [Numerical check, not for credit: Plugging in $E_b = 2.3$ MJ/kg should give a result of 230 km.] √
(b) Show that the units of your answer in part a come out right based on the units given for E_b.

7-m8 Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where F is the force required to pull the string back, and x is the distance between the string and the center of the bow's body. It is not a linear Hooke's-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can't be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a 26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.)

√



Problem 7-m8.

7-m9 The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

heat 1 g of ice by 1°C	2.05 J
melt 1 g of ice	333 J
heat 1 g of water by 1°C	4.19 J
boil 1 g of water	2500 J
heat 1 g of steam by 1°C	2.01 J

(a) How much energy is required in order to convert 1.00 g of ice at -20 °C into steam at 137 °C?

√

(b) What is the minimum amount of hot water that could melt 1.00 g of ice?

√

7-m10 Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.]

√

7-m11 Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you'll need to discuss what specific type of energy is involved in each case.

7-p1 At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to "try" many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.)

√

7-p2 In the earth's atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the O_2 molecules is the same as the average KE of the N_2 molecules. (a) If the mass of an O_2 molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 11-s1 first, for insight. ✓

7-p3 Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it's important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats' stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

7-p4 A car starts from rest at $t = 0$, and starts speeding up with constant acceleration. (a) Find the car's kinetic energy in terms of its mass, m, acceleration, a, and the time, t. (b) Your answer in the previous part also equals the amount of work, W, done from $t = 0$ until time t. Take the derivative of the previous expression to find the power expended by the car at time t. (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.) ✓

7-p5 While in your car on the freeway, you're travelling at a constant speed of 55 miles/hour, requiring a power output of 50 horsepower from the engine. Almost all of the energy provided by the engine is used to fight air resistance, which is proportional in magnitude to the square of the speed of the car. If you step on the gas pedal all the way and increase the power output to 100 horsepower, what final speed will you reach? Note that this problem can be done without any conversions or knowledge of US units. ✓

7-s1 A soccer ball of mass m is moving at speed v when you kick it in the same direction it is moving. You kick it with constant force F, and you want to triple the ball's speed. Over what distance must your foot be in contact with the ball? ✓

7-s2 A laptop of mass m and a desktop computer of mass $3m$ are both dropped from the top of a building. The laptop has kinetic energy K when it reaches the ground.
(a) Find the kinetic energy of the desktop machine on impact, in terms of K, m, or both. ✓
(b) Find its speed in terms of the same variables. ✓

7-s3 A girl picks up a stone of mass m from the ground and throws it at speed v, releasing the stone from a height h above the ground. If the maximum power output of the girl is P, how many stones could she throw in a time T?

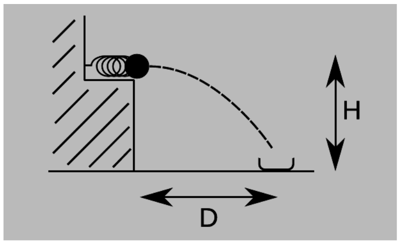
7-s4 A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will "burn rubber" (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.
(a) Show that if a force F is applied to an object moving at speed v, the power required is given by $P = vF$.
(b) Find the speed v at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car's power P, its mass m, the coefficient of static friction μ_s, and g. ✓
(c) Show that your answer to part b has units that make sense.
(d) Show that the dependence of your answer on

each of the four variables makes sense physically.
(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of 3.23×10^5 W (433 horsepower) and a mass (including a 50-kg driver) of 2.03×10^3 kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at v.) Rubber on asphalt has $\mu_s \approx 0.9$. Find v for this car. Answer: 18 m/s, or about 40 miles per hour.
(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to v^2. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine's maximum power is being used to overcome air resistance when the car is moving at the speed v found in part e. Answer: 1%

7-s5 A piece of paper of mass m is dropped from a height H above the ground. Because of air resistance, the paper lands with only 1/5th the speed that it would have landed with had there been no air resistance.
(a) What is the work on the paper due to gravity? √
(b) What is the work on the paper due to the drag force? √

7-s6 A block of mass m is at the top of a ramp of length L inclined at angle θ with respect to the horizontal. The block slides down the ramp, and the coefficient of friction between the two surfaces is μ_k. Parts a-c are about finding the speed of the block when it reaches the bottom of the ramp.
(a) Based on units, infer as much as possible about the form of the answer.
(b) Find the speed. √
(c) Check that the dependence of the result on the variables make sense. Under what conditions is the result unphysical?
(d) The speed of the block at the bottom of the ramp is only half of what it would have been without friction. Knowing this, what is the coefficient of friction μ_k in terms of the other given quantities? √

7-s7 Some kids are playing a game where they shoot a ball of mass m off a spring into a cup that is a distance D away from the base of the table (see figure). The ball starts at a height H, and the spring has spring constant k. The goal of the problem is to find the distance you should compress the spring so that the ball lands in the cup.
(a) Infer as much as possible about the form of the result based on units.
(b) Find the result. √

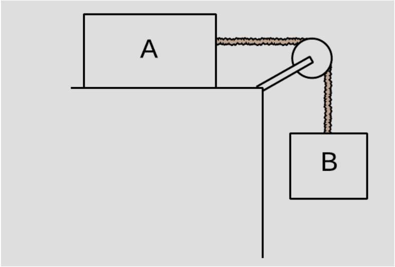


Problem 7-s7.

7-s8 A person on a bicycle is to coast down a ramp of height h and then pass through a circular loop of radius r. What is the smallest value of h for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.) √

7-s9 Suppose that the cyclist in problem 7-s8 wants to have a little extra security, passing through the top of the loop at twice the minimum speed that would have theoretically been required.
(a) What is this speed? √
(b) Find the height of the ramp that is needed. √

7-s10 Consider the system shown in the figure. Block A has mass M_A and block B has mass M_B. There is a coefficient of kinetic friction μ_k between block A and the table. The system is released from rest and block B drops a distance D.
(a) What is the work done by gravity on block B? √
(b) What is the tension in the string? √
(c) What is the work done on block B by the string? √
(d) By adding your results in parts a and c, find the speed of block B. √
(e) Show that the sum of the works done on block A equals the change in KE of block A.

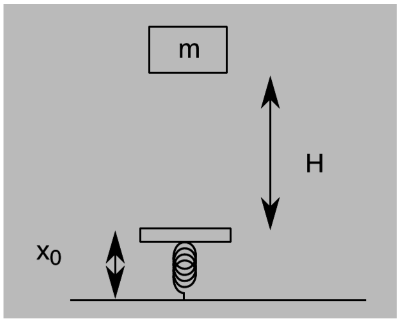


Problem 7-s10.

7-s11 (a) A circular hoop of mass m and radius r spins like a wheel while its center remains at rest. Its period (time required for one revolution) is T. Show that its kinetic energy equals $2\pi^2 mr^2/T^2$.
(b) If such a hoop rolls with its center moving at velocity v, its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in the first part of this problem. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

7-s12 As shown in the figure, a box of mass m is dropped from a height H above a spring. The platform has negligible mass. The spring has spring constant k.
(a) By what amount is the spring compressed? Your answer requires the quadratic equation; to choose the correct root, note that x needs to be positive. √
(b) What is the speed of the box when it is first in contact with the platform? √
(c) What is the maximum speed of the box? *Hint: the box speeds up until the force from the spring equals the gravitational force.*

√ ⋆



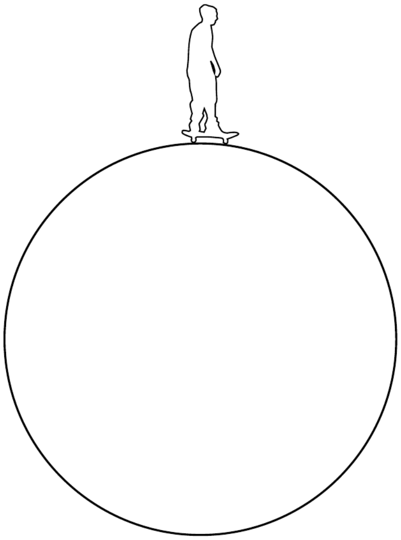
Problem 7-s12.

7-s13 A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe.

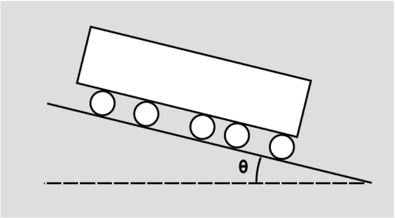
⋆

7-s14 The figure shows a slab of mass M rolling freely down an inclined plane inclined at an angle θ to the horizontal. The slab is on top of a set of rollers, each of radius r, that roll without slipping at their top and bottom surfaces. The rollers may for example be cylinders, or spheres such as ball bearings. Each roller's center of mass coincides with its geometrical center. The sum of the masses of the rollers is m, and the sum of their moments of inertia (each about its own center) is I. Find the acceleration of the slab, and verify that your expression has the correct behavior in interesting limiting cases.

√ ⋆



Problem 7-s13.



Problem 7-s14.

7-v1 The magnitude of the force between two magnets separated by a distance r can be approximated as kr^{-3} for large values of r. The constant k depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance r_i, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance r_f? (Ignore friction.) ✓

7-v2 An object's potential energy is described by the function $U(x) = -\alpha x^2 + \beta x^4$, where α and β are positive constants.
(a) For what positive value of x is the force on the object equal to zero? ✓
(b) What is the force on the particle when $x = 2\sqrt{\alpha/\beta}$? ✓

7-v3 The potential energy of a particle moving in a certain one-dimensional region of space is

$$U(x) = (1.00\ \mathrm{J/m^3})x^3 - (7.00\ \mathrm{J/m^2})x^2 + (10.0\ \mathrm{J/m})x.$$

(a) Determine the force $F(x)$ acting on the particle as a function of position.
(b) Is the force you found in part a conservative, or non-conservative? Explain.
(c) Let "$\mathcal{R}$" refer to the region $x = -1.00$ m to $x = +6.00$ m. Draw $U(x)$ on $\mathcal{R}$ (label your axes). On your plot, label all points of stable and unstable equilibrium on $\mathcal{R}$, and find their locations.
(d) What is the maximum force (in magnitude) experienced by a particle on $\mathcal{R}$?
(e) The particle has mass 1.00 kg and is released from rest at $x = 2.00$ m. Describe the subsequent motion. What is the maximum KE that the particle achieves?

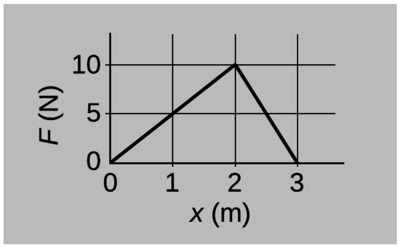
7-v4 The potential energy of a particle moving in a certain one-dimensional region of space is

$$U(x) = (1.00\ \mathrm{J/m^4})x^4 - (4.00\ \mathrm{J/m^2})x^2 + (1.00\ \mathrm{J/m})x.$$

You might want to plot the function $U(x)$ using a graphing calculator or an online utility such as desmos.com. An object of mass 2.00 kg is released from rest at $x = 2.00$ m.
(a) At what position does the object have maximum speed? ✓
(b) What is the maximum speed of the object? ✓
(c) What is the velocity of the object when it reaches the unstable equilibrium point? ✓
(d) What is the lowest value of x that the object gets to?

√

7-v5 A banana starts at rest and is subject to the force shown. This force is the only force acting on the banana.
(a) What is the work done on the banana as it moves from $x = 0$ m to 2 m? √
(b) From 2 m to 3 m? √
(c) If this force acts on the banana for 0.5 s, what is the average power delivered to the banana? √



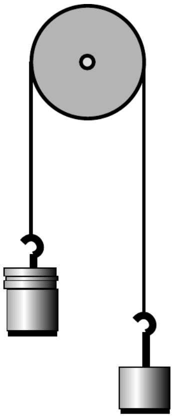
Problem 7-v5.

7-v6 An object of mass m is moving with speed v in the $+x$-direction when, starting at $x = 0$, it is subjected to the position-dependent force $F(x) = -kx^2$, where k is a positive constant. What is the maximum x-coordinate the object will reach? √

7-v7 In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance r, was $PE(r) = gr^{-1}e^{-r/a}$, where g parametrizes the strength of the interaction, e is the base of natural logarithms, and a is about 10^{-15} m. Find the force between two nucleons that would be consistent with this equation for the potential energy. √

7-v8 A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F = be^{-cx}$, where x is the altitude, b and c are constants, and e is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.) √

7-v9 The figure shows two unequal masses, M and m, connected by a string running over a pulley. This system was analyzed previously in problem 5-p2 on p. 63, using Newton's laws.
(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance h. √
(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem. √



Problem 7-v9.

7-v10 A mass moving in one dimension is attached to a horizontal spring. It slides on the surface below it, with equal coefficients of static and kinetic friction, $\mu_k = \mu_s$. The equilibrium position is $x = 0$. If the mass is pulled to some initial position and released from rest, it will complete some number of oscillations before friction brings it to a stop. When released from $x = a$ ($a > 0$), it completes exactly 1/4 of an oscillation, i.e., it stops precisely at $x = 0$. Similarly, define $b > 0$ as the greatest x from which it could be released and comlete 1/2 of an oscillation, stopping on the far side and not coming back toward equilibrium. Find b/a. Hint: To keep the algebra simple, set every fixed parameter of the system equal to 1.

√

7-v11 In 2003, physicist and philosopher John Norton came up with the following apparent paradox, in which Newton's laws, which appear deterministic, can produce nondeterministic results. Suppose that a bead moves frictionlessly on a curved wire under the influence of gravity. The shape of the wire is defined by the function $y(x)$, which passes through the origin, and the bead is released from rest at the origin. For convenience of notation, choose units such that $g = 1$, and define $\dot{y} = \mathrm{d}y/\,\mathrm{d}t$ and $y' = \mathrm{d}y/\,\mathrm{d}x$.

(a) Show that the equation of motion is

$$y = -\frac{1}{2}\dot{y}^2\left(1 + y'^{-2}\right).$$

(b) To simplify the calculations, assume from now on that $y' \ll 1$. Find a shape for the wire such that $x = t^4$ is a solution. (Ignore units.) √

(c) Show that not just the motion assumed in part b, but any motion of the following form is a solution:

$$x = \begin{cases} 0 & \text{if } t \le t_0 \\ (t - t_0)^4 & \text{if } t \ge t_0 \end{cases}$$

This is remarkable because there is no physical principle that determines t_0, so if we place the bead at rest at the origin, there is no way to predict when it will start moving.

⋆

8 Conservation of momentum

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

8.1 Momentum: a conserved vector

Consider the hockey puck in figure 8.1. If we release it at rest, we expect it to remain at rest. If it did start moving all by itself, that would be strange: it would have to pick some direction in which to move, and why would it have such a deep desire to visit the region of space on one side rather than the other? Such behavior, which is not actually observed, would suggest that the laws of physics differed between one region of space and another.

Figure 8.1: A hockey puck is released at rest. Will it start moving in some direction?

The laws of physics are in fact observed to be the same everywhere, and this symmetry leads to a conservation law, conservation of *momentum.* The momentum of a material object, notated **p** for obscure reasons, is given by the product of its mass and its momentum,

$$\mathbf{p} = m\mathbf{v}. \tag{8.1}$$

From the definition, we see that momentum is a vector. That's important because up until now, the only conserved quantities we'd encountered were mass and energy, which are both scalars. Clearly the laws of physics would be incomplete if we never had a law of physics that related to the fact that the universe has three dimensions of space. For example, it wouldn't violate conservation of mass or energy if an object was moving in a certain direction and then suddenly changed its direction of motion, while maintaining the same speed.

If we differentiating the equation for momentum with respect to time and apply Newton's second law, we obtain

$$\mathbf{F}_{\text{total}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{8.2}$$

We can also see from this equation that in the special case of a system of particles, conservation of momentum is closely related to Newton's third law.

8.2 Collisions

A collision is an interaction between particles in which the particles interact over some period of time and then stop interacting. It is assumed that external forces are negligible. Often the result of a collision can be uniquely predicted by simultaneously imposing conservation of energy and conservation of momentum,

$$\sum E_j = \sum E'_j \tag{8.3}$$

$$\sum \mathbf{p}_k = \sum \mathbf{p}'_k, \tag{8.4}$$

where j runs over the types of energy and k over the particles.

Some collisions are highly elastic, meaning that little or no kinetic energy is transformed into other forms, such as heat or sound. Other collisions, for example car crashes, are highly inelastic. Unless we have some reason to believe that the collision is elastic, we cannot assume that the total *kinetic* energy is conserved.

8.3 The center of mass

Figure 4.4 on p. 44 showed two ice skaters, initially at rest, pushing off from each other in opposite directions. If their masses are equal, then the average of their positions, $(x_1 + x_2)/2$, remains at rest. Generalizing this to more than one dimension, more than two particles, and possibly unequal masses, we define the *center of mass* of a system of particles to be

$$\mathbf{x}_{\text{cm}} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}, \tag{8.5}$$

which is a weighted average of all the position vectors $\mathbf{x}_i$. The velocity $\mathbf{v}_{\text{cm}}$ with which this point moves is related to the total momentum of the system by

$$\mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}}. \tag{8.6}$$

If no external force acts on the system, it follows that $\mathbf{v}_{\text{cm}}$ is constant. Often problems can be simplified by adopting the center of mass frame of reference, in which $\mathbf{v}_{\text{cm}} = 0$.

Problems

8-a1 When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer *power* through its radiator coils, but not *force*?

▷ Solution, p. 299

8-a2 A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

▷ Solution, p. 299

8-a3 Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

8-a4 Two blobs of putty collide head-on and stick. The collision is completely symmetric: the blobs are of equal mass, and they collide at equal speeds. What becomes of the energy the blobs had before the collision? The momentum?

8-d1 Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass.

√

8-d2 Show that for a body made up of many *equal* masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8-d3 Objects of mass m and $4m$ are dropped from the top of a building (both starting from rest). When it hits the ground, the object of mass m has momentum p. What is the momentum of the heavier object when it hits the ground?

8-d4 The force acting on an object is $F = Ae^{-t/\tau}$, where A and τ are positive constants. The object is at rest at time $t = 0$.
(a) What is the momentum of the object at time $t = \tau$? √
(b) What is the final momentum of the object? √

8-d5 The force acting on an object is $F = At^2$. The object is at rest at time $t = 0$. What is its momentum at $t = T$?

√

8-g1 Decide whether the following statements about one-dimensional motion are true or false:
(a) The momentum transferred to an object is equal to the final momentum of the object.
(b) The momentum delivered to an object by a force F is equal to the average force on the object multiplied by the time over which the force acts on the object.
(c) Momentum transfer has the same dimensions as force (SI units of newtons).
(d) The area underneath a momentum-vs-time graph gives the average force delivered to an object.

8-g2 Can the result of a collision always be determined by the condition that both energy and momentum are conserved? If your answer is no, give a counterexample.

8-g3 The big difference between the equations for momentum and kinetic energy is that one is proportional to v and one to v^2. Both, however, are proportional to m. Suppose someone tells you that there's a third quantity, funkosity, defined as $f = m^2v$, and that funkosity is conserved. How do you know your leg is being pulled?

▷ Solution, p. 300 ⋆

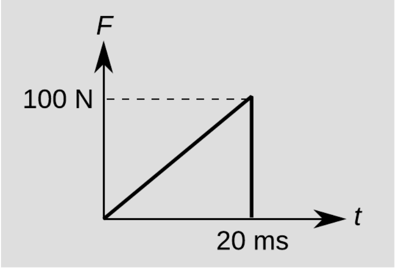
8-j1 A mass m moving at velocity v collides with a stationary target having the same mass m. Find the maximum amount of energy that can be released as heat and sound.

√

8-j2 A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.

(a) Find the bullet's final velocity. √
(b) Find the bullet's final momentum. √
(c) Find the momentum of the recoiling gun.
(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter.
√

8-j3 The figure shows the force acting on a 58.5 g tennis ball as a function of time.
(a) What is the momentum transferred to the tennis ball? √
(b) What is the final speed of the tennis ball if it is initially at rest? √
(c) What is the final speed of the tennis ball if its initial velocity is -25m/s?
√



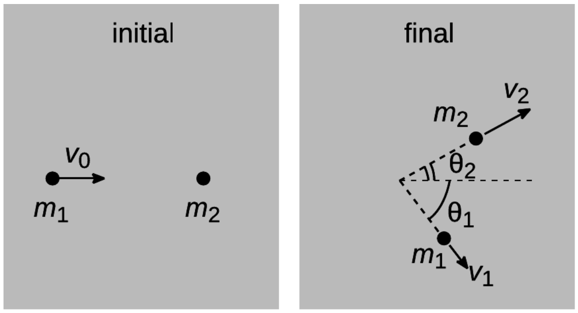
Problem 8-j3.

8-j4 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet's mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.)
√

8-j5 A 1000 kg car approaches an intersection traveling north at 20.0 m/s. A 1200 kg car approaches the same intersection traveling east at 22.0 m/s. The two cars collide at the intersection and lock together. The drivers probably wish it was all over now, but they're still moving. What is the velocity of the cars immediately after the collision?
√

8-j6 Two equal masses travel at equal speeds and collide in a perfectly inelastic collision. The final velocity of the two masses is 1/3 the initial speed. What was the angle between the velocity vectors of the two masses when they collided? (Give an exact expression, not a decimal approximation.)
√

8-j7 A ball of mass m_1 is moving to the right at speed v_0 when it collides with a ball of mass m_2 initially at rest. After the collision, m_1 loses 75% of its initial kinetic energy and has a velocity at an angle $\theta_1 = 60°$ below the horizontal, as shown.
(a) What is the speed of ball 1 after the collision?
√
(b) What are the x and y components of the momentum of ball 1 after the collision? √
(c) What are the x and y components of the momentum of ball 2 after the collision? √
(d) What fraction of the initial kinetic energy does ball 2 have after the collision? √
(e) Is this collision elastic, or inelastic? Explain.

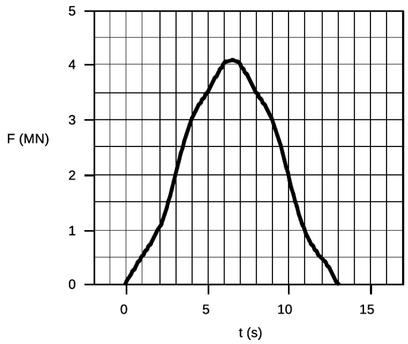


Problem 8-j7.

hw-collision-2d-given-one-angle

8-j8 The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket's mass is 4000 kg, at what speed is the rocket moving when the engine stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.

✓



Problem 8-j8.

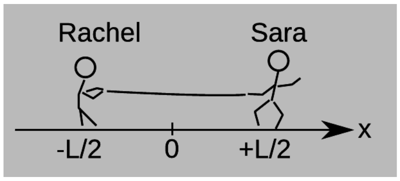
hw-rocket

8-j9 Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein's special theory of relativity. Don't worry about that, though.)

8-j10 A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

8-m1 A student of mass M is traveling on his skateboard of mass m. They are both moving at speed v, when suddenly the student kicks the board back so that it is immediately at rest relative to the ground. How fast is the student moving after kicking back the skateboard?

8-m2 Rachel and Sara are playing a game of tug-of-war on frictionless ice. They are separated by a distance L, and the coordinate system is given in the figure (with Sara at $x = +L/2$ and Rachel at $x = -L/2$). Rachel has a mass M_R, Sara has a mass M_S, and $M_R > M_S$. Where will they meet?

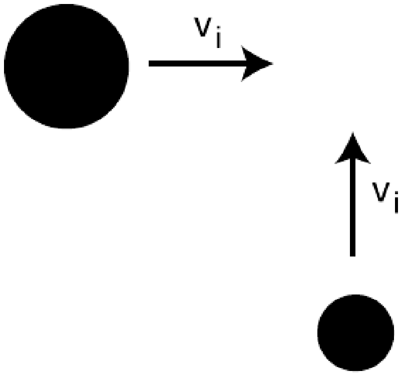


Problem 8-m2.

8-m3 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, v_i, before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.

▷ Solution, p. 302

8-m4 The moon doesn't really just orbit the Earth. By Newton's third law, the moon's gravitational force on the earth is the same as the earth's force on the moon, and the earth must respond to the moon's force by accelerating. If we consider the earth and moon in isolation and ignore outside forces, then Newton's first law says



Problem 8-m3.

their common center of mass doesn't accelerate, i.e., the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon's mass is 81 times smaller than the earth's. Compare the kinetic energies of the earth and moon.

8-m5 A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$. [Hint: Use the center-of-mass frame of reference.]

8-m6 An ice puck of mass m, traveling with speed v, hits another identical ice puck that is sitting at rest. The collision is 1-dimensional.
(a) If the collision is perfectly elastic, what is the final speed of the puck that was initially at rest? √
(b) If the collision is perfectly inelastic, what is the final speed of the two pucks after the collision? √
(c) If the collision is perfectly inelastic, what fraction of the total energy was lost during the collision? √
(d) If one-fourth of the initial kinetic energy was lost during the collision, what is the final speed of the puck that was initially at rest? √

8-m7 A bullet of mass m strikes a block of mass M which is hanging by a string of length L from the ceiling. It is observed that, after the sticky collision, the maximum angle that the string makes with the vertical is θ. This setup is called a ballistic pendulum, and it can be used to measure the speed of the bullet.
(a) What vertical height does the block reach? √
(b) What was the speed of the block just after the collision? √
(c) What was the speed of the bullet just before it struck the block? √

8-m8 A car of mass M and a truck of mass $2M$ collide head-on with equal speeds v, and the collision is perfectly inelastic, i.e., the maximum possible amount of kinetic energy is transformed into heat and sound, consistent with conservation of momentum.
(a) What is the magnitude of the change in momentum of the car? √
(b) What is the magnitude of the change in momentum of the truck? √
(c) What is the final speed of the two vehicles? √
(d) What fraction of the initial kinetic energy was lost as a result of the collision? √

8-m9 A 5.00 kg firework is launched straight up into the air. When it reaches its maximum height of $H = 140$ m, it explodes into two fragments that fly off in horizontal directions. The explosion is very quick, and only lasts 15 ms. One of the two fragments (fragment A, with mass $M_A = 2.00$ kg) lands 290 meters away from the initial launch position.
(a) Find the speed of fragment A just after the explosion. √
(b) By using conservation of momentum and your answer from part a, find the speed of the other fragment (call this fragment B) just after the explosion. √
(c) Calculate the magnitude of the momentum transferred to fragment A due to the explosion.

$\checkmark$

(d) Calculate the magnitude of the impulse delivered by gravity to fragment A over the course of the explosion. $\checkmark$

(e) How far away from the initial launch position does fragment B land? *Hint: the center of mass of the two fragments lands at the location where the firework was initially launched.*

$\checkmark$

8-m10 An object of mass m, moving at velocity u, undergoes a one-dimensional elastic collision with a mass km that is initially at rest. Let the positive direction be in the direction of the initial motion, so that $u > 0$. (a) What is the final velocity of mass m? $\checkmark$

(b) What is the final velocity of the mass km?

$\checkmark$ $\star$

8-m11 Two blocks, each of mass M, are connected by a thread and moving with speed v_0. Between them is also a spring of spring constant k, and it is compressed a distance x (so that the tension in the thread is kx). Suddenly, the thread breaks, and the spring relaxes to its equilibrium length. Find the speed of the block that is pushed forward by the spring.

$\star$

8-m12 Suppose a system consisting of pointlike particles has a total kinetic energy K_{cm} measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion.

(a) Prove that in a different frame of reference, moving with velocity $\mathbf{u}$ relative to the center-of-mass frame, the total kinetic energy equals $K_{cm} + M|\mathbf{u}|^2/2$, where M is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.]

(b) Use this to prove that if energy is conserved in one frame of reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles' interactions with each other, which we assume depends only on the distance between particles.

8-m13 A flexible rope of mass m and length L slides without friction over the edge of a table. Let x be the length of the rope that is hanging over the edge at a given moment in time.

(a) Show that x satisfies the equation of motion $\mathrm{d}^2 x/\mathrm{d}t^2 = gx/L$. [Hint: Use $F = \mathrm{d}p/\mathrm{d}t$, which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]

(b) Give a physical explanation for the fact that a larger value of x on the right-hand side of the equation leads to a greater value of the acceleration on the left side.

(c) When we take the second derivative of the function $x(t)$ we are supposed to get essentially the same function back again, except for a constant out in front. The function e^x has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form $x = be^{ct}$, where b and c are constants. Show that this does indeed provide a solution for two specific values of c (and for any value of b).

(d) Show that the sum of any two solutions to the equation of motion is also a solution.

(e) Find the solution for the case where the rope starts at rest at $t = 0$ with some nonzero value of x.

$\star$

8-m14 A rocket ejects exhaust with an exhaust velocity u. The rate at which the exhaust mass is used (mass per unit time) is b. We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket's initial mass (fuel plus the body and payload) be m_i, and m_f be its final mass, after all the fuel is used up. (a) Find the rocket's final velocity, v, in terms of u, m_i, and m_f. Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won't work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is fairly small compared to c, so it's not an unreasonable approximation to ignore relativity.) $\surd$ $\star$

9 Conservation of angular momentum

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

9.1 Angular momentum

We have discarded Newton's laws of motion and begun the process of rebuilding the laws of mechanics from scratch using conservation laws. So far we have encountered conservation of energy and momentum.

It is not hard to come up with examples to show that this list of conservation laws is incomplete. The earth has been rotating about its own axis at very nearly the same speed (once every 24 hours) for all of human history. Why hasn't the planet's rotation slowed down and come to a halt? Conservation of energy doesn't protect us against this unpleasant scenario. The kinetic energy tied up in the earth's spin could be transformed into some other type of energy, such as heat. Nor is such a deceleration prevented by conservation of momentum, since the total momentum of the earth due to its rotation cancels out.

There is a third important conservation law in mechanics, which is conservation of *angular momentum.* A spinning body such as the earth has angular momentum. Conservation of angular momentum arises from symmetry of the laws of physics with respect to rotation. That is, there is no special direction built into the laws of physics, such as the direction toward the constellation Sagittarius. Suppose that a non-spinning asteroid were to gradually start spinning. Even if there were some source of energy to initiate this spin (perhaps the heat energy stored in the rock), it wouldn't make sense for the spin to start spontaneously, because then the axis of spin would have to point in some direction, but there is no way to determine why one particular direction would be preferred.

As a concrete example, suppose that a bike wheel of radius r and mass m is spinning at a rate such that a point on the rim (where all the mass is concentrated) moves at speed v. We then define the wheel's angular momentum L to have magnitude mvr.

More generally, we define the angular momentum of a system of particles to be the sum of the quantity $\mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position of a particle relative to an arbitrarily chosen point called the axis, $\mathbf{p}$ is the particle's momentum vector, and $\times$ represents the vector cross product. This definition is chosen both because experiments show that this is the quantity that is conserved. It follows from the definition that angular momentum is a vector, and that its direction is defined by the same right-hand rule used to define the cross product.

9.2 Rigid-body dynamics

In the special case where a rigid body rotates about an axis of symmetry and its center of mass is at rest, the dimension parallel to the axis becomes irrelevant both kinematically and dynamically. We can imagine squashing the system flat so that the object rotates in a two-dimensional plane about a fixed point. The kinetic energy in this situation is

$$K = \frac{1}{2} I \omega^2$$

and the angular momentum about the center of mass is

$$L = I\omega,$$

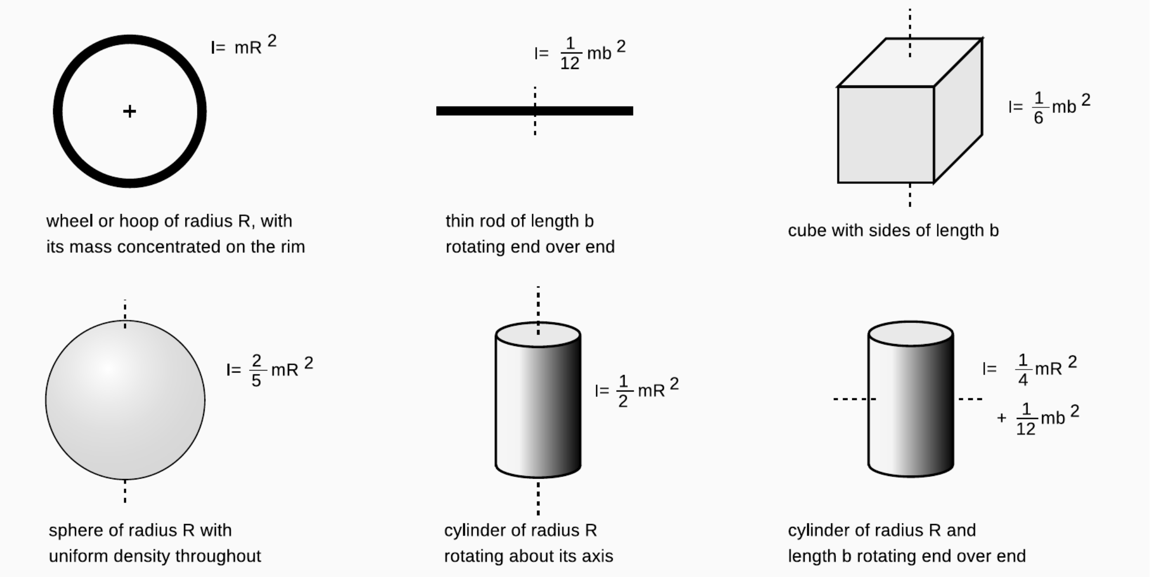

$I= mR^2$
wheel or hoop of radius R, with its mass concentrated on the rim
$I= \frac{1}{12} mb^2$
thin rod of length b rotating end over end
$I= \frac{1}{6} mb^2$
cube with sides of length b
$I= \frac{2}{5} mR^2$
sphere of radius R with uniform density throughout
$I= \frac{1}{2} mR^2$
cylinder of radius R rotating about its axis
$I= \frac{1}{4} mR^2 + \frac{1}{12} mb^2$
cylinder of radius R and length b rotating end over end

Figure 9.1: Moments of inertia of some geometric shapes.

where I is a constant of proportionality, called the *moment of inertia*. These relations are analogous to $K = (1/2)mv^2$ and $\mathbf{p} = m\mathbf{v}$ for motion of a particle. The angular velocity can also be made into a vector $\boldsymbol{\omega}$, which points along the axis in the right-handed direction. We then have $\mathbf{L} = I\boldsymbol{\omega}$.

For a system of particles, the moment of inertia is given by

$$I = \sum m_i r_i^2,$$

where r_i is the ith particle's distance from the axis. For a continuous distribution of mass,

$$I = \int r^2 \, \mathrm{d}m.$$

Figure **??** gives the moments of inertia of some commonly encountered shapes.

9.3 Torque

For each of the three conserved quantities we have encountered so far, we can define a rate of transfer or transformation:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} \qquad \text{[power]}$$
$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \qquad \text{[force]}$$
$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \qquad \text{[torque].}$$

The torque exerted by a force $\mathbf{F}$ can be expressed as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

For rigid-body rotation about an axis of symmetry, we have

$$\alpha = \frac{\boldsymbol{\tau}}{I},$$

which is analogous to Newton's second law.

9.4 Statics

If an object is to be in *equilibrium*, both the total torque and the total force acting on it must be zero:

$$\sum \mathbf{F} = 0$$
$$\sum \boldsymbol{\tau} = 0$$

We consider an object to be in equilibrium when these conditions apply, even if it is moving.

An equilibrium can be stable, like a marble at the bottom of a bowl, unstable, like a marble placed on top of a hemispherical hill, or neutral, like a marble placed on a flat tabletop.

Problems

9-a1 A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

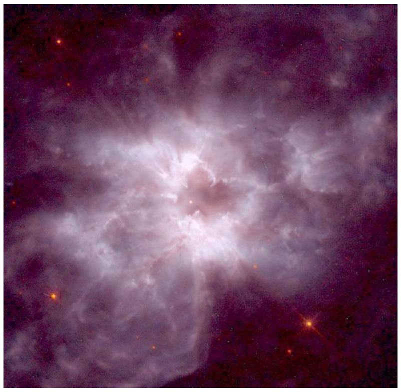
9-a2 The earth moves about the sun in an elliptical orbit where angular momentum about the sun is conserved. Earth's distance from the sun ranges from 0.983 AU (at perihelion, the closest point) to 1.017 AU (at aphelion, the farthest point). The AU is a unit of distance. If the orbital speed of Earth at aphelion is v, what is the orbital speed of Earth at perihelion?

9-a3 (a) Alice says Cathy's body has zero momentum, but Bob says Cathy's momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.
(b) Alice and Bob agree that Dong's body has nonzero momentum, but disagree about Dong's angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

9-a4 Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]

9-a5 Find the angular momentum of a particle whose position is $\mathbf{r} = 3\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in meters) and whose momentum is $\mathbf{p} = -2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in kg·m/s). ✓

9-d1 The sun turns on its axis once every 26.0 days. Its mass is 2.0×10^{30} kg and its radius is 7.0×10^{8} m. Assume it is a rigid sphere of uniform density.
(a) What is the sun's angular momentum? ✓
In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes 5.8×10^{6} m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)
(b) What will its angular momentum be?
(c) How long will it take to turn once on its axis? ✓


Problem 9-d1.

9-d2 Give a numerical comparison of the two molecules' moments of inertia for rotation in the plane of the page about their centers of mass. ✓

9-d3 A baseball pitcher can throw a curveball toward home plate at 138 km/hr with a spin of 2500 r.p.m. What percentage of the total KE of the baseball is in rotational kinetic energy? Treat the 145-gram baseball as a uniform sphere of radius 3.7 cm. ✓

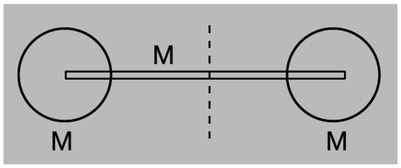
9-d4 A merry-go-round consists of a uniform disc of mass M. It spins around at N revolutions per minute. Mary, who also has mass M, is thinking about jumping on.
(a) What is the initial angular velocity of the merry-go-round (in terms of N alone)? ✓
(b) Suppose Mary jumps on the merry-go-round at the edge with negligible initial velocity. How many revolutions per minute does the merry-go-round make? ✓
(c) Suppose that, instead of jumping on the edge with no initial velocity, Mary wants to jump onto the edge (tangent to the disc) such that the merry-go-round is at rest right after jumping on. Express your answer in terms of the radius of the merry-go-round, R, and the initial angular velocity of the merry-go-round, ω. ✓

9-d5 A circular, solid disc of radius R and mass $2M$ is rotating with angular velocity ω. A second disc of radius R and mass M is dropped onto the rotating disc, and the two slide against each other until they reach the same final angular velocity.
(a) What is the final angular velocity of the discs? ✓
(b) What percentage of the initial KE was lost during the collision? ✓

9-d6 In a physics lecture, a student holds a bicycle wheel of moment of inertia I while sitting on a stool that can spin. With her feet on the ground so as not to move, she starts the wheel spinning with angular velocity ω in the counterclockwise direction when looking down from above, so that the angular velocity vector points towards the ceiling. She then picks up her feet and turns the wheel over so that the rotation is reversed. This causes her to start rotating about the stool's axis of rotation. In the following, take angular momenta to be positive if pointing vertically upwards and negative if pointing vertically downwards.
(a) What is the final (spin) angular momentum of the bicycle wheel? ✓
(b) What is the final angular momentum of rest of the system, assuming there is no external frictional torque during the flip? ✓

9-d7 Show that a sphere of radius R that is rolling without slipping has angular momentum and momentum in the ratio $L/p = (2/5)R$.

9-d8 A dumbbell consists of two solid, spherical masses (each of mass M) attached to the two ends of a bar, also of mass M. The length of the bar is L.
(a) Find the moment of inertia of the mass distribution about an axis through the center of mass and perpendicular to the bar, assuming the balls are pointlike. ✓
(b) Now find the moment of inertia assuming the spherical masses have radius r. You will need to use the parallel-axis theorem for this. To simplify the math and make it easier to compare the answers to parts a and b, take the densities to add in the places where the bar's volume overlaps with the volume of a ball. ✓
(c) Let $r = L/5$. By what percentage is your result from part a off from that in part b? Give your answer to two significant figures. ✓

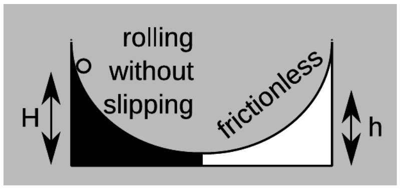


Problem 9-d8.

9-d9 A solid sphere of weight W rolls without slipping up an incline at an angle of θ (with respect to the horizontal). At the bottom of the incline the center of mass of the sphere has translational speed v. How far along the ramp does the sphere travel before coming to rest (not vertical height, but distance along the ramp)? ✓

9-d10 A sphere of mass M and radius R is not necessarily solid or hollow. It has moment of inertia $I = cMR^2$. As shown in the figure, the sphere starts from rest and rolls without slipping down a ramp from height H. It then moves back up the other side, but now with no friction at all between the sphere and the ramp. What height does the sphere reach?

✓



Problem 9-d10.

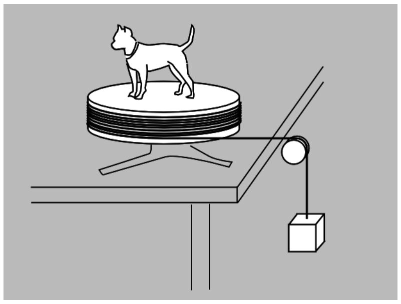
9-d11 You race a hoop, a solid sphere, and a solid cylinder down an incline of angle θ with respect to the horizontal. Each object rolls without slipping.
(a) What is the linear acceleration of the center of mass of the hoop? ✓
(b) The solid sphere? ✓
(c) The solid cylinder?Note this problem is easier if you don't do each part separately, but rather say that $I = cMR^2$, and plug in different values of c at the very end of the calculation.

✓

9-d12 The figure shows a tabletop experiment that can be used to determine an unknown moment of inertia. A rotating platform of radius R has a string wrapped around it. The string is threaded over a pulley and down to a hanging weight of mass m. The mass is released from rest, and its downward acceleration a ($a > 0$) is measured. Find the total moment of inertia I of the platform plus the object sitting on top of it. (The moment of inertia of the object itself can then be found by subtracting the value for the empty platform.)

✓

9-d13 Show that when a thin, uniform ring rotates about a diameter, the moment of inertia is half as big as for rotation about the axis of symmetry.

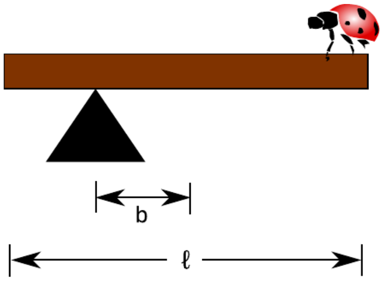


Problem 9-d12.

▷ Solution, p. 310

9-d14 A bug stands at the right end of a rod of length ℓ, which is initially at rest in a horizontal position. The rod rests on a fulcrum which is at a distance b to the left of the rod's center, so that when the rod is released from rest, the bug's end will drop. For what value of b will the bug experience apparent weightlessness at the moment when the rod is released?

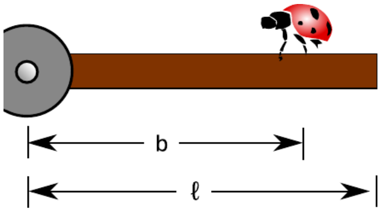
✓



Problem 9-d14.

9-d15 The figure shows a trap door of length ℓ, which is released at rest from a horizontal position and swings downward under its own weight. The bug stands at a distance b from the hinge. Because the bug feels the floor dropping out from under it with some acceleration, it feels a change in the apparent acceleration of gravity from g to some value g_a, at the moment when the door is released. Find g_a.

✓



Problem 9-d15.

9-e1 The nucleus ^{168}Er (erbium-168) contains 68 protons (which is what makes it a nucleus of the element erbium) and 100 neutrons. It has an ellipsoidal shape like an American football, with one long axis and two short axes that are of equal diameter. Because this is a subatomic system, consisting of only 168 particles, its behavior shows some clear quantum-mechanical properties. It can only have certain energy levels, and it makes quantum leaps between these levels. Also, its angular momentum can only have certain values, which are all multiples of 2.109×10^{-34} kg $\cdot$ m^2/s. The table shows some of the observed angular momenta and energies of ^{168}Er, in SI units (kg $\cdot$ m^2/s and joules).

$L \times 10^{34}$	$E \times 10^{14}$
0	0
2.109	1.2786
4.218	4.2311
6.327	8.7919
8.437	14.8731
10.546	22.3798
12.655	31.135
14.764	41.206
16.873	52.223

(a) These data can be described to a good approximation as a rigid end-over-end rotation. Estimate a single best-fit value for the moment of inertia from the data, and check how well the data agree with the assumption of rigid-body rotation. √

(b) Check whether this moment of inertia is on the right order of magnitude. The moment of inertia depends on both the size and the shape of the nucleus. For the sake of this rough check, ignore the fact that the nucleus is not quite spherical. To estimate its size, use the fact that a neutron or proton has a volume of about 1 fm^3 (one cubic femtometer, where 1 fm $= 10^{-15}$ m), and assume they are closely packed in the nucleus.

9-e2 When we talk about rigid-body rotation, the concept of a perfectly rigid body can only be an idealization. In reality, any object will compress, expand, or deform to some extent when subjected to the strain of rotation. However, if we let it settle down for a while, perhaps it will reach a new equilibrium. As an example, suppose we fill a centrifuge tube with some compressible substance like shaving cream or Wonder Bread. We can model the contents of the tube as a one-dimensional line of mass, extending from $r = 0$ to $r = \ell$. Once the rotation starts, we expect that the contents will be most compressed near the "floor" of the tube at $r = \ell$; this is both because the inward force required for circular motion increases with r for a fixed ω, and because the part at the floor has the greatest amount of material pressing "down" (actually outward) on it. The linear density $\mathrm{d}m/\,\mathrm{d}r$, in units of kg/m, should therefore increase as a function of r. Suppose that we have $\mathrm{d}m/\,\mathrm{d}r = \mu e^{r/\ell}$, where μ is a constant. Find the moment of inertia.

√

9-e3 (a) As suggested in the figure, find the area of the infinitesimal region expressed in polar coordinates as lying between r and $r+\mathrm{d}r$ and between θ and $\theta + \mathrm{d}\theta$. √

(b) Generalize this to find the infinitesimal element of volume in cylindrical coordinates (r, θ, z), where the Cartesian z axis is perpendicular to the directions measured by r and θ. √

(c) Find the moment of inertia for rotation about its axis of a cone whose mass is M, whose height is h, and whose base has a radius b.

√

9-e4 Find the moment of inertia of a solid rectangular box of mass M and uniform density, whose sides are of length a, b, and c, for rotation about an axis through its center parallel to the edges of length a.

✓

9-e5 (a) Prove the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ by expanding the product in terms of its components. Note that because the x, y, and z components are treated symmetrically in the definitions of the vector cross product, it is only necessary to carry out the proof for the x component of the result.
(b) Applying this to the angular momentum of a rigidly rotating body, $L = \int \mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})\,\mathrm{d}m$, show that the diagonal elements of the moment of inertia tensor can be expressed as, e.g., $I_{xx} = \int (y^2+z^2)\,\mathrm{d}m$.
(c) Find the diagonal elements of the moment of inertia matrix of an ellipsoid with axes of lengths a, b, and c, in the principal-axis frame, and with the axis at the center.

✓

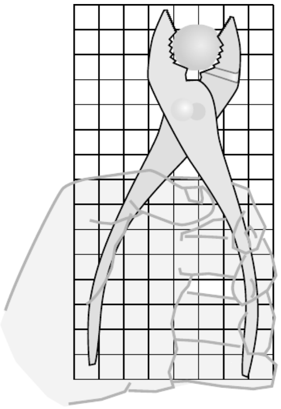
9-e6 Let two sides of a triangle be given by the vectors $\mathbf{A}$ and $\mathbf{B}$, with their tails at the origin, and let mass m be uniformly distributed on the interior of the triangle. (a) Show that the distance of the triangle's center of mass from the intersection of sides $\mathbf{A}$ and $\mathbf{B}$ is given by $\frac{1}{3}|\mathbf{A}+\mathbf{B}|$.
(b) Consider the quadrilateral with mass $2m$, and vertices at the origin, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$. Show that its moment of inertia, for rotation about an axis perpendicular to it and passing through its center of mass, is $\frac{m}{6}(A^2+B^2)$.
(c) Show that the moment of inertia for rotation about an axis perpendicular to the plane of the original triangle, and passing through its center of mass, is $\frac{m}{18}(A^2+B^2-\mathbf{A}\cdot\mathbf{B})$. Hint: Combine the results of parts a and b with the result of problem **??**.

9-e7 In this problem we investigate the notion of division by a vector.
(a) Given a nonzero vector $\mathbf{a}$ and a scalar b, suppose we wish to find a vector $\mathbf{u}$ that is the solution of $\mathbf{a}\cdot\mathbf{u} = b$. Show that the solution is not unique, and give a geometrical description of the solution set.
(b) Do the same thing for the equation $\mathbf{a}\times\mathbf{u} = \mathbf{c}$.
(c) Show that the *simultaneous* solution of these two equations exists and is unique.

Remark: This is one motivation for constructing the number system called the quaternions. For a certain period around 1900, quaternions were more popular than the system of vectors and scalars more commonly used today. They still have some important advantages over the scalar-vector system for certain applications, such as avoiding a phenomenon known as gimbal lock in controlling the orientation of bodies such as spacecraft.

⋆

9-g1 The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person's hand.
▷ Solution, p. 314



Problem 9-g1.

9-g2 Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

9-g3 An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

9-g4 You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long.

Problem 9-g2.

If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

√

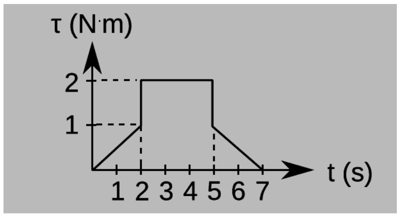
9-g5 A bicycle wheel with moment of inertia 0.15 kg·m^2 takes 30 seconds to come to rest from an initial angular velocity of 90 r.p.m. What is the magnitude of the average frictional torque over this deceleration?

9-g6 The graph shows the torque on a rigid body as a function of time. The body is at rest at $t = 2$ s. Later, at $t = 5$ s, it is spinning at angular velocity 90 s^{-1}.
(a) What is the moment of inertia of the body?

√

(b) What is its angular velocity at $t = 7$ s?

√



Problem 9-g6.

9-g7 A disk starts from rest and rotates about a fixed axis, subject to a constant torque. The work done by the torque during the first revolution is W. What is the work done by the torque during the second revolution?

√

9-g8 A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.)

√

9-g9 Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]

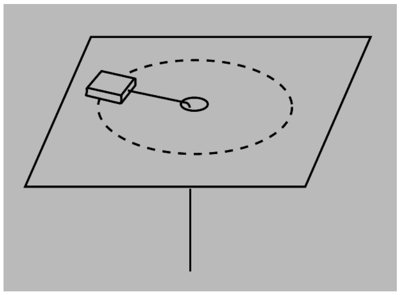
9-g10 Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux's angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.

9-j1 A massless rod of length ℓ has weights, each of mass m, attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance b from the rod's center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

9-j2 A solid rectangular door of moment of inertia I is initially open and at rest when a piece of sticky clay of mass m and velocity v_0 strikes the door perpendicularly at a distance d from the axis of the hinges.
(a) Find the angular speed of the door just after the sticky collision. ✓
(b) What fraction of the initial KE was lost during the sticky collision? ✓
(c) Suppose the door comes to rest after rotating $\Delta\theta > 0$ because of a constant frictional torque. What is the value of this frictional torque? ✓

9-j3 A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

9-j4 A book is spinning around a circle of radius R on top of a flat tabletop. The speed of the book is initially v_0. The inward force is provided by a string attached to the book, going through a hole at the center of the circle ($r = 0$), and passing underneath to a person holding the string. The person pulls on the string so that the book spirals inward, eventually cutting the radius of the circular motion in half.
(a) What is the speed of the book at $r = R/2$? ✓
(b) How much work is done by the person pulling on the string as the book moves from $r = R$ to $r = R/2$? ✓
(The string's force exerts no torque on the book, since it is always in the direction towards the hole. The force, however, can have a non-zero component tangential to the book's motion, because the book spirals in toward the hole. This is why the answer to part b is nonzero.)

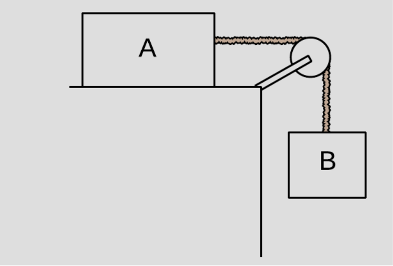


Problem 9-j4.

9-j5 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals 5/7 of its initial velocity. [Hint: You'll need the result of problem 9-d7.]

9-j6 A yo-yo of total mass m consists of two solid cylinders of radius R, connected by a small spindle of negligible mass and radius r. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1 + R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau = \Delta L/\Delta t$ and $F = ma$ to determine these two unknowns.]

9-m1 Block A (of mass M_A) rests on a frictionless horizontal table. It is connected via a light string to block B (of mass M_B) hanging over the edge of the table. The pulley itself, a solid disc, has non-negligible mass M_C. The light string does not slip over the pulley. What is the magnitude of the linear acceleration of the system?



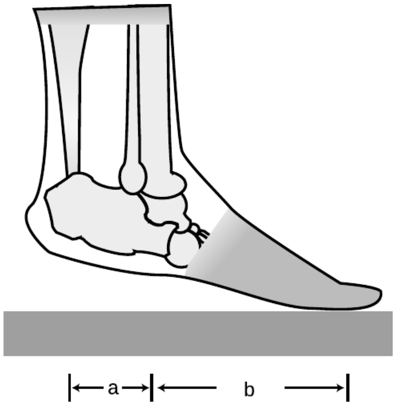
Problem 9-m1.

9-m2 A rod of length b and mass m stands upright. We want to strike the rod at the bottom, causing it to fall and land flat.
(a) Find the momentum, p, that should be delivered, in terms of m, b, and g. [Hint: the solution is nearly determined by units.] √
(b) Can this really be done without having the rod scrape on the floor?
⋆

9-m3 (a) Find the moment of inertia of a uniform square of mass m and with sides of length b, for rotation in its own plane, about one of its corners. √
(b) The square is balanced on one corner on a frictionless surface. An infinitesimal perturbation causes it to topple. Find its angular velocity at the moment when its side slaps the surface.
√ ⋆

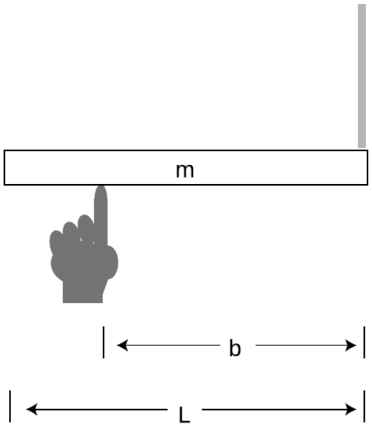
9-p1 An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

9-p2 A person of weight W stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of W, a, and b. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.
√



Problem 9-p2.

9-p3 The rod in the figure is supported by the finger and the string.
(a) Find the tension, T, in the string, and the force, F, from the finger, in terms of m, b, L, and g. √
(b) Comment on the cases $b = L$ and $b = L/2$.
(c) Are any values of b unphysical?



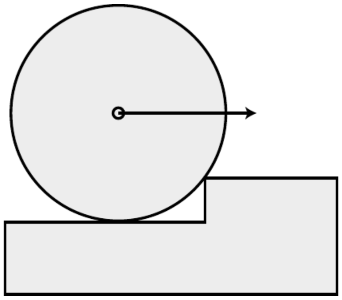
Problem 9-p3.

9-p4 Two atoms will interact through electrical forces between their protons and electrons. One fairly good approximation to the electrical energy is the Lennard-Jones formula,

$$U(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right],$$

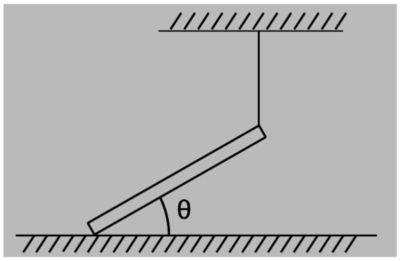
where r is the center-to-center distance between the atoms and k is a positive constant. Show that (a) there is an equilibrium point at $r = a$,
(b) the equilibrium is stable, and
(c) the energy required to bring the atoms from their equilibrium separation to infinity is k.

9-s1 (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]
(b) Under what circumstances does your result become infinite? Give a physical interpretation.



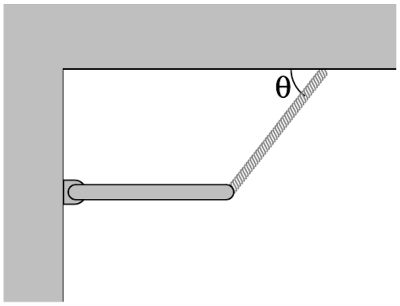
Problem 9-s1.

9-s2 A uniform bar of mass M and length L is held up by a vertical rope attached to the ceiling, as shown.
(a) What is the tension in the rope? √
(b) What is the normal force provided by the ground on the bar? √
(c) What is the static frictional force provided by the ground on the bar? √



Problem 9-s2.

9-s3 (a) The bar of mass m is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, T, in the cable in terms of the angle θ. √
(b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable?
(c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpet this physically. ⋆

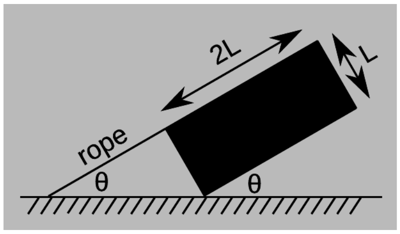


Problem 9-s3.

9-s4 A block of mass m (uniformly distributed) and dimensions $2L \times L$ is prevented from toppling over by a rope, making the same angle with respect to the horizontal as the long end of the block. ($\theta < \arctan(2) \approx 63.4°$, which is another way of saying that the block's center of mass lies to the right of the point of contact with the ground.) (a) What is the tension in the rope? √
(b) What is the normal force provided by the ground? √
(c) What is the minimum coefficient of friction required for this configuration to be in static

equilibrium? Evaluate your expression for $\theta = 30°$.

√ ⋆



Problem 9-s4.

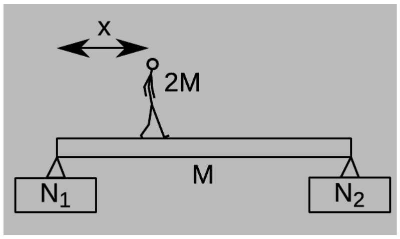
9-s5 A meter-stick balances on a fulcrum placed at the 50.0 cm mark only when a 4.0 g weight is placed at the 80.0 cm mark. Without the weight, the fulcrum needs to be placed at the 48.0 cm mark to balance the stick. What is the mass of the meter-stick?

9-s6 Julia, whose mass is $2M$, is walking across a uniform bar of mass M and length L, as shown. The bar is supported by scales at its left and right ends. The readings on the scales are N_1 and N_2, respectively.

(a) What is the maximum distance x from the left end of the bar that she can reach so that N_2 does not exceed Mg? √

(b) What is the reading N_1 when she reaches this maximum distance?

√



Problem 9-s6.

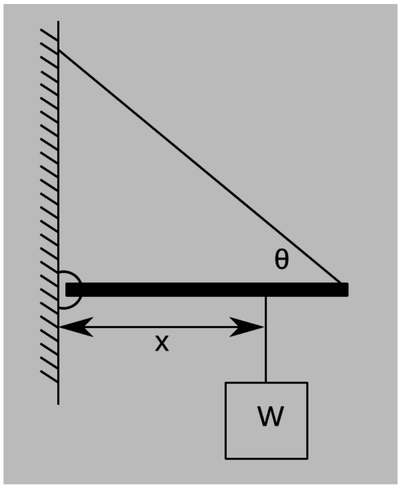
9-s7 A bar of mass m and length L is supported by a hinge on one end and by a rope on the other end. The rope makes an angle θ with respect to the horizontal. In addition, a weight W is hung from the bar at a distance x away from the hinge. Take the $+x$ direction to the right, and $+y$ vertically upwards.

(a) What is the tension in the rope? √

(b) What is the x component of the force from the hinge on the bar? √

(c) What is the y component of the force from the hinge on the bar?

√



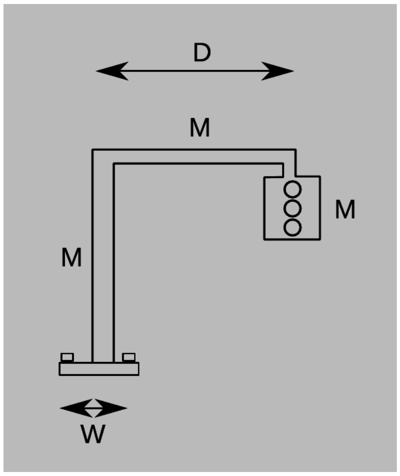
Problem 9-s7.

9-s8 A stoplight consists of three pieces of mass M: a thin, vertical bar mounted at the center of a base of width w, a horizontal bar of length D, and the stoplight fixture itself. (The two bars have a uniform mass distribution.) The structure is screwed to the ground at the base. For simplicity, we take there to be only two screws, one on the left and one on the right side of the base, which has width $w < (D/2)$. Assume that the entire normal force from the ground acts on the right-hand side of the base. (This is where the structure would naturally pivot.)

(a) By taking the right side of the base as your pivot point, you should be able to easily see that the screw on the left must provide a downwards force to keep the stoplight in static equilibrium.

What is the magnitude of this force? ✓
(b) Find the upward normal force acting on the base.

✓

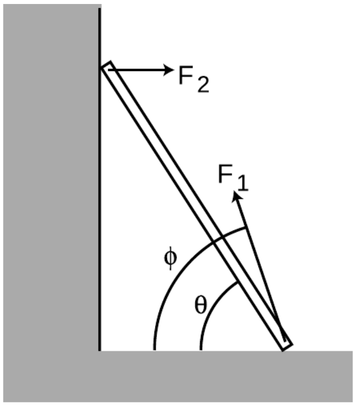


Problem 9-s8.

9-s9 A uniform ladder of mass m and length L leans against a smooth wall, making an angle θ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude F_1 at an angle ϕ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be F_2.
(a) Explain why ϕ must be greater than θ. No math is needed.
(b) Choose any numerical values you like for m and L, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta = 45.00°$ and $\phi = 63.43°$.

9-v1 Continuing problem 9-s9, find an equation for ϕ in terms of θ, and show that m and L do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a-b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.)

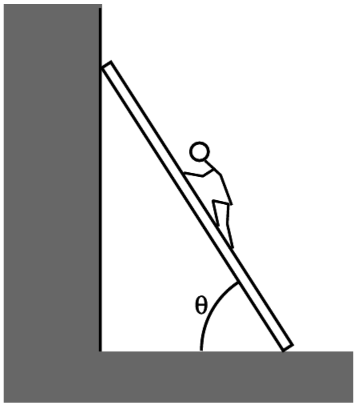
✓ ⋆



Problem 9-v1.

9-v2 A person of mass M is climbing a ladder of length L and negligible mass, propped up against a wall making an angle θ with respect to the horizontal. There is no friction between the ladder and the wall, but there is a coefficient of static friction μ_s between the ladder and the ground. What is the maximum distance along the ladder that the person can reach before the ladder starts to slide?

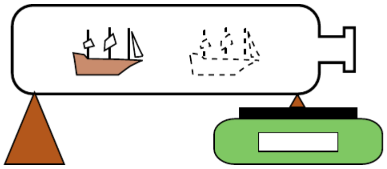
⋆



Problem 9-v2.

9-v3 You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can't determine the position of the ship's center of mass just by looking at it, and likewise for the bottle. In particular, you can't just say, "position the ship right on top of the fulcrum" or "position it right on top of the balance."

⋆



Problem 9-v3.

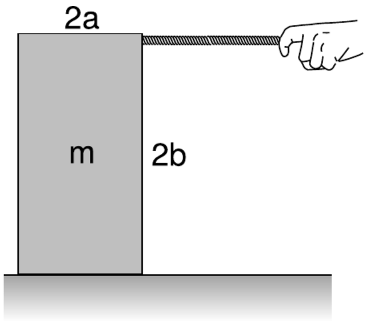
9-v4 The box shown in the figure is being accelerated by pulling on it with the rope.

(a) Assume the floor is frictionless. What is the maximum force that can be applied without causing the box to tip over? √

(b) Repeat part a, but now let the coefficient of friction be μ. √

(c) What happens to your answer to part b when the box is sufficiently tall? How do you interpret this?

⋆

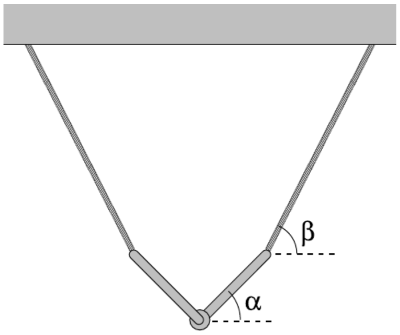


Problem 9-v4.

9-v5 (a) The two identical rods are attached to one another with a hinge, and are supported by the two massless cables. Find the angle α in terms of the angle β, and show that the result is a purely geometric one, independent of the other variables involved. √

(b) Using your answer to part a, sketch the configurations for $\beta \to 0$, $\beta = 45°$, and $\beta = 90°$. Do your results make sense intuitively?

⋆



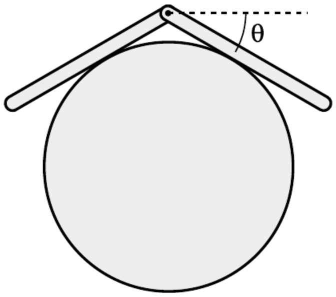
Problem 9-v5.

9-v6 Two bars of length L are connected with a hinge and placed on a frictionless cylinder of radius r. (a) Show that the angle θ shown in the figure is related to the unitless ratio r/L by the equation

$$\frac{r}{L} = \frac{\cos^2\theta}{2\tan\theta}.$$

(b) Discuss the physical behavior of this equation for very large and very small values of r/L.

⋆



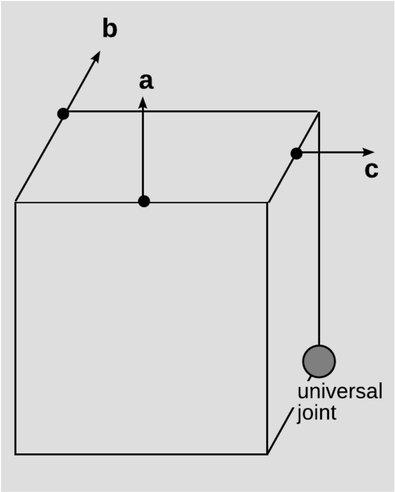
Problem 9-v6.

9-v7 We have n identical books of width w, and we wish to stack them at the edge of a table so that they extend the maximum possible distance L_n beyond the edge. Surprisingly, it is possible to have values of L_n that are greater than w, even with fairly small n. For large n, however, L_n begins to grow very slowly. Our goal is to find L_n for a given n. We adopt the restriction that only one book is ever used at a given height.[1] (a) Use proof by induction to find L_n, expressing your result as a sum. (b) Find a sufficiently tight lower bound on this sum, as a closed-form expression, to prove that 1,202,604 books suffice for $L > 7w$.

$\star$

9-v8 The uniform cube has unit weight and sides of unit length. One corner is attached to a universal joint, i.e., a frictionless bearing that allows any type of rotation. If the cube is in equilibrium, find the magnitudes of the forces **a**, **b**, and **c**.

$\checkmark$



Problem 9-v8.

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[1] When this restriction is lifted, the calculation of L_n becomes a much more difficult problem, which was partially solved in 2009 by Paterson, Peres, Thorup, Winkler, and Zwick.

10 Fluids

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

10.1 Statics

Fluids

In physics, the term *fluid* is used to mean either a gas or a liquid. The important feature of a fluid can be demonstrated by comparing with a cube of jello on a plate. The jello is a solid. If you shake the plate from side to side, the jello will respond by shearing, i.e., by slanting its sides, but it will tend to spring back into its original shape. A solid can sustain shear forces, but a fluid cannot. A fluid does not resist a change in shape unless it involves a change in volume.

Pressure

We begin by restricting ourselves to the case of fluid statics, in which the fluid is at rest and in equilibrium. A small chunk or "parcel" of the fluid has forces acting on it from the adjacent portions of the fluid. We have assumed that the parcel is in equilibrium, and if no external forces are present then these forces must cancel. By the definition of a fluid these forces are perpendicular to any part of the imaginary boundary surrounding the parcel. Since force is an additive quantity, the force the fluid exerts on any surface must be proportional to the surface's area. We therefore define the *pressure* to be the (perpendicular) force per unit area,

$$P = \frac{F_\perp}{A}. \tag{10.1}$$

If the forces are to cancel, then this force must be the same on all sides on an object such as a cube, so it follows that pressure has no direction: it is a scalar. The SI units of pressure are newtons per square meter, which can be abbreviated as pascals, 1 Pa = 1 N/m^2. The pressure of the earth's atmosphere at sea level is about 100 kPa.

Only pressure *differences* are ordinarily of any importance. For example, your ears hurt when you fly in an airplane because there is a pressure difference between your inner ear and the cabin; once the pressures are equalized, the pain stops.

Variation of pressure with depth

If a fluid is subjected to a gravitational field and is in equilibrium, then the pressure can only depend on depth (figure 10.1).

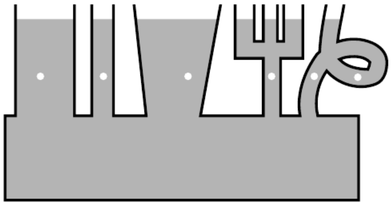


Figure 10.1: The pressure is the same at all the points marked with dots.

To find the variation with depth, we consider the vertical forces acting on a tiny, cubical parcel of the fluid having infinitesimal height dy, where positive y is up. By requiring equilibrium, we find that the difference in pressure between the top and bottom is d$P = -\rho g\,$dy. A more elegant way of writing this is in terms of a dot product,

$$\mathrm{d}P = \rho \mathbf{g} \cdot \mathrm{d}\mathbf{y} \tag{10.2}$$

which automatically takes care of the plus or minus sign, and avoids any requirements about the coordinate system. By integrating this equation, we can find the change in pressure ΔP corresponding to any change in depth Δy.

Archimedes' principle

A helium balloon or a submarine experiences unequal pressure above and below, due to the variation of pressure with depth. The total force of the surrounding fluid does not vanish, and is called the buoyant force. In a fluid in equilibrium that does not contain any foreign object, any parcel of fluid evidently has its weight canceled out by the buoyant force on it. This buoyant force is unchanged if another object is substituted for the parcel of fluid, so the buoyant force on a submerged object is upward and equal to the weight of the displaced fluid. This is called *Archimedes' principle.*

10.2 Dynamics

Continuity

We now turn to fluid dynamics, eliminating the restriction to cases in which the fluid is at rest and in equilibrium. Mass is conserved, and this constrains the ways in which a fluid can flow. For example, it is not possible to have a piece of pipe with water flowing *out* of it at each end indefinitely. The principle of continuity states that when a fluid flows steadily (so that the velocity at any given point is constant over time), mass enters and leaves a region of space at equal rates.

Liquids are highly incompressible, so that it is often a good approximation to assume that the density is the same everywhere. In the case of incompressible flow, we can frequently relate the rate of steady flow to the cross-sectional area, as in figure 10.2. Because the water is incompressible, the rate at which mass flows through a perpendicular cross-section depends only on the product of the velocity and the cross-sectional area. Therefore as the water falls and accelerates, the cross-sectional area goes down.

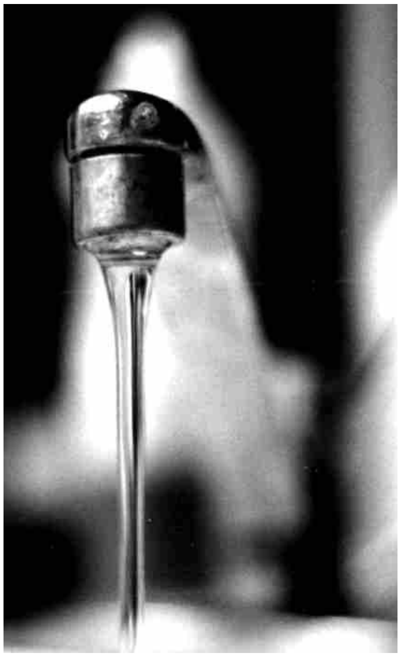


Figure 10.2: Due to conservation of mass, the stream of water narrows.

Bernoulli's equation

Consider a parcel of fluid as it flows from one place to another. If it accelerates or decelerates, then its kinetic energy changes. If it rises or falls, its potential energy changes as well. If there is a net change in $KE + PE$, then this must be accomplished through forces from the surrounding fluid. For example, if water is to move uphill at constant speed, then there must be a pressure difference, such as one produced by a pump. Based on these considerations, one can show that along a streamline of the flow,

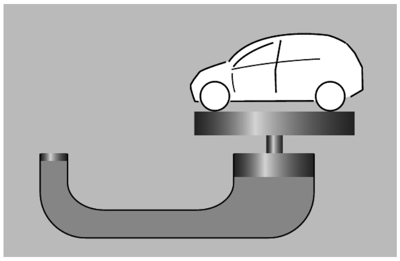
$$\rho g y + \frac{1}{2}\rho v^2 + P = \text{constant}, \qquad (10.3)$$

which is *Bernoulli's principle.*

Problems

10-a1 The pressure at 11,000 meters (where most commercial airliners fly) is about a quarter of atmospheric pressure at sea level. Suppose you're a passenger on a plane, flying at such an altitude but with the cabin pressurized to 1 atm, and you notice a gremlin on the wing. Suppose that the window near your seat isn't screwed in and is held in place only by the difference in pressure. What force would the gremlin need to exert on your 40 cm × 50 cm window in order to get in? $\surd$

10-a2 The figure shows a schematic diagram of a car on a hydraulic lift. The small piston has diameter d and the large one D. Any difference in height between the two sides is not enough to create a significant difference in pressure due to gravity.
(a) If we would like to lift a car of mass M, what force is required at the piston end? $\surd$
(b) If we would like to lift the car by a distance L, how far does the piston need to move? Assume that the fluid is incompressible, so that its volume is conserved. $\surd$
(c) By considering the work done in lifting the car, verify that your answers to part a and b are consistent with conservation of energy.



Problem 10-a2.

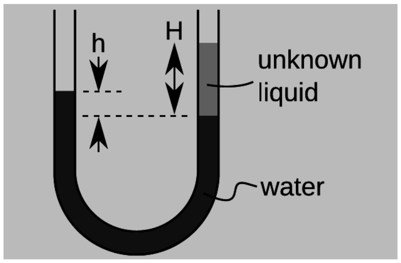
10-d1 A drinking straw works because, when you suck air out of the straw, there is decreased pressure. The equilibrium water level inside the straw is such that the pressure at the water level outside the straw (atmospheric pressure) is equal to the pressure at the same height inside the straw. Inside the straw, there are two contributions: the air pressure inside the straw plus the column of water above this height.
(a) Suppose you find that you can only suck water up to a height of 1.0 m. What is the minimum pressure inside the straw? $\surd$
(b) Superman, strong as he is, can suck out all the air from a (strong-walled) straw. What is the tallest straw through which Superman can drink water out of a lake? $\surd$

10-d2 The first transatlantic telegraph cable was built in 1858, lying at a depth of up to 3.2 km. What is the pressure at this depth, in atmospheres?

10-d3 One way to measure the density of an unknown liquid is by using it as a barometer. Suppose you have a column of length L of the unknown liquid (inside a vacuum tube), which provides the same pressure as atmospheric pressure P_0.
(a) What is the density of the unknown liquid? $\surd$
(b) Mercury barometers have $L = 760$ mm at standard atmospheric pressure $P_0 = 1.013 \times 10^5$ Pa. Given these data, what is the density of mercury to three significant figures? $\surd$

10-d4 A U-shaped tube, with both ends exposed to the atmosphere, has two immiscible liquids in it: water, and some unknown liquid. The unknown liquid sits on top of the water on the right side of the tube in a column of height H. Also, the water extends a height h above the unknown-water interface. If the density of water is ρ_w, what is the density of the unknown liquid? $\surd$

10-d5 Typically the atmosphere gets colder with increasing altitude. However, sometimes there is an *inversion layer*, in which this trend is reversed, e.g., because a less dense mass of warm air moves into a certain area, and rises above the



Problem 10-d4.

denser colder air that was already present. Suppose that this causes the pressure P as a function of height y to be given by a function of the form $P = P_o e^{-ky}(1 + by)$, where constant temperature would give $b = 0$ and an inversion layer would give $b > 0$. (a) Infer the units of the constants P_o, k, and b. (b) Find the density of the air as a function of y, of the constants, and of the acceleration of gravity g. (c) Check that the units of your answer to part b make sense.

▷ Solution, p. 325

10-d6 Estimate the pressure at the center of the Earth, assuming it is of constant density throughout. The gravitational field g is not constant with respect to depth. It equals Gmr/b^3 for r, the distance from the center, less than b, the earth's radius. Here m is the mass of the earth, and G is Newton's universal gravitational constant, which has units of $\text{N}\cdot\text{m}^2/\text{kg}^2$.

(a) State your result in terms of G, m, and b. ✓

(b) Show that your answer from part a has the right units for pressure.

(c) Evaluate the result numerically. ✓

(d) Given that the earth's atmosphere is on the order of one thousandth the earth's radius, and that the density of the earth is several thousand times greater than the density of the lower atmosphere, check that your result is of a reasonable order of magnitude.

10-g1 A uniform, solid sphere floats in a liquid of known density. We measure its draft, i.e., the depth to which it is submerged. From this measurement, we want to find the density of the sphere.

(a) Based on units, infer as much as possible about the form of the result.

(b) Find the density of the sphere in terms of the relevant variables.

✓

10-g2 Suppose we want to send a space probe to Venus and have it release a balloon that can float over the landscape and collect data. The venerian atmosphere is hot and corrosive, so it would destroy the kind of mylar or rubber balloon we use on earth. The purpose of this problem is to get a feel for things by estimating whether an aluminum beer can full of helium would float on Venus. Here are some data:

density of atmosphere	67 kg/m^3
density of helium	6.0 kg/m^3
mass of beer can	15 g
volume of beer can	330 cm^3

Find the ratio F_B/F_g of the buoyant force to the can's weight. Does it float? ✓

10-g3 Aluminum and lead have densities $2.8\rho_\text{w}$ and $11.3\rho_\text{w}$, respectively, where ρ_w is the density of water. If the maximum mass of the lead that you can lift while underwater is M, what is the maximum for aluminum?

✓

10-g4 A block of wood is floating in water. It has density $k\rho_\text{w}$, where ρ_w is the density of water. When you apply a downward force of magnitude F, the block becomes fully submerged. What is the mass of the block?

✓

10-g5 Gina has a mass of 62 kg.

(a) Estimate her volume, assuming that her density is the same as that of water. ✓

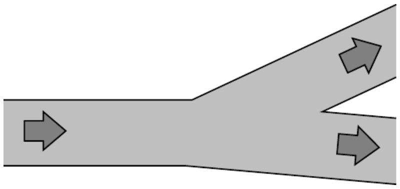
(b) Gina is in air, which has density 1.2 kg/m^3. What is the buoyant force on her? ✓

(c) Without the air pushing up slightly, she would weigh more on a standard scale. How much more would the scale claim her mass was?

✓

10-g6 An object of mass m has an apparent weight $(3/5)mg$ when submerged in water, which has density ρ_w. If the object is placed in bromine, which has density $3.1\rho_\text{w}$, it floats. What fraction of the object is submerged in bromine? ✓

10-k1 A river with a certain width and depth splits into two parts, each of which has the same width and depth as the original river. What can you say about the speed of the current after the split?



Problem 10-k1.

10-k2 At one cross-section of the Mississippi River, the width is 150 m, the depth is 12 m, and the speed of the water flowing is 1.2 m/s.

(a) What is the rate of flow of the Mississippi River at this location, measured as volume per unit time? ✓

(b) What is the corresponding rate of flow of mass? ✓

(c) If the river narrows to 140 m with a depth of 20 m, what is the new speed of the water? ✓

10-k3 The firehose shown in the figure has radius a, and water flows through it at speed u. The conical nozzle narrows the radius down to b. Find the speed v at which the water leaves the nozzle. ✓

Problem 10-k3.

10-k4 A pipe of slow-moving water is basically at atmospheric pressure, P_0. As the pipe narrows (without changing height), the water speeds up, and the pressure decreases. How fast would the water need to move in order for the pressure in the pipe to be $0.75P_0$? ✓

11 Gravity

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

11.1 Kepler's laws

Johannes Kepler (1571-1630) studied newly available high-precision data on the motion of the planets, and discovered the following three empirical laws:

Kepler's elliptical orbit law: The planets orbit the sun in elliptical orbits with the sun at one focus.

Kepler's equal-area law: The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

Kepler's law of periods: The time required for a planet to orbit the sun, called its period, is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

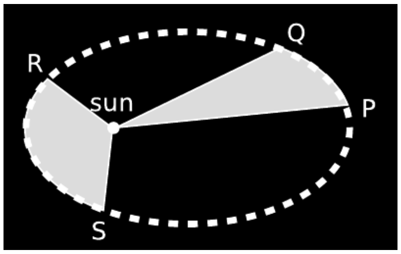


Figure 11.1: If the time interval taken by the planet to move from P to Q is equal to the time interval from R to S, then according to Kepler's equal-area law, the two shaded areas are equal.

11.2 Newton's law of gravity

Kepler's laws were a beautifully simple explanation of what the planets did, but they didn't address why they moved as they did. Once Newton had formulated his laws of motion and taught them to some of his friends, they began trying to connect them to Kepler's laws. It was clear now that an inward force would be needed to bend the planets' paths. Since the outer planets were moving slowly along more gently curving paths than the inner planets, their accelerations were apparently less. This could be explained if the sun's force was determined by distance, becoming weaker for the farther planets. In the approximation of a circular orbit, it is not difficult to show that Kepler's law of periods requires that this weakening with distance vary according to $F \propto 1/r^2$. We know that objects near the earth's surface feel a gravitational force that is also in proportion to their masses. Newton therefore hypothesized a universal law of gravity,

$$F = \frac{Gm_1m_2}{r^2},$$

which states that any two massive particles, anywhere in the universe, attract each other with a certain amount of force. The universal constant G is equal to $6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$.

11.3 The shell theorem

Newton proved the following theorem, known as the *shell theorem*:

If an object lies outside a thin, spherical shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if the shell's mass had been concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

The earth is nearly spherical, and the density in each concentric spherical shell is nearly constant. Therefore for terrestrial gravity, each shell acts as though its mass was at the center, and the result is the same as if the whole mass was there.

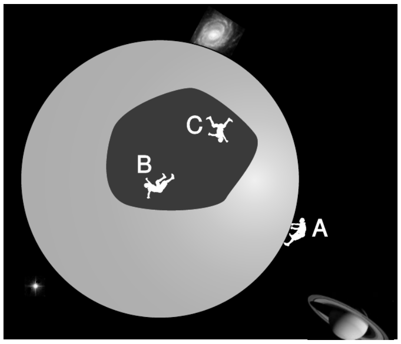


Figure 11.2: Cut-away view of a spherical shell of mass. A, who is outside the shell, feels gravitational forces from every part of the shell — stronger forces from the closer parts, and weaker ones from the parts farther away. The shell theorem states that the vector sum of all the forces is the same as if all the mass had been concentrated at the center of the shell. B, at the center, is clearly weightless, because the shell's gravitational forces cancel out. Surprisingly, C also feels exactly zero gravitational force.

11.4 Universality of free fall

Suppose that masses m_1 and m_2 interact gravitationally, and m_1 is fixed in place, or has so much mass that its inertia makes its acceleration negligible. For example, m_1 could be the earth, and m_2 a rock. When we combine Newton's law of gravity with Newton's second law, we find that m_2's acceleration equals Gm_1/r^2, which is completely independent of the mass m_2. (We assume that no other forces act.) That is, if we give an object a certain initial position and velocity in an ambient gravitational field, then its later motion is independent of its mass: free fall is *universal.*

This fact had first been demonstrated empirically a generation earlier by Galileo, who dropped a cannonball and a musketball simultaneously from the leaning tower of Pisa, and observed that they hit the ground at nearly the same time. This contradicted Aristotle's long-accepted idea that heavier objects fell faster. Modern experiments have verified the universality of free fall to the phenomenal precision of about one part in 10^{11}.

11.5 Current status of Newton's theory

Newton's theory of gravity, according to which masses act on one another *instantaneously* at a distance, is not consistent with Einstein's theory of relativity, which requires that all physical influences travel no faster than the speed of light. Einstein generalized Newton's description of gravity in his general theory of relativity. Newton's theory is a good approximation to the general theory when the masses that interact move at speeds small compared to the speed of light, when the gravitational fields are weak, and when the distances involved are small in cosmological terms. General relativity is needed in order to discuss phenomena such as neutron stars and black holes, the big bang and the expansion of the universe. The effects of general relativity also become important, for example, in the Global Positioning System (GPS), where extremely high precision is required, so that even extremely small deivations from the Newtonian picture are important.

General relativity shares with Newtonian gravity the prediction that free fall is universal. High-precision tests of this universality are therefore stringent tests of both theories.

11.6 Energy

Newton's law of gravity can be restated as a description of energy rather than force. Taking the indefinite integral of the force law gives the expression

$$PE = -\frac{GMm}{r},$$

where for convenience the constant of integration is taken to be zero.

Problems

11-a1 Astronomers have detected a solar system consisting of three planets orbiting the star Upsilon Andromedae. The planets have been named b, c, and d. Planet b's average distance from the star is 0.059 A.U., and planet c's average distance is 0.83 A.U., where an astronomical unit or A.U. is defined as the distance from the Earth to the sun. For technical reasons, it is possible to determine the ratios of the planets' masses, but their masses cannot presently be determined in absolute units. Planet c's mass is 3.0 times that of planet b. Compare the star's average gravitational force on planet c with its average force on planet b. [Based on a problem by Arnold Arons.]

▷ Solution, p. 327

11-a2 The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets' orbital radii? ✓

11-d1 Ceres, the largest asteroid in our solar system, is a spherical body with a mass 6000 times less than the earth's, and a radius which is 13 times smaller. If an astronaut who weighs 400 N on earth is visiting the surface of Ceres, what is her weight?

▷ Solution, p. 327

11-d2 (a) A certain vile alien gangster lives on the surface of an asteroid, where his weight is 0.20 N. He decides he needs to lose weight without reducing his consumption of princesses, so he's going to move to a different asteroid where his weight will be 0.10 N. The real estate agent's database has asteroids listed by mass, however, not by surface gravity. Assuming that all asteroids are spherical and have the same density, how should the mass of his new asteroid compare with that of his old one?
(b) Jupiter's mass is 318 times the Earth's, and its gravity is about twice Earth's. Is this consistent with the results of part a? If not, how do you explain the discrepancy?

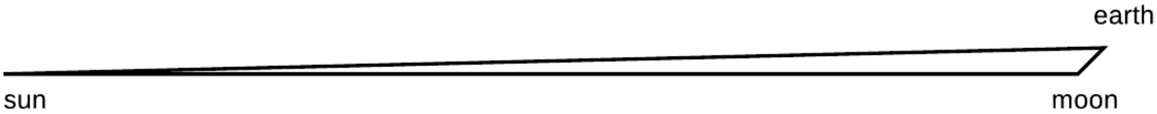
▷ Solution, p. 327

11-d3 Roy has a mass of 60 kg. Laurie has a mass of 65 kg. They are 1.5 m apart.
(a) What is the magnitude of the gravitational force of the earth on Roy?
(b) What is the magnitude of Roy's gravitational force on the earth?
(c) What is the magnitude of the gravitational force between Roy and Laurie?
(d) What is the magnitude of the gravitational force between Laurie and the sun? ✓

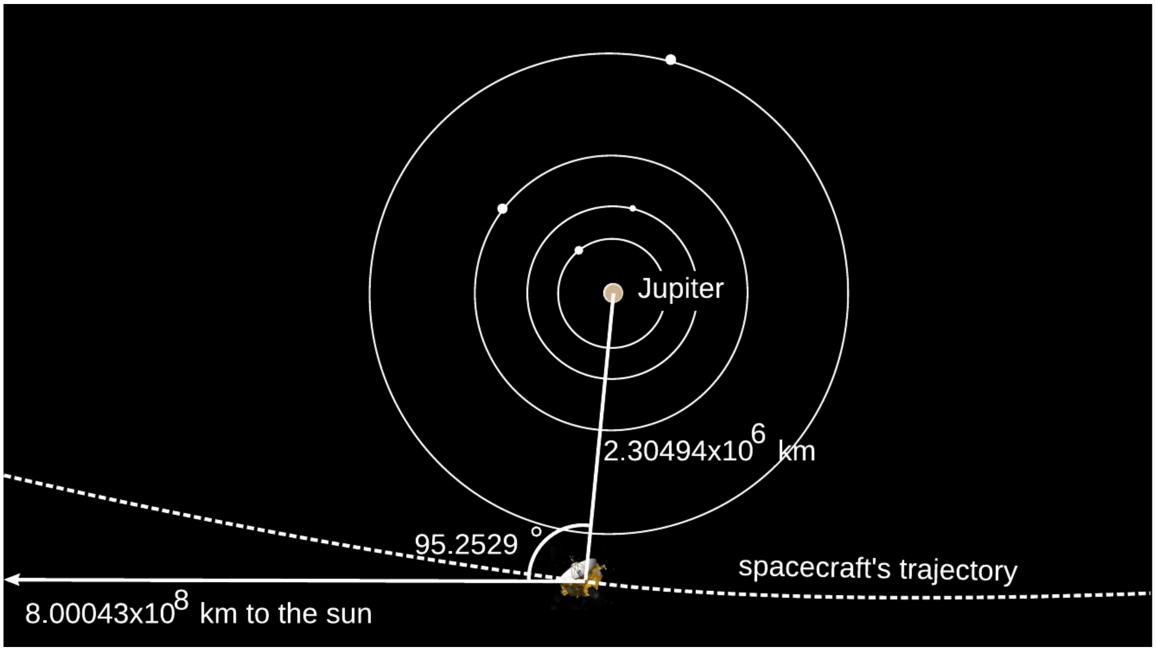
11-d4 The planet Uranus has a mass of 8.68×10^{25} kg and a radius of 2.56×10^4 km. The figure shows the relative sizes of Uranus and Earth.
(a) Compute the ratio g_U/g_E, where g_U is the strength of the gravitational field at the surface of Uranus and g_E is the corresponding quantity at the surface of the Earth. ✓
(b) What is surprising about this result? How do you explain it?

11-d5 The International Space Station orbits at an average altitude of about 370 km above sea level. Compute the value of g at that altitude. ✓

11-d6 Two spherical objects, both of mass m and radius R, have center-to-center separation $4R$ and are initially at rest. No external forces act. The spheres accelerate towards one another until they collide.
(a) What is the speed of each object just before the collision? Find the exact answer using energy methods. ✓
(b) Using the final speed from part (a), and assuming the acceleration is roughly constant and equal to the initial acceleration, calculate an upper limit on the time it takes for the two objects to collide. (The actual time will be shorter, because the acceleration increases as they get closer.) ✓
(c) Calculate a numerical value of the time from part b, in the case of two bowling balls having $m = 7$ kg and $R = 0.1$ m. ✓



Problem 11-d9.

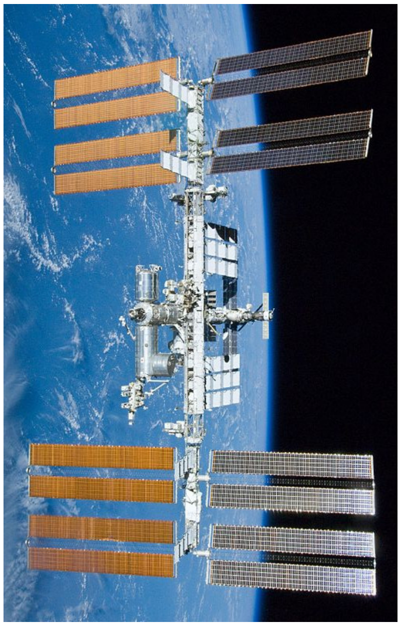


Problem 11-d10.

Problem 11-d4.

11-d7 The figure shows the International Space Station (ISS). One of the purposes of the ISS is supposed to be to carry out experiments in microgravity. However, the following factor limits this application. The ISS orbits the earth once every 92.6 minutes. It is desirable to keep the same side of the station always oriented toward the earth, which means that the station has to rotate with the same period. In the photo, the direction of orbital motion is left or right on the page, so the rotation is about the axis shown as up and down on the page. The greatest distance of any pressurized compartment from the axis of rotation is 36.5 meters. Find the acceleration due to the rotation at this point, and the apparent weight of a 60 kg astronaut at that location. √

11-d8 During a solar eclipse, the moon, earth and sun all lie on the same line, with the moon between the earth and sun. Define your coordinates so that the earth and moon lie at greater x values than the sun. For each force, give the correct sign as well as the magnitude. (a) What force is exerted on the moon by the sun? (b) On the moon by the earth? (c) On the earth by the sun? (d) What total force is exerted on the sun? (e) On the moon? (f) On the earth? √



Problem 11-d7.

11-d9 Suppose that on a certain day there is a crescent moon, and you can tell by the shape of the crescent that the earth, sun and moon form a triangle with a 135° interior angle at the moon's corner. What is the magnitude of the total gravitational force of the earth and the sun on the moon? (If you haven't done problem 11-d8 already, you might want to try it first, since it's easier, and some of its results can be recycled in this problem.) √

11-d10 On Feb. 28, 2007, the New Horizons space probe, on its way to a 2015 flyby of Pluto, passed by the planet Jupiter for a gravity-assisted maneuver that increased its speed and changed its course. The dashed line in the figure shows the spacecraft's trajectory, which is curved because of three forces: the force of the exhaust gases from the probe's own engines, the sun's gravitational force, and Jupiter's gravitational force. Find the magnitude of the total gravitational force acting on the probe. You will find that the sun's force is much smaller than Jupiter's, so that the magnitude of the total force is determined almost entirely by Jupiter's force. However, this is a high-precision problem, and you will find that the total force is slightly different from Jupiter's force.

√

11-g1 Tidal interactions with the earth are causing the moon's orbit to grow gradually larger. Laser beams bounced off of a mirror left on the moon by astronauts have allowed a measurement of the moon's rate of recession, which is about 4 cm per year. This means that the gravitational force acting between earth and moon is decreasing. By what fraction does the force decrease with each 27-day orbit?

[Based on a problem by Arnold Arons.]

▷ Solution, p. 329

11-g2 How high above the Earth's surface must a rocket be in order to have 1/100 the weight it would have at the surface? Express your answer in units of the radius of the Earth.

√

11-g3 You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth's orbit at one end and tangent to Mars' orbit at the other. Your spacecraft's engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (The orbits of Earth and Mars are nearly circular, and Mars's is bigger by a factor of 1.52.)

√

11-g4 Where would an object have to be located so that it would experience zero total gravitational force from the earth and moon?

√

11-j1 In a Star Trek episode, the Enterprise is in a circular orbit around a planet when something happens to the engines. Spock then tells Kirk that the ship will spiral into the planet's surface unless they can fix the engines. Is this scientifically correct? Why?

11-j2 Astronomers have recently observed stars orbiting at very high speeds around an unknown object near the center of our galaxy. For stars orbiting at distances of about 10^{14} m from the object, the orbital velocities are about 10^{6} m/s. Assuming the orbits are circular, estimate the mass of the object, in units of the mass of the sun, 2×10^{30} kg. If the object was a tightly packed cluster of normal stars, it should be a very bright source of light. Since no visible light is detected coming from it, it is instead believed to be a supermassive black hole.

√

11-j3 (a) A geosynchronous orbit is one in which the satellite orbits above the equator, and has an orbital period of 24 hours, so that it is always above the same point on the spinning earth. Calculate the altitude of such a satellite. √

(b) What is the gravitational field experienced by the satellite? Give your answer as a percentage in relation to the gravitational field at the earth's surface. √

11-j4 (a) Suppose a rotating spherical body such as a planet has a radius r and a uniform density ρ, and the time required for one rotation is T. At the surface of the planet, the apparent acceleration of a falling object is reduced by the acceleration of the ground out from under it. Derive an equation for the apparent acceleration of gravity, g, at the equator in terms of r, ρ, T, and G. √

(b) Applying your equation from a, by what fraction is your apparent weight reduced at the equator compared to the poles, due to the Earth's rotation? √

(c) Using your equation from a, derive an equation giving the value of T for which the apparent acceleration of gravity becomes zero, i.e., objects can spontaneously drift off the surface of the planet. Show that T only depends on ρ, and not on r. √

(d) Applying your equation from c, how long would a day have to be in order to reduce the apparent weight of objects at the equator of the Earth to zero? [Answer: 1.4 hours]

(e) Astronomers have discovered objects they called pulsars, which emit bursts of radiation at regular intervals of less than a second. If a pulsar is to be interpreted as a rotating sphere beaming

out a natural “searchlight” that sweeps past the earth with each rotation, use your equation from c to show that its density would have to be much greater than that of ordinary matter.
(f) Astrophysicists predicted decades ago that certain stars that used up their sources of energy could collapse, forming a ball of neutrons with the fantastic density of $\sim 10^{17}$ kg/m^3. If this is what pulsars really are, use your equation from c to explain why no pulsar has ever been observed that flashes with a period of less than 1 ms or so.

Problem 11-j5.

11-j5 The figure shows an image from the Galileo space probe taken during its August 1993 flyby of the asteroid Ida. Astronomers were surprised when Galileo detected a smaller object orbiting Ida. This smaller object, the only known satellite of an asteroid in our solar system, was christened Dactyl, after the mythical creatures who lived on Mount Ida, and who protected the infant Zeus. For scale, Ida is about the size and shape of Orange County, and Dactyl the size of a college campus. Galileo was unfortunately unable to measure the time, T, required for Dactyl to orbit Ida. If it had, astronomers would have been able to make the first accurate determination of the mass and density of an asteroid. Find an equation for the density, ρ, of Ida in terms of Ida's known volume, V, the known radius, r, of Dactyl's orbit, and the lamentably unknown variable T. (This is the same technique that was used successfully for determining the masses and densities of the planets that have moons.)

▷ Solution, p. 331

11-j6 On an airless body such as the moon, there is no atmospheric friction, so it should be possible for a satellite to orbit at a very low altitude, just high enough to keep from hitting the mountains. (a) Suppose that such a body is a smooth sphere of uniform density ρ and radius r. Find the velocity required for a ground-skimming orbit. $\checkmark$
(b) A typical asteroid has a density of about 2 g/cm^3, i.e., twice that of water. (This is a lot lower than the density of the earth's crust, probably indicating that the low gravity is not enough to compact the material very tightly, leaving lots of empty space inside.) Suppose that it is possible for an astronaut in a spacesuit to jump at 2 m/s. Find the radius of the largest asteroid on which it would be possible to jump into a ground-skimming orbit.

$\checkmark$

11-j7 If a bullet is shot straight up at a high enough velocity, it will never return to the earth. This is known as the escape velocity. In this problem, you will analyze the motion of an object of mass m whose initial velocity is *exactly* equal to escape velocity. We assume that it is starting from the surface of a spherically symmetric planet of mass M and radius b. The trick is to guess at the general form of the solution, and then determine the solution in more detail. Assume (as is true) that the solution is of the form $r = kt^p$, where r is the object's distance from the center of the planet at time t, and k and p are constants.
(a) Find the acceleration, and use Newton's second law and Newton's law of gravity to determine k and p. You should find that the result is independent of m. $\checkmark$
(b) What happens to the velocity as t approaches infinity?
(c) Determine escape velocity from the Earth's surface.

$\checkmark$

11-j8 Planet X rotates, as the earth does, and is perfectly spherical. An astronaut who weighs 980.0 N on the earth steps on a scale at the north pole of Planet X and the scale reads 600.0 N; at the equator of Planet X, the scale only reads 500.0 N. The distance from the north pole to the equator is 20,000 km, measured along the surface of Planet X.
(a) Explain why the astronaut appears to weigh more at the north pole of planet X than at the equator. Which is the "actual" weight of the astronaut? Analyze the forces acting on an astronaut standing on a scale, providing one analysis for the north pole, and one for the equator.
(b) Find the mass of planet X. Is planet X more massive than the earth, or less massive? The radius of the earth is 6370 km, and its mass is 5.97×10^{24} kg. $\surd$
(c) If a 30,000 kg satellite is orbiting the planet very close to the surface, what is its orbital period? Assume planet X has no atmosphere, so that there's no air resistance. $\surd$
(d) How long is a day on planet X? Is this longer than, or shorter than, the period of the satellite in part c? $\surd$

11-j9 A 20.0 kg satellite has a circular orbit with a period of 2.40 hours and a radius of 8.00×10^6 m around planet Z. The magnitude of the gravitational acceleration on the surface of the planet is 8.00 m/s^2.
(a) What is the mass of planet Z? $\surd$
(b) What is the radius of planet Z? $\surd$
(c) Find the KE and the PE of the satellite. What is the ratio PE/KE (including both magnitude and sign)? You should get an integer. This is a special case of something called the *virial theorem.* $\surd$
(d) Someone standing on the surface of the planet sees a moon orbiting the planet (a circular orbit) with a period of 20.0 days. What is the distance between planet Z and its moon? $\surd$

11-m1 Astronomers calculating orbits of planets often work in a nonmetric system of units, in which the unit of time is the year, the unit of mass is the sun's mass, and the unit of distance is the astronomical unit (A.U.), defined as half the long axis of the earth's orbit. In these units, find an exact expression for the gravitational constant, G. $\surd$

11-m2 Suppose that we inhabited a universe in which, instead of Newton's law of gravity, we had $F = k\sqrt{m_1 m_2}/r^2$, where k is some constant with different units than G. (The force is still attractive.) However, we assume that $a = F/m$ and the rest of Newtonian physics remains true, and we use $a = F/m$ to define our mass scale, so that, e.g., a mass of 2 kg is one which exhibits half the acceleration when the same force is applied to it as to a 1 kg mass.
(a) Is this new law of gravity consistent with Newton's third law?
(b) Suppose you lived in such a universe, and you dropped two unequal masses side by side. What would happen?
(c) Numerically, suppose a 1.0-kg object falls with an acceleration of 10 m/s^2. What would be the acceleration of a rain drop with a mass of 0.1 g? Would you want to go out in the rain?
(d) If a falling object broke into two unequal pieces while it fell, what would happen?
(e) Invent a law of gravity that results in behavior that is the opposite of what you found in part b. [Based on a problem by Arnold Arons.]

11-m3 The structures that we see in the universe, such as solar systems, galaxies, and clusters of galaxies, are believed to have condensed from clumps that formed, due to gravitational attraction, in preexisting clouds of gas and dust. Observations of the cosmic microwave background radiation suggest that the mixture of hot hydrogen and helium that existed soon after the Big Bang was extremely uniform, but not perfectly so. We can imagine that any region that started out a little more dense would form a natural center for the collapse of a clump. Suppose that we have a spherical region with density ρ and radius r, and for simplicity let's just assume that it's surrounded by vacuum. (a) Find the acceleration of the material at the edge of the cloud. To what power of r is it proportional? $\surd$

(b) The cloud will take a time t to collapse to some fraction of its original size. Show that t is independent of r.

Remark: This result suggests that structures would get a chance to form at all scales in the universe. That is, solar systems would not form before galaxies got to, or vice versa. It is therefore physically natural that when we look at the universe at essentially all scales less than a billion light-years, we see structure.

11-m4 You have a fixed amount of material with a fixed density. If the material is formed into some shape S, then there will be some point in space at which the resulting gravitational field attains its maximum value g_S. What shape maximizes g_S?

11-m5 The *escape velocity* of a massive body is the speed for which the total energy of a projectile is zero: the projectile has just enough KE to move infinitely far away from the massive body, with no left-over KE. The escape velocity depends on the distance from which the projectile is launched — often the body's surface.

The Schwarzschild radius (R_s) of a massive body is the radius where the escape velocity is equal to the speed of light, $c = 3.00 \times 10^8$ m/s. An object is called a *black hole* if it has a Schwarzschild radius.

An object must be very compact to be a black hole. For example, escape velocity from the surface of the earth is tens of thousands of times less than c, as is the escape velocity for a projectile launched from the center of the earth through a hypothetical radial, evacuated tunnel.

In this problem we will make some numerical estimates of how compact an object has to be in order to be a black hole. We will use Newtonian gravity, which is a poor approximation for such strong gravitational fields, so we expect these estimates to be rough.

(a) For an object of mass M, what would its radius have to be if all of its mass was to fit within the Schwarzschild radius? $\surd$

(b) Evaluate your equation from part a for M equal to the masses of the earth and the sun. If these bodies were compressed to approximately these sizes, they would become black holes. (Because these are rough estimates, treat them as having no more than 1 significant figure.) $\surd$

11-m6 Problems 11-m6-11-m8 all investigate the following idea. Cosmological surveys at the largest observable distance scales have detected structures like filaments. As an idealization of such a structure, consider a uniform mass distribution lying along the entire x axis, with mass density λ in units of kg/m. The purpose of this problem is to find the gravitational field created by this structure at a distance y.

(a) Determine as much as possible about the form of the solution, based on units.

(b) To evaluate the actual result, find the contribution $\mathrm{d}g_y$ to the y component of the field arising from the mass $\mathrm{d}m$ lying between x and $x + \mathrm{d}x$, then integrate it.

▷ Solution, p. 334

11-m7 Let us slightly change the physical situation described in problem 11-m6, letting the filament have a finite size, while retaining its symmetry under rotation about the x axis. The details don't actually matter very much for our purposes, but if we like, we can take the mass density to be constant within a cylinder of radius b centered on the x axis. Now consider the following two limits:

$$g_1 = \lim_{y\to 0} \lim_{b\to 0} g \qquad \text{and}$$

$$g_2 = \lim_{b\to 0} \lim_{y\to 0} g.$$

Each of these is a limit inside another limit, the only difference being the order of the limits. Either of these could be used as a definition of the field at a point *on* an infinitely thin filament. Do they agree?

11-m8 Suppose we have a mass filament like the one described in problems 11-m6 and 11-m7, but now rather than taking it to be straight, let it have the shape of an arbitrary smooth curve. Locally, "under a microscope," this curve will look like an arc of a circle, i.e., we can describe its shape solely in terms of a radius of curvature. As in problem 11-m7, consider a point P lying *on* the filament itself, taking g to be defined as in definition g_1. Investigate whether g is finite, and also whether it points in a specific direction. To clarify the mathematical idea, consider the following two limits:

$$A = \lim_{x\to 0} \frac{1}{x} \qquad \text{and}$$
$$B = \lim_{x\to 0} \frac{1}{x^2}.$$

We say that $A = \infty$, while $B = +\infty$, i.e., both diverge, but B diverges with a definite sign. For a straight filament, as in problem 11-m6, with an infinite radius of curvature, symmetry guarantees that the field at P has no specific direction, in analogy with limit A. For a curved filament, a calculation is required in order to determine whether we get behavior A or B. Based on your result, what is the expected dynamical behavior of such a filament?

11-p1 (a) If the earth was of uniform density, would your weight be increased or decreased at the bottom of a mine shaft? Explain.
(b) In real life, objects weigh slightly more at the bottom of a mine shaft. What does that allow us to infer about the Earth?

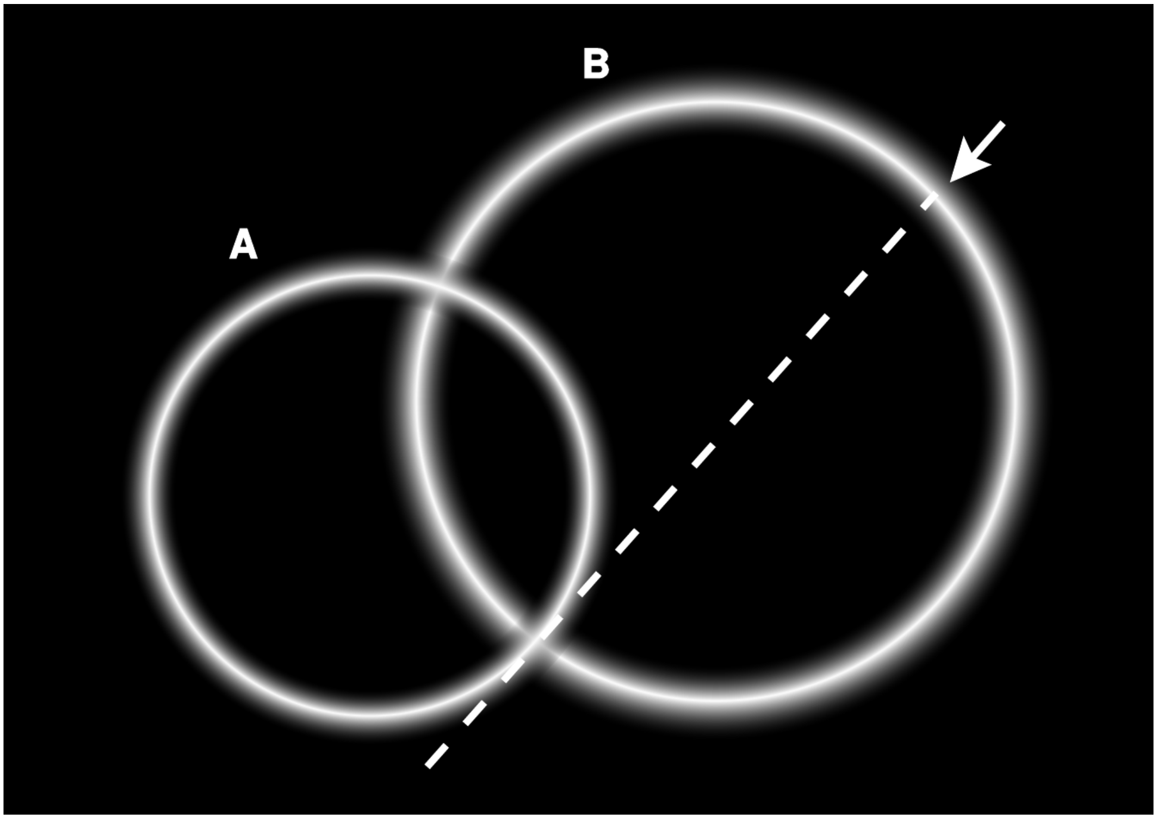
11-p2 Consult a proof of the shell theorem in your textbook or in some other source such as Wikipedia.Prove that the theorem fails if the exponent of r in Newton's law of gravity differs from -2.

11-p3 The shell theorem describes two cases, inside and outside. Show that for an alternative law of gravity $F = GMmr$ (with r^1 rather than r^{-2}), the outside case still holds.

11-p4 The figure shows a region of outer space in which two stars have exploded, leaving behind two overlapping spherical shells of gas, which we assume to remain at rest. The figure is a cross-section in a plane containing the shells' centers. A space probe is released with a very small initial speed at the point indicated by the arrow, initially moving in the direction indicated by the dashed line. Without any further information, predict as much as possible about the path followed by the probe and its changes in speed along that path.

11-p5 Approximate the earth's density as being constant. (a) Find the gravitational field at a point P inside the earth and half-way between the center and the surface. Express your result as a ratio g_P/g_S relative to the field we experience at the surface. (b) As a check on your answer, make sure that the same reasoning leads to a reasonable result when the fraction 1/2 is replaced by the value 0 (P being the earth's center) or the value 1 (P being a point on the surface).

11-p6 The earth is divided into solid inner core, a liquid outer core, and a plastic mantle. Physical properties such as density change discontinuously at the boundaries between one layer and the next. Although the density is not completely constant within each region, we will approximate it as being so for the purposes of this problem. (We neglect the crust as well.) Let R be the radius of the earth as a whole and M its mass. The following table gives a model of some properties of the three layers, as determined by methods such as the observation of earthquake waves that have propagated from one side of the planet to the other.



Problem 11-p4.

region	*outer radius/R*	*mass/M*
mantle	1	0.69
outer core	0.55	0.29
inner core	0.19	0.017

The boundary between the mantle and the outer core is referred to as the Gutenberg discontinuity. Let g_s be the strength of the earth's gravitational field at its surface and g_G its value at the Gutenberg discontinuity. Find g_G/g_s.

$\surd$

11-s1 Starting at a distance r from a planet of mass M, how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

$\surd$

11-s2 A certain binary star system consists of two stars with masses m_1 and m_2, separated by a distance b. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, v, symbolically in terms of b, m_1, m_2, and fundamental constants.

$\surd$

12 Oscillations

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

12.1 Periodic motion

The sine function has the property that $\sin(x+2\pi)=\sin x$, so that whetever the function is doing at one point, it is guaranteed to do the same thing again at a point 2π to the right. Such a function is called periodic. When an object's position as a function of time is periodic, we say that it exhibits periodic motion. Examples include uniform circular motion and a mass vibrating back and forth frictionlessly on a spring. The time from one repetition of the motion to the next is called the *period*, T. The inverse of the period is the *frequency*,

$$f=\frac{1}{T},$$

and it is also convenient to define the angular frequency

$$\omega=\frac{2\pi}{T}.$$

Either f or ω can be referred to simply as frequency, when context makes it clear or the distinction isn't important. The units of frequency are s^{-1}, which can also be abbreviated as hertz, $1\ \mathrm{Hz}=1\ \mathrm{s}^{-1}$.

12.2 Simple harmonic motion

When an object is displaced from equilibrium, it can oscillate around the equilibrium point. Let the motion be one-dimensional, let the equilibrium be at $x=0$, and let friction be negligible. Then by conservation of energy, the oscillations are periodic, and they extend from some negative value of x on the left to a value on the right that is the same except for the sign. We describe the size of the oscillations as their *amplitude*, A.

When the oscillations are small enough, Hooke's law $F=-kx$ is a good approximation, because any function looks linear close up. Therefore all such oscillations have a universal character, differing only in amplitude and frequency. Such oscillations are referred to as *simple harmonic motion.*

For simple harmonic motion, Newton's second law gives $x''=-(k/m)x$. This is a type of equation referred to as a differential equation, because it relates the function $x(t)$ to its own (second) derivative. The solution of the equation is $x=A\sin(\omega t+\delta)$, where

$$\omega=\sqrt{\frac{k}{m}}$$

is independent of the amplitude.

12.3 Damped oscillations

The total energy of an oscillation is proportional to the square of the amplitude. In the simple harmonic oscillator, the amplitude and energy are constant. Unlike this idealization, real oscillating systems have mechanisms such as friction that dissipate energy. These mechanisms are referred to as damping. A simple mathematical model that incorporates this behavior is to incorporate a frictional force that is proportional to velocity. The equation of motion then becomes $mx''+bx'+kx=0$. In the most common case, where $b<2\sqrt{km}$. In this *underdamped* case, the solutions are decaying exponentials of the form

$$x=Ae^{-ct}\sin\omega t,$$

where $c = b/2m$ and $\omega = [k/m - b^2/4m^2]^{1/2}$.

It is customary to describe the amount of damping with a quantity called the *quality factor*, Q, defined as the number of cycles required for the energy to fall off by a factor of $e^{2\pi} \approx 535$. The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

Underdamped motion occurs for $Q > 1/2$. For the case $Q < 1/2$, referred to as *overdamped*, there are no oscillations, and the motion is a decaying exponential.

12.4 Driven oscillations

It is often of interest to consider an oscillator that is driven by an oscillating force. Examples would be a mother pushing a child on a playground swing, or the ear responding to a sound wave. We assume for simplicity that the driving force oscillates sinusoidally with time, although most of the same results are qualitatively correct when this requirement is relaxed. The oscillator responds to the driving force by gradually settling down into a steady, sinusoidal pattern of vibration called the steady state. Figure 12.1 shows the bell-shaped curve that results when we graph the energy of the steady-state response against the frequency of the driving force. We have the following results.

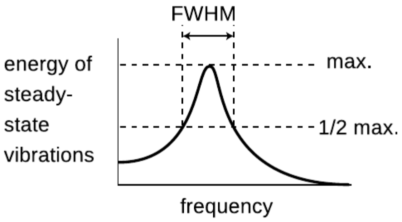


Figure 12.1: The response of an oscillator to a driving force, showing the definition of the full width at half maximum (FWHM).

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

(2) A vibrating system *resonates* at its own natural frequency.[1] That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q.

(4) The FWHM of a resonance, defined in figure 12.1, is related to its Q and its resonant frequency f_{res} by the equation

$$\text{FWHM} = \frac{f_{res}}{Q}.$$

(This equation is only a good approximation when Q is large.)

---

[1] This is an approximation, which is valid in the usual case where Q is significantly greater than 1.

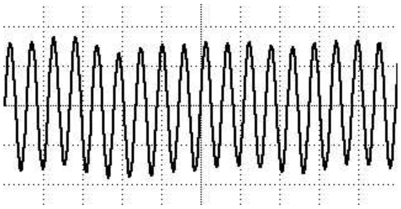
Problems

12-a1 Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail's vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond?

√

12-a2 The figure shows the oscillation of a microphone in response to the author whistling the musical note "A." The horizontal axis, representing time, has a scale of 1.0 ms per square. Find the period T, the frequency f, and the angular frequency ω.

√



Problem 12-a2.

12-d1 Show that the equation $T = 2\pi\sqrt{m/k}$ has units that make sense.

12-d2 (a) Pendulum 2 has a string twice as long as pendulum 1. If we define x as the distance traveled by the bob along a circle away from the bottom, how does the k of pendulum 2 compare with the k of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of x.]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

12-d3 A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by $F_{air} = ax^{-\beta}$, where $\beta = 1.4$ and a is a constant with funny units of $\text{N}\cdot\text{m}^{1.4}$. For simplicity, assume the air only supports the weight mg of the piston itself, although in practice this device is used to support some other object. The equilibrium position, x_0, is where mg equals $-F_{air}$. (Note that in the main text I have assumed the equilibrium position to be at $x = 0$, but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Find the angular frequency of oscillation.

√

12-d4 Verify that energy is conserved in simple harmonic motion.

12-d5 Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new "chunk" of water. (a) Show that a cube of mass m with edges of length b floating upright (not tilted) in a fluid of density ρ will have a draft (depth to which it sinks below the waterline) h given at equilibrium by $h_0 = m/b^2\rho$. (b) Find the total force on the cube when its draft is h, and verify that plugging in $h - h_0$ gives a total force of zero. (c) Find the cube's period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of and g only.

√

12-d6 A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826 m/s^2. This change, amounting to 0.046 m/s^2, is greater than the 0.022 m/s^2 effect to be expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator?

√

12-d7 A certain mass, when hung from a certain spring, causes the spring to stretch by an amount h compared to its equilibrium length. If the mass is displaced vertically from this equilibrium, it will oscillate up and down with a period T_{osc}. Give a numerical comparison between T_{osc} and T_{fall}, the time required for the mass to fall from rest through a height h, when it isn't attached to the spring. (You will need the result of problem **??**).

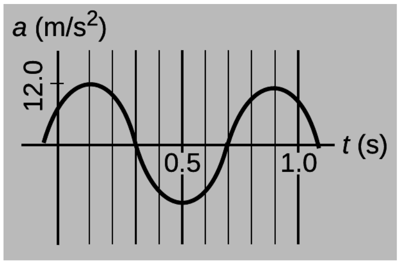
√

12-d8 An object undergoing simple harmonic motion oscillates with position $x(t) = (35\text{ cm})\cos[(25\text{ s}^{-1})t + \pi]$.
(a) Find the period, angular frequency, and frequency. √
(b) What is the initial velocity? What is the maximum speed? √
(c) What is the initial acceleration? What is the maximum in magnitude of the acceleration? √
(d) Find the location and velocity of the object at 1.00 s.

√

12-d9 On a frictionless, horizontal air track, a glider oscillates at the end of an ideal spring of force constant 150 N/m. The graph shows the acceleration of the glider as a function of time.
(a) Find the period of oscillations and the mass of the glider. √
(b) Notice that the graph says that the maximum acceleration of the glider is 12.0 m/s^2. Use this to find the amplitude of oscillations, A. √
(c) When the glider is $A/3$ away from its equilibrium position, what are its kinetic and potential energies? √
(d) Suppose $x(t) = A\cos(\omega t + \phi)$, where $0 \le \phi < 2\pi$. Find ϕ.

√

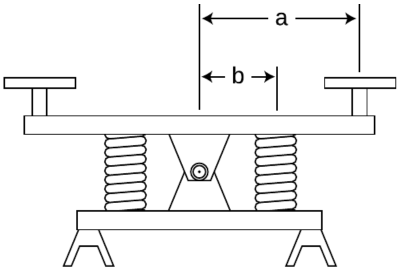


Problem 12-d9.

12-g1 Consider the same pneumatic piston described in problem 12-d3, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of x vs t, showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the curvature of the $x - t$ graph, so if the force is greater, the graph should curve around more quickly.]

12-g2 The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant k, and a kid of mass m sits on each seat. (a) Find the period of vibration in terms of the variables k, m, a, and b. (b) Discuss the special case where $a = b$, rather than $a > b$ as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of $b = 0$.

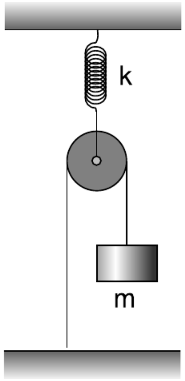
√



Problem 12-g2.

12-g3 Find the period of vertical oscillations of the mass m. The spring, pulley, and ropes have negligible mass.

√

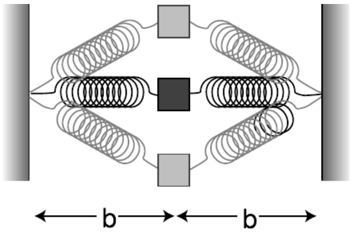


Problem 12-g3.

12-g4 The equilibrium length of each spring in the figure is b, so when the mass m is at the center, neither spring exerts any force on it. When the mass is displaced to the side, the springs stretch; their spring constants are both k.

(a) Find the energy, U, stored in the springs, as a function of y, the distance of the mass up or down from the center. √

(b) Show that the period of small up-down oscillations is infinite.



Problem 12-g4.

12-g5 For a one-dimensional harmonic oscillator, the solution to the energy conservation equation,

$$U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant},$$

is an oscillation with frequency $\omega = \sqrt{k/m}$. Now consider an analogous system consisting of a bar magnet hung from a thread, which acts like a magnetic compass. A normal compass is full of water, so its oscillations are strongly damped, but the magnet-on-a-thread compass has very little friction, and will oscillate repeatedly around its equilibrium direction. The magnetic energy of the bar magnet is

$$U = -Bm\cos\theta,$$

where B is a constant that measures the strength of the earth's magnetic field, m is a constant that parametrizes the strength of the magnet, and θ is the angle, measured in radians, between the bar magnet and magnetic north. The equilibrium occurs at $\theta = 0$, which is the minimum of U.

(a) By making analogies between rotational and linear motion, translate the equation defining the linear quantity k to one that defines an analogous angular one κ (Greek letter kappa). Applying this to the present example, find an expression for κ. (Assume the thread is so thin that its stiffness does not have any significant effect compared to earth's magnetic field.) √

(b) Find the frequency of the compass's vibrations.

√

12-j1 A mass m on a spring oscillates around an equilibrium at $x = 0$. Any function $F(x)$ with an equilibrium at $x = 0$, $F(0) = 0$, can be approximated as $F(x) = -kx$, and if the spring's behavior is symmetric with respect to positive and negative values of x, so that $F(-x) = -F(x)$, then the next level of improvement in such an approximation would be $F(x) = -kx - bx^3$. The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like sin or e^x, it's using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of b gets stiffer when stretched strongly than an "ideal" spring with $b = 0$. A spring with a negative b is like a person who cracks under stress — when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter no matter how much it is stretched. For example, there has to be some point at which it breaks.

Do a numerical simulation of the oscillation of a mass on a spring whose force has a nonvanishing b. Is the period still independent of amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of an x^3 term with $b > 0$ tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

12-j2 An idealized pendulum consists of a pointlike mass m on the end of a massless, rigid rod of length L. Its amplitude, θ, is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of m, L, and θ. Examine the effect of changing each variable while manipulating the others.

Problems 12-k1 through 12-k4 require specific knowledge of the properties of simple and physical pendulums.

12-k1 A simple pendulum of length L is released from from angle θ. Solve for the maximum speed of the pendulum bob two ways:
(a) Exactly, by using conservation of energy. ✓
(b) Approximately, by assuming $\theta \ll 1$, using $|v_{\text{max}}| = A\omega$, and writing A and ω in terms of the given quantities. Your result is the first nonzero term in the Taylor expansion of the exact answer around $\theta = 0$.

✓

12-k2 A pendulum of length L has period T on Earth. If a pendulum of length $2L$ has a period $4T$ on planet W, then what is the acceleration due to gravity on planet W? Give your answer to two significant figures.

✓

12-k3 A uniform rod of length L is hung at one end. What is the period of oscillations for this physical pendulum?

✓

12-k4 A pendulum with length L has period T when a very small mass is placed at the end of it (with size much less than L). Suppose we do not want to ignore the size of the bob. Consider a spherical bob with radius xL (x is a dimensionless constant, and L is the length of the string, connecting the pivot to the center of the bob). The period of motion of this physical pendulum is $T = 2\pi\sqrt{L/g}f(x)$. What is $f(x)$? Your expression for $f(x)$ should satisfy $f(0) = 1$. (Why?)

✓

12-m1 If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

12-m2 What fraction of the total energy of an object undergoing SHM is kinetic at time $t = T/3$ (where T is the period of motion) if:
(a) the object is at maximum displacement from

equilibrium at $t = 0$; ✓
(b) the object is at equilibrium at $t = 0$. ✓

12-m3 An object undergoing simple harmonic motion has amplitude A and angular frequency ω. What is the speed of the object when it is at a distance $x = A/4$ from equilibrium? ✓

12-m4 A spring is attached to a wall as shown (the horizontal surface is frictionless). One fact is known about the spring: when compressed a distance 11.0 cm, the spring holds 1.00 J of elastic potential energy.
(a) What mass M must be attached to the spring so that it will oscillate with a period of 1.00 s? ✓
(b) If the amplitude of the motion is 5.00 cm and the period is that specified in part a, where is the object (relative to equilibrium) and in what direction is it moving 0.35 s after it has passed the equilibrium position, moving to the left? ✓
(c) At the instant described in part b, what are the kinetic and potential energies of the system? ✓
(d) What force (magnitude and direction) does the spring exert on mass M when it is 3.00 cm to the right of the equilibrium position, moving to the right?

12-p1 (a) Let W be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy E that remains in the oscillations after n cycles of motion.

(b) From this, prove the equation

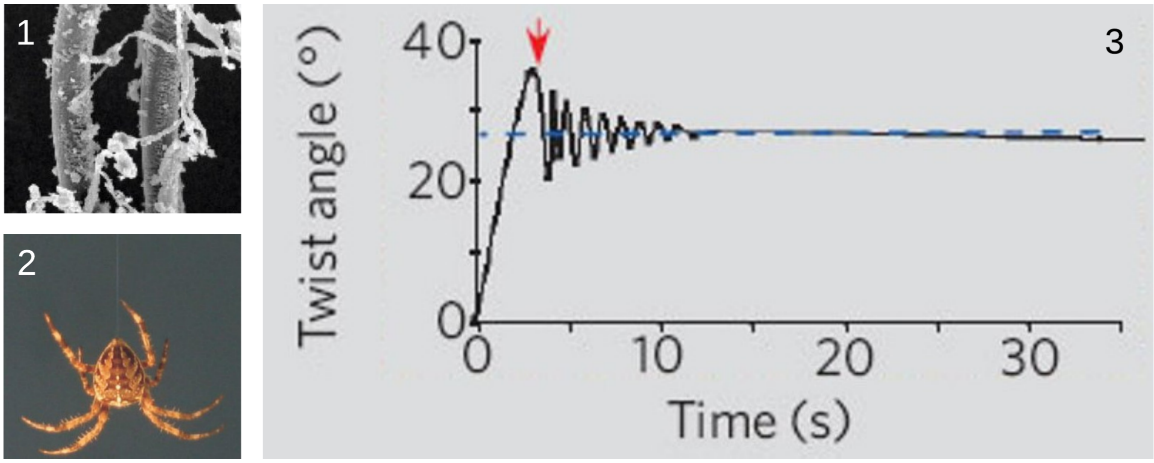
$$\left(1-\frac{W}{E}\right)^Q = e^{-2\pi}$$

(recalling that the number 535 in the definition of Q is $e^{2\pi}$).

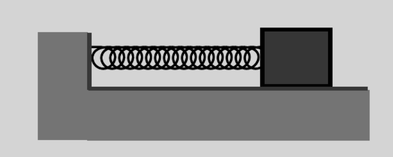
(c) Use this to prove the approximation $1/Q \approx (1/2\pi)W/E$. (Hint: Use the approximation $\ln(1+x) \approx x$, which is valid for small values of x, as shown on p. **??**.)

12-p2 (a) We observe that the amplitude of a certain free oscillation decreases from A_o to A_o/Z after n oscillations. Find its Q. ✓
(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope's image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it's probably a bad thing for the spider if it twists back and forth while hanging like this. (We're referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn't be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low Q, so the vibrations damp out quickly.
2. It doesn't become brittle with repeated twisting as a copper wire would.
3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than ±35 degrees, settling down to a new equilibrium at 27 degrees.
4. Over much longer time scales (hours), the thread eventually resets itself to its original equilbrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials



Problem 12-p2.

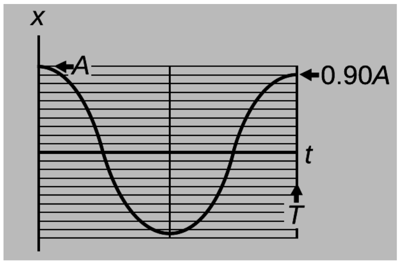


Problem 12-m4.

> have this "memory" property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the Q of spider silk from the graph. √

12-p3 An object has underdamped motion as depicted in the figure, where $T = 2\pi/\omega$, and, as described in the text, ω differs from $\omega_0 = \sqrt{k/m}$.
(a) What fraction of the energy was lost during this first cycle? This fraction is lost in every ensuing cycle. √
(b) Where will the object be (in terms of A) after the second full oscillation? √
(c) By assuming $\omega \approx \omega_0$, what is the value of b? To express your answer, write $b = C\sqrt{km}$, and solve for the unitless constant C. √
(d) Use this value of b to find the percentage increase in the period of the motion as compared to the undamped case. You should get an answer much less than 1%, which means the approximation made in part c was justified. √ ⋆



Problem 12-p3.

12-s1 Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish's buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder's Q is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

✓

12-s2 As noted in section **??**, it is only approximately true that the amplitude has its maximum at the natural frequency $(1/2\pi)\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_\text{o} = (1/2\pi)\sqrt{k/m}$ and f_res for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2 \left(f^2 - f_0^2\right)^2 + b^2 f^2}}.$$

Show that the maximum occurs not at f_o but rather at

$$f_\text{res} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2}\text{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.

12-s3 An oscillator with sufficiently strong damping has its maximum response at $\omega = 0$. Using the result derived on page **??** , find the value of Q at which this behavior sets in.

12-s4 The goal of this problem is to refine the proportionality FWHM $\propto f_{res}/Q$ into the equation FWHM $= f_{res}/Q$, i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force $F = -bv$ over one cycle of steady-state motion equals $W_{damp} = -2\pi^2 bfA^2$. Hint: It is less confusing to calculate the work done over half a cycle, from $x = -A$ to $x = +A$, and then double it.

(b) Show that the fraction of the undriven oscillator's energy lost to damping over one cycle is $|W_{damp}|/E = 4\pi^2 bf/k$.

(c) Use the previous result, combined with the result of problem 4, to prove that Q equals $k/2\pi bf$.

(d) Combine the preceding result for Q with the equation FWHM $= b/2\pi m$ from section **??** to prove the equation FWHM $= f_{res}/Q$.

12-s5 An oscillator has Q=6.00, and, for convenience, let's assume $F_m = 1.00$, $\omega_\text{o} = 1.00$, and $m = 1.00$. The usual approximations would give

$$\omega_{res} = \omega_\text{o},$$
$$A_{res} = 6.00, \qquad \text{and}$$
$$\Delta\omega = 1/6.00.$$

Determine these three quantities numerically using the result derived on page **??** , and compare with the approximations.

13 Waves

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

13.1 Free waves

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 6.1
- OpenStax University Physics, openstax.org, v. 1, sec. 16.1-4

13.2 Bounded waves

References:

- Crowell, *Simple Nature*, lightandmatter.com, v. 1, sec. 6.2

Problems

13-a1 The following is a graph of the height of a water wave as a function of *position*, at a certain moment in time.

Trace this graph onto another piece of paper, and then sketch below it the corresponding graphs that would be obtained if

(a) the amplitude and frequency were doubled while the velocity remained the same;

(b) the frequency and velocity were both doubled while the amplitude remained unchanged;

(c) the wavelength and amplitude were reduced by a factor of three while the velocity was doubled.

Explain all your answers. [Problem by Arnold Arons.]

13-a2 (a) The graph shows the height of a water wave pulse as a function of position. Draw a graph of height as a function of time for a specific point on the water. Assume the pulse is traveling to the right.
(b) Repeat part a, but assume the pulse is traveling to the left.
(c) Now assume the original graph was of height as a function of time, and draw a graph of height as a function of position, assuming the pulse is traveling to the right.
(d) Repeat part c, but assume the pulse is traveling to the left.
Explain all your answers. [Problem by Arnold Arons.]

13-a3 The figure shows one wavelength of a steady sinusoidal wave traveling to the right along a string. Define a coordinate system in which the positive x axis points to the right and the positive y axis up, such that the flattened string would have $y = 0$. Copy the figure, and label with $y = 0$ all the appropriate parts of the string. Similarly, label with $v = 0$ all parts of the string whose velocities are zero, and with $a = 0$ all parts whose accelerations are zero. There is more than one point whose velocity is of the greatest magnitude. Pick one of these, and indicate the direction of its velocity vector. Do the same for a point having the maximum magnitude of acceleration. Explain all your answers.

[Problem by Arnold Arons.]

13-a4 (a) Find an equation for the relationship between the Doppler-shifted frequency of a wave and the frequency of the original wave, for the case of a stationary observer and a source moving directly toward or away from the observer. ✓
(b) Check that the units of your answer make sense.
(c) Check that the dependence on v_s makes sense.

13-a5 Suggest a quantitative experiment to look for any deviation from the principle of superposition for surface waves in water. Make it simple and practical.

13-a6 The musical note middle C has a frequency of 262 Hz. What are its period and wavelength?

✓

13-a7 Singing that is off-pitch by more than about 1% sounds bad. How fast would a singer have to be moving relative to the rest of a band to make this much of a change in pitch due to the Doppler effect?

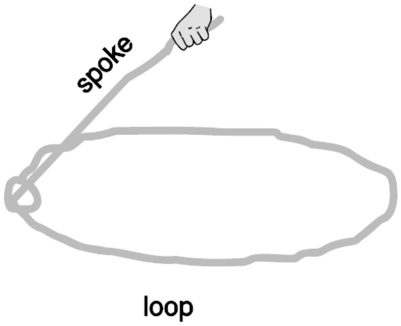
✓

13-a8 In section **??**, we saw that the speed of waves on a string depends on the ratio of T/μ, i.e., the speed of the wave is greater if the string is under more tension, and less if it has more inertia. This is true in general: the speed of a mechanical wave always depends on the medium's inertia in relation to the restoring force (tension, stiffness, resistance to compression,...). Based on these ideas, explain why the speed of sound in air is significantly greater on a hot day, while the speed of sound in liquids and solids shows almost no variation with temperature.

13-a9 At a particular moment in time, a wave on a string has a shape described by $y = 3.5\cos(0.73\pi x + 0.45\pi t + 0.37\pi)$. The stuff inside the cosine is in radians. Assume that the units of the numerical constants are such that x, y, and t are in SI units.
(a) Is the wave moving in the positive x or the negative x direction?
(b) Find the wave's period, frequency, wavelength.
(c) Find the wave's velocity.
(d) Find the maximum velocity of any point on the string, and compare with the magnitude and direction of the wave's velocity.
√

13-a10 The simplest trick with a lasso is to spin a flat loop in a horizontal plane. The whirling loop of a lasso is kept under tension mainly due to its own rotation. Although the spoke's force on the loop has an inward component, we'll ignore it. The purpose of this problem, which is based on one by A.P. French, is to prove a cute fact about wave disturbances moving around the loop. As far as I know, this fact has no practical implications for trick roping! Let the loop have radius r and mass per unit length μ, and let its angular velocity be ω.
(a) Find the tension, T, in the loop in terms of r, μ, and ω. Assume the loop is a perfect circle, with no wave disturbances on it yet. √
(b) Find the velocity of a wave pulse traveling around the loop. Discuss what happens when the pulse moves is in the same direction as the rotation, and when it travels contrary to the rotation.
√

13-a11 A string hangs vertically, free at the bottom and attached at the top.
(a) Find the velocity of waves on the string as a function of the distance from the bottom. √
(b) Find the acceleration of waves on the string.
(c) Interpret your answers to parts a and b for the case where a pulse comes down and reaches the end of the string. What happens next? Check your answer against experiment and conservation of energy.



Problem 13-a10.

13-d1 Light travels faster in warmer air. On a sunny day, the sun can heat a road and create a layer of hot air above it. Let's model this layer as a uniform one with a sharp boundary separating it from the cooler air above. Use this model to explain the formation of a mirage appearing like the shiny surface of a pool of water.

13-d2 (a) Compute the amplitude of light that is reflected back into air at an air-water interface, relative to the amplitude of the incident wave. Assume that the light arrives in the direction directly perpendicular to the surface.The speeds of light in air and water are 3.0×10^8 and 2.2×10^8 m/s, respectively.
(b) Find the energy of the reflected wave as a fraction of the incident energy.
√

13-d3 A concert flute produces its lowest note, at about 262 Hz, when half of a wavelength fits inside its tube. Compute the length of the flute.

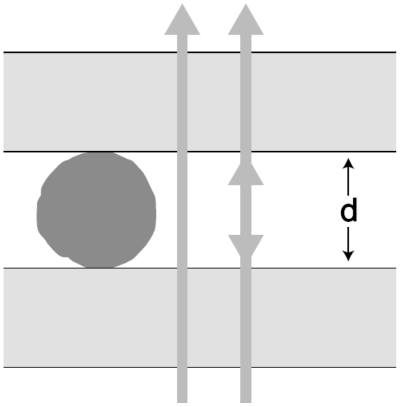
13-d4 (a) A good tenor saxophone player can play all of the following notes without changing her fingering, simply by altering the tightness of her lips: E♭ (150 Hz), E♭ (300 Hz), B♭ (450 Hz), and E♭ (600 Hz). How is this possible? (I'm not asking you to analyze the coupling between the lips, the reed, the mouthpiece, and the air column, which is very complicated.)
(b) Some saxophone players are known for their

Problem 13-d3.

ability to use this technique to play "freak notes," i.e., notes above the normal range of the instrument. Why isn't it possible to play notes below the normal range using this technique?

13-d5 The table gives the frequencies of the notes that make up the key of F major, starting from middle C and going up through all seven notes.
(a) Calculate the first four or five harmonics of C and G, and determine whether these two notes will be consonant or dissonant. (Recall that harmonics that differ by about 1-10% cause dissonance.)
(b) Do the same for C and B♭.

13-d6 A Fabry-Perot interferometer, shown in the figure being used to measure the diameter of a thin filament, consists of two glass plates with an air gap between them. As the top plate is moved up or down with a screw, the light passing through the plates goes through a cycle of constructive and destructive interference, which is mainly due to interference between rays that pass straight through and those that are reflected twice back into the air gap. (Although the dimensions in this drawing are distorted for legibility, the glass plates would really be much thicker than the length of the wave-trains of light, so no interference effects would be observed due to reflections within the glass.)
(a) If the top plate is cranked down so that the thickness, d, of the air gap is much less than the wavelength λ of the light, i.e., in the limit $d \to 0$, what is the phase relationship between the two rays? (Recall that the phase can be inverted by a reflection.) Is the interference constructive, or destructive?
(b) If d is now slowly increased, what is the first value of d for which the interference is the same as at $d \to 0$? Express your answer in terms of λ.
(c) Suppose the apparatus is first set up as shown in the figure. The filament is then removed, and n cycles of brightening and dimming are counted while the top plate is brought down to $d = 0$. What is the thickness of the filament, in terms of n and λ?
Based on a problem by D.J. Raymond.



Problem 13-d6.

13-d7 (a) A wave pulse moves into a new medium, where its velocity is greater by a factor α. Find an expression for the fraction, f, of the wave energy that is transmitted, in terms of α. Note that, as discussed in the text, you cannot simply find f by squaring the amplitude of the transmitted wave.
(b) Suppose we wish to transmit a pulse from one medium to another, maximizing the fraction of the wave energy transmitted. To do so, we sandwich another layer in between them, so that the wave moves from the initial medium, where

its velocity is v_1, through the intermediate layer, where it is v_2, and on into the final layer, where it becomes v_3. What is the optimal value of v_2? (Assume that the middle layer is thicker than the length of the pulse, so there are no interference effects. Also, although there will be later echoes that are transmitted after multiple reflections back and forth across the middle layer, you are only to optimize the strength of the transmitted pulse that is first to emerge. In other words, it's simply a matter of applying your answer from part a twice to find the amount that finally gets through.)

13-d8 The expressions for the amplitudes of reflected and transmitted waves depend on the unitless ratio v_2/v_1 (or, more generally, on the ratio of the impedances). Call this ratio α. (a) Show that changing α to $1/\alpha$ (e.g., by interchanging the roles of the two media) has an effect on the reflected amplitude that can be expressed in a simple way, and discuss what this means in terms of inversion and energy. (b) Find the two values of α for which $|R| = 1/2$.

14 Electrical interactions

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

14.1 Charge and Coulomb's law

It appears superficially that nature has many different types of forces, such as frictional forces, normal forces, sticky forces, the force that lets bugs walk on water, the force that makes gunpowder explode, and the force that causes honey to flow so slowly. Actually, all of the forces on this list are manifestations of electrical interactions at the atomic scale. Like gravity, electricity is a $1/r^2$ force. The electrical counterpart of Newton's law of gravity is Coulomb's law,

$$F = \frac{k|q_1||q_2|}{r^2}, \tag{14.1}$$

where F is the magnitude of the force, k is a universal constant, r is the distance between the two interacting objects, and q_1 and q_2 are properties of the objects called their electric charges. This equation is often written using the alternate form of the constant $\epsilon_0 = 1/(4\pi k)$.

Electric charge is measured in units of Coulombs, C. Charge is to electricity as mass is to gravity. There are two types of charge, which are conventionally labeled positive and negative. Charges of the same type repel one another, and opposite charges attract. Charge is conserved.

Charge is quantized, meaning that all charges are integer multiples of a certain fundamental charge e. (The quarks that compose neutrons and protons have charges that come in thirds of this unit, but quarks are never observed individually, only in clusters that have integer multiples of e.) The electron has charge $-e$, the proton $+e$.

14.2 The electric field

Newton conceived of forces as acting instantaneously at a distance. We now know that if masses or charges in a certain location are moved around, the change in the force felt by a distant mass or charge is delayed. The effect travels at the speed of light, which according to Einstein's theory of relativity represents a maximum speed at which cause and effect can propagate, built in to the very structure of space and time. Because time doesn't appear as a variable in Coulomb's law, Coulomb's law cannot fundamentally be a correct description of electrical interactions. It is only an approximation, which is valid when charges are not moving (the science of electrostatics) or when the time lags in the propagation of electrical interactions are negligible. These considerations imply logically that while an electrical effect is traveling through space, it has its own independent physical reality. We think of space as being permeated with an electric field, which varies dynamically according to its own rules, even if there are no charges nearby. Phenomena such as visible light and radio waves are ripples in the electric (and magnetic) fields. For now we will study only static electric fields (ones that don't change with time), but fields come into their own when their own dynamics are important.

The electric field $\mathbf{E}$ at a given point in space can be defined in terms of the electric force $\mathbf{F}$ that would be exerted on a hypothetical test charge q_t inserted at that point:

$$\mathbf{E} = \frac{\mathbf{F}}{q_t}. \tag{14.2}$$

By a test charge, we mean one that is small enough so that its presence doesn't disturb the situation that we're trying to measure. From the definition, we see that the electric field is a vector with units of newtons per coulumb, N/C. Its gravitational counterpart is the familiar **g**, whose magnitude on earth is about 9.8 $\mathrm{m/s^2}$. Because forces combine according to the rules of vector addition, it follows that the electric field of a combination of charges is the vector sum of the fields that would have been produced individually by those charges.

The electric field contains energy. The electrical energy contained in an infinitesimal volume $\mathrm{d}v$ is given by $\mathrm{d}U_e = (1/8\pi k)E^2\,\mathrm{d}v$.

14.3 Conductors and insulators

Some materials, such as metals, are good electrical conductors, meaning that they contain charges that are free to move. A material like dry wood is an insulator, because it contains few such free charges. When a perfect conductor is in static equilibrium, any net charge is on the surface, and the electric field is zero on its interior. The electric field at the surface is perpendicular to the surface.

14.4 The electric dipole

When an unbalanced distribution of charges is subject to an external electric field **E**, it experiences a torque $\boldsymbol{\tau}$. We define the electric dipole moment **d** to be the vector such that

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}. \qquad (14.3)$$

When the total charge is zero, this relation uniquely defines **d**, regardless of the point chosen as the axis. In the simplest case, of charges $+q$ and $-q$ at opposite ends of a stick of length ℓ, the dipole moment has magnitude $q\ell$ and points from the negative charge to the positive one. The potential energy of a dipole in an external field is

$$U = -\mathbf{d} \cdot \mathbf{E}. \qquad (14.4)$$

14.5 The field of a continuous charge distribution

The field of a continuous charge distribution can be found by integrating the contribution to the field from each infinitesimal part of the distribution.

14.6 Gauss's law

When we look at the "sea of arrows" representation of a field, 14.1/1, there is a natural visual tendency to imagine connecting the arrows as in 14.1/2. The curves formed in this way are called field lines, and they have a direction, shown by the arrowheads.

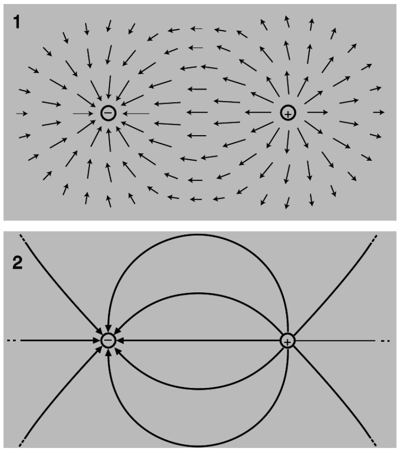


Figure 14.1: Two different representations of an electric field.

Electric field lines originate from positive charges and terminate on negative ones. We can choose a constant of proportionality that fixes how coarse or fine the "grain of the wood" is, but once this choice is made the strength of each charge is shown by the number of lines that begin or end on it. For example, figure 14.1/2 shows eight lines at each charge, so we know that $q_1/q_2 = (-8)/8 = -1$. Because lines never begin or end except on a charge, we can always find the total charge inside any given region by subtracting the number of lines that go in from the number that come out and multiplying by the appropriate constant of proportionality. Ignoring the constant, we can apply this technique to figure 14.2 to find $q_A = -8$, $q_B = 2 - 2 = 0$, and $q_C = 5 - 5 = 0$.

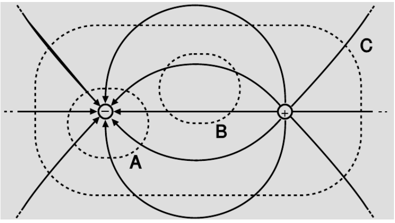


Figure 14.2: The number of field lines coming in and out of each region depends on the total charge it encloses.

Let us now make this description more mathematically precise. Given a smooth, closed surface such as the ones in figure 14.2, we have an inside and an outside, so that at any point on the surface we can define a unit normal $\hat{\mathbf{n}}$ (i.e., a vector with magnitude 1, perpendicular to the surface) that points outward. Given an infinitesimally small piece of the surface, with area $\mathrm{d}A$, we define an area vector $\mathrm{d}\mathbf{A} = \hat{\mathbf{n}}\,\mathrm{d}A$. The infinitesimal flux $\mathrm{d}\Phi$ through this infinitesimal patch of the surface is defined as $\mathrm{d}\Phi = \mathbf{E} \cdot \mathrm{d}\mathbf{A}$, and integrating over the entire surface gives the total flux $\Phi = \int \mathrm{d}\Phi = \int \mathbf{E} \cdot \mathrm{d}\mathbf{A}$. Intuitively, the flux measures how many field lines pierce the surface. Gauss's law states that

$$q_{\text{in}} = \frac{\Phi}{4\pi k}, \tag{14.5}$$

where q_{in} is the total charge inside a closed surface, and Φ is the flux through the surface. (In terms of the constant $\epsilon_0 = 1/(4\pi k)$, we have $q_{\text{in}} = \epsilon_0 \Phi$.)

Unlike Coulomb's law, Gauss's law holds in all circumstances, even when there are charges moving in complicated ways and electromagnetic waves flying around. Gauss's law can be thought of as a definition of electric charge.

14.7 Gauss's law in differential form

Gauss' law is a bit spooky. It relates the field on the Gaussian surface to the charges inside the surface. What if the charges have been moving around, and the field at the surface right now is the one that was created by the charges in their previous locations? Gauss' law — unlike Coulomb's law — still works in cases like these, but it's far from obvious how the flux and the charges can still stay in agreement if the charges have been moving around.

For this reason, it would be more physically attractive to restate Gauss' law in a different form, so that it related the behavior of the field at one point to the charges that were actually present at that point. We define the *divergence* of the electric field,

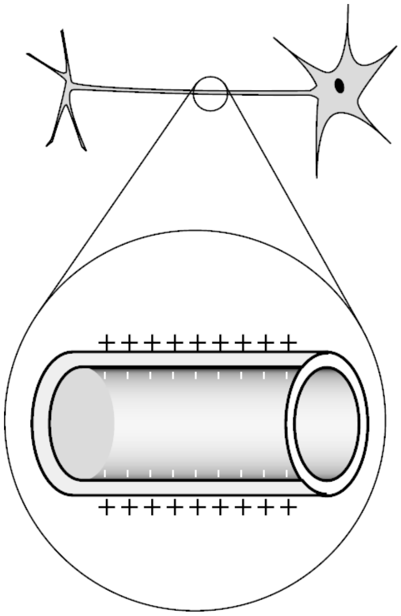
$$\operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

Gauss's law in differential form is

$$\operatorname{div} \mathbf{E} = 4\pi k \rho.$$

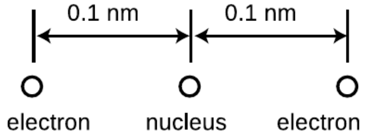
Problems

14-a1 The figure shows a neuron, which is the type of cell your nerves are made of. Neurons serve to transmit sensory information to the brain, and commands from the brain to the muscles. All this data is transmitted electrically, but even when the cell is resting and not transmitting any information, there is a layer of negative electrical charge on the inside of the cell membrane, and a layer of positive charge just outside it. This charge is in the form of various ions dissolved in the interior and exterior fluids. Why would the negative charge remain plastered against the inside surface of the membrane, and likewise why doesn't the positive charge wander away from the outside surface?



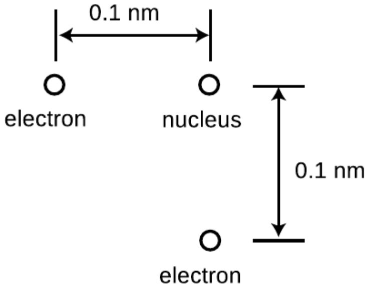
Problem 14-a1.

14-a2 A helium atom finds itself momentarily in this arrangement. Find the direction and magnitude of the force acting on the right-hand electron. The two protons in the nucleus are so close together (~ 1 fm) that you can consider them as being right on top of each other. ✓



Problem 14-a2.

14-a3 The helium atom of problem 14-a2 has some new experiences, goes through some life changes, and later on finds itself in the configuration shown here. What are the direction and magnitude of the force acting on the bottom electron? (Draw a sketch to make clear the definition you are using for the angle that gives direction.) ✓



Problem 14-a3.

14-a4 Suppose you are holding your hands in front of you, 10 cm apart.
(a) Estimate the total number of electrons in each hand. ✓
(b) Estimate the total repulsive force of all the electrons in one hand on all the electrons in the other. ✓
(c) Why don't you feel your hands repelling each other?
(d) Estimate how much the charge of a proton could differ in magnitude from the charge of an electron without creating a noticeable force between your hands.

14-a5 Suppose that a proton in a lead nucleus wanders out to the surface of the nucleus, and experiences a strong nuclear force of about 8 kN from the nearby neutrons and protons pulling it back in. Compare this numerically to the repulsive electrical force from the other protons, and verify that the net force is attractive. A lead nucleus is very nearly spherical, is about 6.5 fm in radius, and contains 82 protons, each with a charge of $+e$, where $e = 1.60 \times 10^{-19}$ C.

√

14-a6 The subatomic particles called muons behave exactly like electrons, except that a muon's mass is greater by a factor of 206.77. Muons are continually bombarding the Earth as part of the stream of particles from space known as cosmic rays. When a muon strikes an atom, it can displace one of its electrons. If the atom happens to be a hydrogen atom, then the muon takes up an orbit that is on the average 206.77 times closer to the proton than the orbit of the ejected electron. How many times greater is the electric force experienced by the muon than that previously felt by the electron?

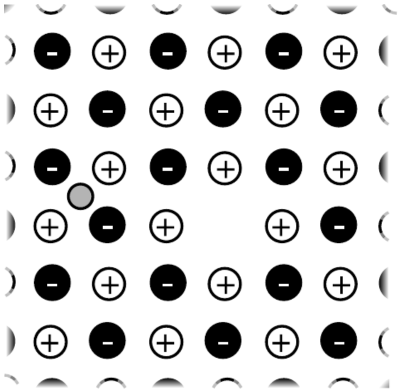
√

14-a7 The Earth and Moon are bound together by gravity. If, instead, the force of attraction were the result of each having a charge of the same magnitude but opposite in sign, find the quantity of charge that would have to be placed on each to produce the required force.

√

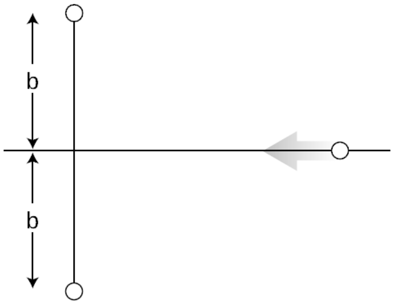
14-a8 The figure shows one layer of the three-dimensional structure of a salt crystal. The atoms extend much farther off in all directions, but only a six-by-six square is shown here. The larger circles are the chlorine ions, which have charges of $-e$, where $e = 1.60 \times 10^{-19}$ C. The smaller circles are sodium ions, with charges of $+e$. The center-to-center distance between neighboring ions is about 0.3 nm. Real crystals are never perfect, and the crystal shown here has two defects: a missing atom at one location, and an extra lithium atom, shown as a grey circle, inserted in one of the small gaps. If the lithium atom has a charge of $+e$, what is the direction and magnitude of the total force on it? Assume there are no other defects nearby in the crystal besides the two shown here.

√


Problem 14-a8.

14-a9 In the semifinals of an electrostatic croquet tournament, Jessica hits her positively charged ball, sending it across the playing field, rolling to the left along the x axis. It is repelled by two other positive charges. These two equal charges are fixed on the y axis at the locations shown in the figure. (a) Express the force on the ball in terms of the ball's position, x. (b) At what value of x does the ball experience the greatest deceleration? Express you answer in terms of b. [Based on a problem by Halliday and Resnick.]

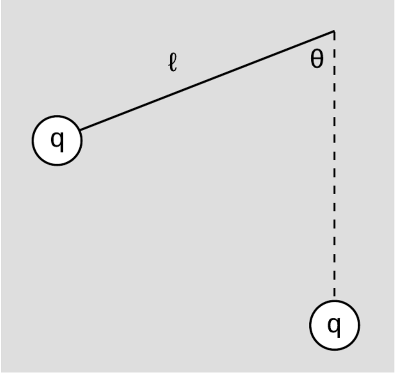
√



Problem 14-a9.

14-a10 As shown in the figure, a particle of mass m and charge q hangs from a string of length ℓ, forming a pendulum fixed at a central point. Another charge q is fixed at the same distance ℓ, directly below the center. Find the equilibrium values of θ and determine whether they are stable or unstable.

⋆



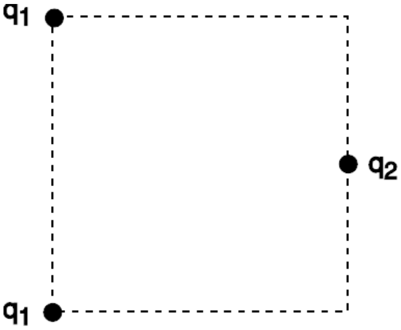
Problem 14-a10.

14-d1 (a) At time $t = 0$, a positively charged particle is placed, at rest, in a vacuum, in which there is a uniform electric field of magnitude E. Write an equation giving the particle's speed, v, in terms of t, E, and its mass and charge m and q. √

(b) If this is done with two different objects and they are observed to have the same motion, what can you conclude about their masses and charges? (For instance, when radioactivity was discovered, it was found that one form of it had the same motion as an electron in this type of experiment.)

14-d2 Three charges are arranged on a square as shown. All three charges are positive. What value of q_2/q_1 will produce zero electric field at the center of the square?

√



Problem 14-d2.

14-d3 In an electrical storm, the cloud and the ground act like a parallel-plate capacitor, which typically charges up due to frictional electricity in collisions of ice particles in the cold upper atmosphere. Lightning occurs when the magnitude of the electric field reaches a critical value E_c, at which air is ionized.

(a) Treat the cloud as a flat square with sides of length L. If it is at a height h above the ground, find the amount of energy released in the lightning strike. √

(b) Based on your answer from part a, which is more dangerous, a lightning strike from a high-altitude cloud or a low-altitude one?

(c) Make an order-of-magnitude estimate of the energy released by a typical lightning bolt, assuming reasonable values for its size and altitude. E_c is about 10^6 N/C.

See problem **??** for a note on how recent research affects this estimate.

Problem 14-d3.

14-d4 The figure shows cross-sectional views of two cubical capacitors, and a cross-sectional view of the same two capacitors put together so that their interiors coincide. A capacitor with the plates close together has a nearly uniform electric field between the plates, and almost zero field outside; these capacitors don't have their plates very close together compared to the dimensions of the plates, but for the purposes of this problem, assume that they still have approximately the kind of idealized field pattern shown in the figure. Each capacitor has an interior volume of 1.00 m^3, and is charged up to the point where its internal field is 1.00 V/m.

(a) Calculate the energy stored in the electric field of each capacitor when they are separate. ✓

(b) Calculate the magnitude of the interior field when the two capacitors are put together in the manner shown. Ignore effects arising from the redistribution of each capacitor's charge under the influence of the other capacitor. ✓

(c) Calculate the energy of the put-together configuration. Does assembling them like this release energy, consume energy, or neither? ✓

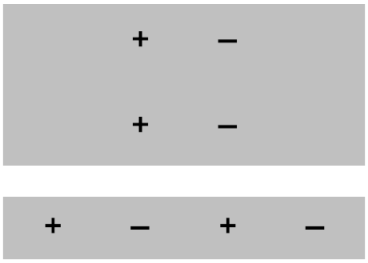
14-g1 The definition of the dipole moment, $\mathbf{D} = \sum q_i \mathbf{r}_i$, involves the vector $\mathbf{r}_i$ stretching from the origin of our coordinate system out to the charge q_i. There are clearly cases where this causes the dipole moment to be dependent on the choice of coordinate system. For instance, if there is only one charge, then we could make the dipole moment equal zero if we chose the origin to be right on top of the charge, or nonzero if we put the origin somewhere else.

(a) Make up a numerical example with two charges of equal magnitude and opposite sign. Compute the dipole moment using two different coordinate systems that are oriented the same way, but differ in the choice of origin. Comment on the result.

(b) Generalize the result of part a to any pair of charges with equal magnitude and opposite sign. This is supposed to be a proof for *any* arrangement of the two charges, so don't assume any numbers.

(c) Generalize further, to n charges.

14-g2 Compare the two dipole moments.



Problem 14-g2.

14-g3 Find an arrangement of charges that has zero total charge and zero dipole moment, but that will make nonvanishing electric fields.

14-g4 This is a one-dimensional problem, with everything confined to the x axis. Dipole A consists of a -1.000 C charge at $x = 0.000$ m and a 1.000 C charge at $x = 1.000$ m. Dipole B has a -2.000 C charge at $x = 0.000$ m and a 2.000 C charge at $x = 0.500$ m.

(a) Compare the two dipole moments.

(b) Calculate the field created by dipole A at $x = 10.000$ m, and compare with the field dipole B would make. Comment on the result. ✓

14-g5 A dipole has a midplane, i.e., the plane that cuts through the dipole's center, and is perpendicular to the dipole's axis. Consider a two-charge dipole made of point charges $\pm q$ located at $z = \pm \ell/2$. Use approximations to find the field at a distant point in the midplane, and show that its magnitude comes out to be kD/R^3 (half what it would be at a point on the axis lying an equal distance from the dipole).

14-j1 Astronomers believe that the mass distribution (mass per unit volume) of some galaxies may be approximated, in spherical coordinates, by $\rho = ae^{-br}$, for $0 \le r \le \infty$, where ρ is the density. Find the total mass. ✓

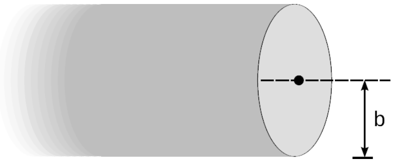
14-j2 A hydrogen atom in a particular state has the charge density (charge per unit volume) of the electron cloud given by $\rho = ae^{-br}z^2$, where r is the distance from the proton, and z is the coordinate measured along the z axis. Given that the total charge of the electron cloud must be $-e$, find a in terms of the other variables.

✓

14-j3 (a) A rod of length L is uniformly charged with charge Q. It can be shown by integration that the field at a point lying in the midplane of the rod at a distance R is $E = k\lambda L/\left[R^2\sqrt{1+L^2/4R^2}\right]$, where λ is the charge per unit length. Starting from this result, take the limit as the length of the rod approaches infinity. Note that λ is not changing, so as L gets bigger, the total charge Q increases.

(b) It can be shown that the field of an infinite, uniformly charged plane is $2\pi k\sigma$. Now you're going to rederive the same result by a different method. Suppose that it is the $x-y$ plane that is charged, and we want to find the field at the point $(0,0,z)$. (Since the plane is infinite, there is no loss of generality in assuming $x=0$ and $y=0$.) Imagine that we slice the plane into an infinite number of straight strips parallel to the y axis. Each strip has infinitesimal width dx, and extends from x to $x+\mathrm{d}x$. The contribution any one of these strips to the field at our point has a magnitude which can be found from part a. By vector addition, prove the desired result for the field of the plane of charge.

14-j4 Consider the electric field created by a uniformly charged cylindrical surface that extends to infinity in one direction.
(a) Show that the field at the center of the cylinder's mouth is $2\pi k\sigma$, which happens to be the same as the field of an infinite *flat* sheet of charge!
(b) This expression is independent of the radius of the cylinder. Explain why this should be so. For example, what would happen if you doubled the cylinder's radius?



Problem 14-j4.

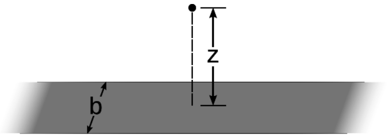
14-j5 (a) Show that the energy in the electric field of a point charge is infinite! Does the integral diverge at small distances, at large distances, or both?
(b) Now calculate the energy in the electric field of a uniformly charged sphere with radius b. Based on the shell theorem, it can be shown that the field for $r > b$ is the same as for a point charge, while the field for $r < b$ is kqr/b^3. (Example **??** shows this using a different technique.)

Remark: The calculation in part a seems to show that infinite energy would be required in order to create a charged, pointlike particle. However, there are processes that, for example, create electron-positron pairs, and these processes don't require infinite energy. According to Einstein's famous equation $E = mc^2$, the energy required to create such a pair should only be $2mc^2$, which is finite.

✓

14-j6 (a) A rod of length L is uniformly charged with charge Q. It can be shown by integration that the field at a point lying in the midplane of the rod at a distance R is $E = k\lambda L/\left[R^2\sqrt{1+L^2/4R^2}\right]$, where λ is the charge per unit length. Show that this field reduces to $E = 2k\lambda/R$ in the limit of $L \to \infty$.
(b) An infinite strip of width b has a surface charge density σ. Find the field at a point at a distance z from the strip, lying in the plane perpendicularly bisecting the strip. ✓
(c) Show that this expression has the correct behavior in the limit where z approaches zero, and also in the limit of $z \gg b$. For the latter, you'll need the result of problem 14-j3a, which is given on page **??**.

14-j7 A solid cylinder of radius b and length ℓ is uniformly charged with a total charge Q. Find the electric field at a point at the center of one of the flat ends.



Problem 14-j6.

14-m1 A sphere of radius b contains a uniform charge density ρ. Use Gauss's law to find the electric field at radius $r \leq b$, and verify that the result is the same as the one obtained using Newton's shell theorem for gravity. Problem by P. Widmann. $\checkmark$

14-m2 A spherical shell of uniform charge density ρ extends from $r = a$ to $r = b$. Find the field in the regions $r \leq a$, $a \leq r \leq b$, and $r \geq b$. Problem by P. Widmann. $\checkmark$

14-m3 A piece of metal contains a hollow spherical cavity of radius a, and at the center of this cavity is a point charge q. Find the charge density on the inner surface of the cavity. Problem by P. Widmann. $\checkmark$

14-m4 An infinite, uniform slab of charge with density ρ extends from $x = 0$ to $x = h$. The distant field on the left is zero. Find the field in the three regions $x \leq 0$, $0 \leq x \leq h$, and $h \leq x$. Problem by P. Widmann. $\checkmark$

14-m5 Assume the earth is an infinite flat sheet, as some persons claim. Flat-earth cosmologies often omit any description of what's on the flip side, so let's assume that the gravitational field is zero there. If the density of the earth is 4 g/cm^3, find the thickness that the earth must have in order to give $g = 9.8$ m/s^2 on our side. Problem by P. Widmann. $\checkmark$

14-m6 (a) Use Gauss' law to find the field inside an infinite cylinder with radius b and uniform charge density ρ. (The external field has the same form as the one in problem **??**.) $\checkmark$
(b) Check that your answer makes sense on the axis.
(c) Check that the units of your answer make sense.

14-m7 In a certain region of space, the electric field is constant (i.e., the vector always has the same magnitude and direction). For simplicity, assume that the field points in the positive x direction. (a) Use Gauss's law to prove that there is no charge in this region of space. This is most easily done by considering a Gaussian surface consisting of a rectangular box, whose edges are parallel to the x, y, and z axes.
(b) If there are no charges in this region of space, what could be making this electric field?

14-m8 (a) In a certain region of space, the electric field is given by $\mathbf{E} = bx\hat{\mathbf{x}}$, where b is a constant. Find the amount of charge contained within a cubical volume extending from $x = 0$ to $x = a$, from $y = 0$ to $y = a$, and from $z = 0$ to $z = a$.
(b) Repeat for $\mathbf{E} = bx\hat{\mathbf{z}}$.
(c) Repeat for $\mathbf{E} = 13bz\hat{\mathbf{z}} - 7cz\hat{\mathbf{y}}$.
(d) Repeat for $\mathbf{E} = bxz\hat{\mathbf{z}}$.

14-m9 Light is a wave made of electric and magnetic fields, and the fields are perpendicular to the direction of the wave's motion, i.e., they're transverse. An example would be the electric field given by $\mathbf{E} = b\hat{\mathbf{x}} \sin cz$, where b and c are constants. (There would also be an associated magnetic field.) We observe that light can travel through a vacuum, so we expect that this wave pattern is consistent with the nonexistence of any charge in the space it's currently occupying. Use Gauss's law to prove that this is true.

14-m10 An electric field is given in cylindrical coordinates (R, ϕ, z) by $E_R = ce^{-u|z|}R^{-1}\cos^2\phi$, where the notation E_R indicates the component of the field pointing directly away from the axis, and the components in the other directions are zero. (This isn't a completely impossible expression for the field near a radio transmitting antenna.) (a) Find the total charge enclosed within the infinitely long cylinder extending from the axis out to $R = b$. (b) Interpret the R-dependence of your answer to part a.

14-m11 An electron in an atom acts like a probability cloud surrounding the nucleus. For a hydrogen atom in its lowest-energy state, the probability falls off exponentially, so we can mock this up with a charge density $\rho = \rho_0 e^{-r/a}$, where r is the distance from the nucleus, and ρ_0 and a are constants. Find the electric field. Problem by P. Widmann.

$\surd$

15 The electric potential

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

15.1 The electric potential

When a test charge q is at a particular position in a static electric field, it has an electrical potential energy U. The electrical potential energy per unit charge, U/q, is called the electric potential, notated V, φ, or Φ, and measured in units of volts, V. Because it is defined in terms of a potential energy, the electric potential is only defined up to an additive constant. In the context of an electric circuit, we often use the synonym "voltage" for the electric potential. Voltage differences are measured by a voltmeter, which reads the difference in potential between its two probes. A voltmeter is wired in parallel with a circuit, and an ideal voltmeter acts like a perfect insulator, so that no charge ever flows through it.

In one dimension, the electric field and electric potential are related by

$$E = -\frac{\mathrm{d}V}{\mathrm{d}x},$$

or equivalently, via the fundamental theorem of calculus,

$$V(x_2) - V(x_1) = -\int_{x_1}^{x_2} E\,\mathrm{d}x.$$

Generalizing to three dimensions,

$$\mathbf{E} = -\nabla V$$

(involving the gradient operator ∇) and

$$\Delta V = -\int \mathbf{E}\cdot\mathrm{d}\mathbf{x}$$

(in terms of a path integral). In electrostatics, the path integral in the latter equation is independent of the path taken.

Since the field of a charge distribution depends additively upon the charges, the same is true of the potential. Given a continuous charge distribution, it is sometimes easier to find the potential by integration than the field, since the potential is a scalar. Having found the potential, one can always take the gradient to find the field.

15.2 Capacitance

A capacitor is a device that stores energy in an electric field. The simplest example consists of two parallel conducting plates. The energy is proportional to the square of the field strength, which is proportional to the charges on the plates. If we assume the plates carry charges that are the same in magnitude, $+q$ and $-q$, then the energy stored in the capacitor must be proportional to q^2. For historical reasons, we write the constant of proportionality as $1/2C$,

$$E = \frac{1}{2C}q^2.$$

The constant C is a geometrical property of the capacitor, called its capacitance. Based on this definition, the units of capacitance must be coulombs squared per joule, and this combination is more conveniently abbreviated as the farad, $1\ \mathrm{F} = 1\ \mathrm{C}^2/\mathrm{J}$.

Voltage is electrical potential energy per unit charge, so the voltage difference across a capacitor is related to the amount by which its energy would increase if we increased the absolute values of the charges on the plates from q to $q+\Delta q$:

$$\begin{aligned} V &= (E_{q+\Delta q} - E_q)/\Delta q \\ &= \frac{\Delta}{\Delta q}\left(\frac{1}{2C}q^2\right) \\ &= \frac{q}{C} \end{aligned}$$

Many books use this as the definition of capacitance. It follows from this relation that capacitances in parallel add, $C_{\text{equivalent}} = C_1 + C_2$, whereas when they are wired in series, it is their inverses that add, $C_{\text{equivalent}}^{-1} = C_1^{-1} + C_2^{-1}$.

15.3 Dielectrics

Many electrically insulating materials fall into a category known as dielectrics. Such materials can be modeled as containing many microscopic dipoles (molecules) that are randomly oriented but can become aligned when subjected to an external field. When we apply Gauss's law to a region of space in which a dielectric is present, the charge can have contributions both from free charges (such as the ones that flow in a circuit) measurable with measuring devices such as ammeters, but also from the bound, microscopic charges inside the dipoles. It can therefore be useful to rewrite Gauss's law as

$$\Phi_D = q_{\text{free}},$$

where

$$\mathbf{D} = \epsilon\mathbf{E}.$$

When the field is constant over time and not too strong, ϵ is approximately constant, and is a property of the material called its permittivity. In a vacuum, $\epsilon = 1/4\pi k$, referred to as ϵ_0, while a dielectric has $\epsilon > \epsilon_0$. With time-varying fields, most materials have permittivities that are highly frequency-dependent. For materials such as crystals, which have special directions defined by the regular atomic lattice, ϵ cannot be modeled as a scalar, and the relation between $\mathbf{D}$ and $\mathbf{E}$ becomes more complicated.

When a capacitor has the space between its electrodes filled with a dielectric, its capacitance is increased by the factor ϵ/ϵ_0.

At a boundary between two different materials, if there is no free charge at the boundary, the components of the fields $D_\perp$ and $E_\parallel$ are continuous.

15.4 Poisson's equation and Laplace's equation

Gauss's law, $\operatorname{div}\mathbf{E} = 4\pi k\rho$, can also be stated in terms of the potential. Since $\mathbf{E} = \nabla V$, we have $\operatorname{div}\nabla V = 4\pi k\rho$. If we work out the combination of operators $\operatorname{div}\nabla$ in a Cartesian coordinate system, we get $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, which is called the Laplacian and notated ∇^2. The version of Gauss's law written in terms of the potential,

$$\nabla^2 V = 4\pi k\rho,$$

is called Poisson's equation, while in the special case of a vacuum, with $\rho = 0$, we have

$$\nabla^2 V = 0,$$

known as Laplace's equation. Many problems in electrostatics can be stated in terms of finding potential that satisfies Laplace's equation, usually with some set of *boundary conditions*. For example, if an infinite parallel-plate capacitor has plates parallel to the x-y plane at certain given potentials, then these plates form a boundary for the region between the plates, and Laplace's equation has a solution in this region of the form $V = az + b$. It's easy to verify that this is a solution of Laplace's equation, since all three of the partial derivatives vanish.

15.5 The method of images

A car's radio antenna is usually in the form of a whip sticking up above its metal roof. This is an example involving radio waves, which are time-varying electric and magnetic fields, but a similar, simpler electrostatic example is the following. Suppose that we position a charge $q > 0$ at a distance ℓ from a conducting plane. What is the resulting electric field? The conductor has charges that are free to move, and due to the field of the charge q, we will end up with a net concentration of negative charge in the part of the plane near q. The field in the vacuum surrounding q will be a sum of fields due to q and fields due to these charges in the conducting plane. The problem can be stated as that of finding a solution to Poisson's equation with the boundary condition that $V = 0$ at the conducting plane. Figure 15.1/1 shows the kind of field lines we expect.

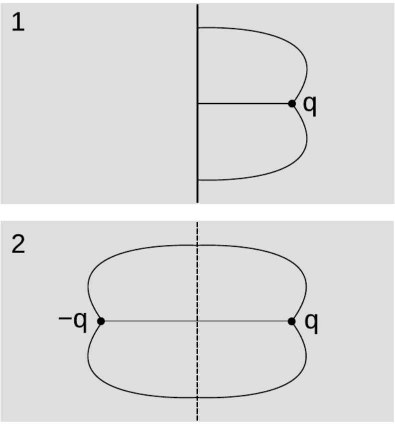


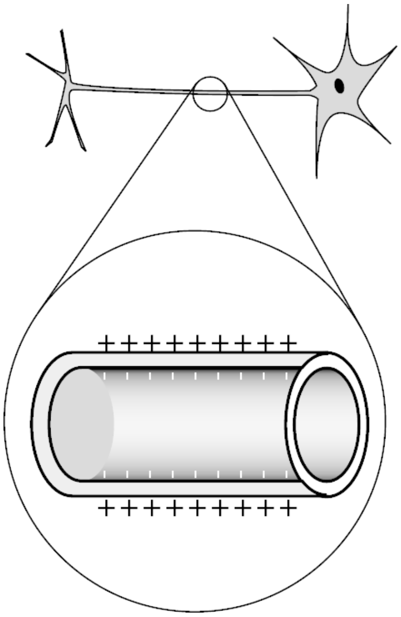
Figure 15.1: The method of images.

This looks like a very complicated problem, but there is trick that allows us to find a simple solution. We can convert the problem into an equivalent one in which the conductor isn't present, but a fictitious *image* charge $-q$ is placed at an equal distance behind the plane, like a reflection in a mirror, as in figure 15.1/2. The field is then simply the sum of the fields of the charges q and $-q$, so we can either add the field vectors or add the potentials. By symmetry, the field lines are perpendicular to the plane, so the plane is an surface of constant potential, as required.

Problems

15-a1 The gap between the electrodes in an automobile engine's spark plug is 0.060 cm. To produce an electric spark in a gasoline-air mixture, an electric field of 3.0×10^6 V/m must be achieved. On starting a car, what minimum voltage must be supplied by the ignition circuit? Assume the field is uniform. √
(b) The small size of the gap between the electrodes is inconvenient because it can get blocked easily, and special tools are needed to measure it. Why don't they design spark plugs with a wider gap?

15-a2 In our by-now-familiar neuron, the voltage difference between the inner and outer surfaces of the cell membrane is about $V_{out} - V_{in} = -70$ mV in the resting state, and the thickness of the membrane is about 6.0 nm (i.e., only about a hundred atoms thick). What is the electric field inside the membrane? √



Problem 15-a2.

15-a3 The neuron in the figure has been drawn fairly short, but some neurons in your spinal cord have tails (axons) up to a meter long. The inner and outer surfaces of the membrane act as the "plates" of a capacitor. (The fact that it has been rolled up into a cylinder has very little effect.) In order to function, the neuron must create a voltage difference V between the inner and outer surfaces of the membrane. Let the membrane's thickness, radius, and length be t, r, and L. (a) Calculate the energy that must be stored in the electric field for the neuron to do its job. (In real life, the membrane is made out of a substance called a dielectric, whose electrical properties increase the amount of energy that must be stored. For the sake of this analysis, ignore this fact.) √
(b) An organism's evolutionary fitness should be better if it needs less energy to operate its nervous system. Based on your answer to part a, what would you expect evolution to do to the dimensions t and r? What other constraints would keep these evolutionary trends from going too far?

15-a4 The figure shows a simplified diagram of an electron gun such as the one that creates the electron beam in a TV tube. Electrons that spontaneously emerge from the negative electrode (cathode) are then accelerated to the positive electrode, which has a hole in it. (Once they emerge through the hole, they will slow down. However, if the two electrodes are fairly close together, this slowing down is a small effect, because the attractive and repulsive forces experienced by the electron tend to cancel.)
(a) If the voltage difference between the electrodes is ΔV, what is the velocity of an electron as it emerges at B? Assume that its initial velocity, at A, is negligible, and that the velocity is nonrelativistic. (If you haven't read ch. 7 yet, don't worry about the remark about relativity.) √
(b) Evaluate your expression numerically for the case where ΔV=10 kV, and compare to the

speed of light. If you've read ch. 7 already, comment on whether the assumption of nonrelativistic motion was justified.

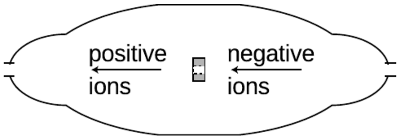
√ ▷ Solution, p. 354

15-a5 The figure shows a simplified diagram of a device called a tandem accelerator, used for accelerating beams of ions up to speeds on the order of 1-10% of the speed of light. (Since these velocities are not too big compared to c, you can use nonrelativistic physics throughout this problem.) The nuclei of these ions collide with the nuclei of atoms in a target, producing nuclear reactions for experiments studying the structure of nuclei. The outer shell of the accelerator is a conductor at zero voltage (i.e., the same voltage as the Earth). The electrode at the center, known as the "terminal," is at a high positive voltage, perhaps millions of volts. Negative ions with a charge of -1 unit (i.e., atoms with one extra electron) are produced offstage on the right, typically by chemical reactions with cesium, which is a chemical element that has a strong tendency to give away electrons. Relatively weak electric and magnetic forces are used to transport these -1 ions into the accelerator, where they are attracted to the terminal. Although the center of the terminal has a hole in it to let the ions pass through, there is a very thin carbon foil there that they must physically penetrate. Passing through the foil strips off some number of electrons, changing the atom into a positive ion, with a charge of $+n$ times the fundamental charge. Now that the atom is positive, it is repelled by the terminal, and accelerates some more on its way out of the accelerator.

(a) Find the velocity, v, of the emerging beam of positive ions, in terms of n, their mass m, the terminal voltage V, and fundamental constants. Neglect the small change in mass caused by the loss of electrons in the stripper foil. √

(b) To fuse protons with protons, a minimum beam velocity of about 11% of the speed of light is required. What terminal voltage would be needed in this case? √

(c) In the setup described in part b, we need a target containing atoms whose nuclei are single protons, i.e., a target made of hydrogen. Since hydrogen is a gas, and we want a foil for our target, we have to use a hydrogen compound, such as a plastic. Discuss what effect this would have on the experiment.



Problem 15-a5.

15-d1 A hydrogen atom is electrically neutral, so at large distances, we expect that it will create essentially zero electric field. This is not true, however, near the atom or inside it. Very close to the proton, for example, the field is very strong. To see this, think of the electron as a spherically symmetric cloud that surrounds the proton, getting thinner and thinner as we get farther away from the proton. (Quantum mechanics tells us that this is a more correct picture than trying to imagine the electron orbiting the proton.) Near the center of the atom, the electron cloud's field cancels out by symmetry, but the proton's field is strong, so the total field is very strong. The potential in and around the hydrogen atom can be approximated using an expression of the form $V = r^{-1}e^{-r}$. (The units come out wrong, because I've left out some constants.) Find the electric field corresponding to this potential, and comment on its behavior at very large and very small r.

▷ Solution, p. 354

15-d2 (a) Given that the on-axis field of a dipole at large distances is proportional to D/r^3, show that its potential varies as D/r^2. (Ignore positive and negative signs and numerical constants of proportionality.)

(b) Write down an exact expression for the potential of a two-charge dipole at an on-axis point, without assuming that the distance is large compared to the size of the dipole. Your expression will have to contain the actual charges and size of the dipole, not just its dipole moment. Now use approximations to show that, at large distances, this is consistent with your answer to part a.

15-d3 A carbon dioxide molecule is structured like O-C-O, with all three atoms along a line. The oxygen atoms grab a little bit of extra negative charge, leaving the carbon positive. The molecule's symmetry, however, means that it has no overall dipole moment, unlike a V-shaped water molecule, for instance. Whereas the potential of a dipole of magnitude D is proportional to D/r^2, (see problem 15-d2), it turns out that the potential of a carbon dioxide molecule at a distant point along the molecule's axis equals b/r^3, where r is the distance from the molecule and b is a constant (cf. problem 14-g3). What would be the electric field of a carbon dioxide molecule at a point on the molecule's axis, at a distance r from the molecule?

√

15-d4 A proton is in a region in which the electric field is given by $E = a + bx^3$. If the proton starts at rest at $x_1 = 0$, find its speed, v, when it reaches position x_2. Give your answer in terms of a, b, x_2, and e and m, the charge and mass of the proton.

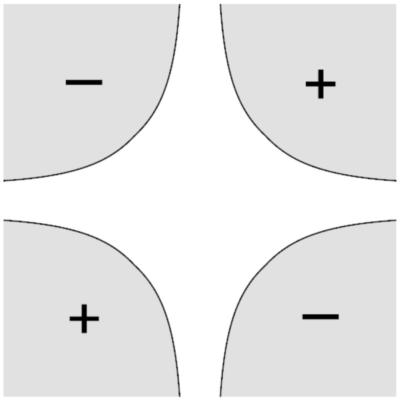
√

15-g1 The figure shows a vacuum chamber surrounded by four metal electrodes shaped like hyperbolas. (Yes, physicists do sometimes ask their university machine shops for things machined in mathematical shapes like this. They have to be made on computer-controlled mills.) We assume that the electrodes extend far into and out of the page along the unseen z axis, so that by symmetry, the electric fields are the same for all z. The problem is therefore effectively two-dimensional. Two of the electrodes are at voltage $+V_o$, and the other two at $-V_o$, as shown. The equations of the hyperbolic surfaces are $|xy| = b^2$, where b is a constant. (We can interpret b as giving the locations $x = \pm b$, $y = \pm b$ of the four points on the surfaces that are closest to the central axis.) There is no obvious, pedestrian way to determine the field or potential in the central vacuum region, but there's a trick that works: with a little mathematical insight, we see that the potential $V = V_o b^{-2} xy$ is consistent with all the given information. (Mathematicians could prove that this solution was unique, but a physicist knows it on physical grounds: if there were two different solutions, there would be no physical way for the system to decide which one to do!)

(a) Find the field in the vacuum region.

(b) Sketch the field as a "sea of arrows."

√



Problem 15-g1.

15-g2 (a) A certain region of three-dimensional space has a potential that varies as $V = br^2$, where r is the distance from the origin. Find the field. √

(b) Write down another potential that gives exactly the same field.

15-j1 Find the capacitance of the surface of the earth, assuming there is an outer spherical "plate" at infinity. (In reality, this outer plate would just represent some distant part of the universe to which we carried away some of the earth's charge in order to charge up the earth.)

√

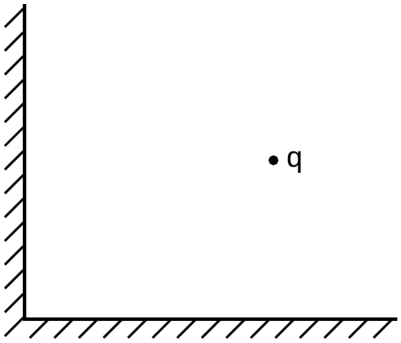
15-j2 Find the energy stored in a capacitor in terms of its capacitance and the voltage difference across it.

$\surd$

15-j3 A capacitor has parallel plates of area A, separated by a distance h. If there is a vacuum between the plates, then Gauss's law gives $E = 4\pi k\sigma = 4\pi kq/A$ for the field between the plates, and combining this with $E = V/h$, we find $C = q/V = (1/4\pi k)A/h$. (a) Generalize this derivation to the case where there is a dielectric between the plates. (b) Suppose we have a list of possible materials we could choose as dielectrics, and we wish to construct a capacitor that will have the highest possible energy density, U_e/v, where v is the volume. For each dielectric, we know its permittivity ϵ, and also the maximum electric field E it can sustain without breaking down and allowing sparks to cross between the plates. Write the maximum energy density in terms of these two variables, and determine a figure of merit that could be used to decide which material would be the best choice.

15-m1 A charged particle of mass m and charge q is below a horizontal conducting plane. We wish to find the distance ℓ between the particle and the plane so that the particle will be in equilibrium, with its weight supported by electrostatic forces.
(a) Determine as much as possible about the form of the answer based on units.
(b) Find the full result for ℓ.
(c) Show that the equilibrium is unstable.

15-m2 A point charge q is situated in the empty space inside a corner formed by two perpendicular half-planes made of sheets of metal. Let the sheets lie in the y-z and x-z planes, so that the charge's distances from the planes are x and y. Both x and y are positive. The charge will accelerate due to the electrostatic forces exerted by the sheets. We wish to find the direction θ in which it will accelerate, expressed as an angle counterclockwise from the negative x axis, so that $0 < \theta < \pi/2$.
(a) Determine as much as possible about the form of the answer based on units.
(b) Find the full result for θ.



Problem 15-m2.

16 Circuits

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

16.1 Current

Electric current is defined as the rate of flow of charge through a boundary, $I = \mathrm{d}q/\,\mathrm{d}t$. Its units of coulombs/second are more conveniently abbreviated as amperes, 1 A=1 C/s. Current is measured by an ammeter, which allows current to flow through itself. An ammeter is wired in series with a circuit, which requires breaking the circuit in order to insert the meter. An ideal ammeter acts like a perfect conductor.

The power dissipated, transformed, or released in an electric circuit element is given by $P = I\Delta V$.

16.2 Resistance

Current will not flow at all through a perfect insulator. When a material is neither a perfect insulator nor a perfect insulator, then current can flow through it, and the result in terms of energy is that electrical energy is transformed into heat. For many materials, under some fairly large range of electric fields, the density of current is proportional to the electric field. When a two-terminal device is formed from such a material, and a voltage difference is applied across it, then the current flowing through it is given by Ohm's law, $I = \Delta V/R$, where R, called the resistance, depends on both the geometry of the device and the material of which it is constructed. Despite the name, Ohm's law is not a law of nature, and it is often violated. Some substances, such as gases, never obey Ohm's law; we say that they are not "ohmic." The units of resistance are abbreviated as ohms, $1\ \Omega = 1\ \text{V/A}$.

Resistances in series add, $R_\text{equivalent} = R_1 + R_2$, while in parallel $R^{-1}_\text{equivalent} = R^{-1}_1 + R^{-1}_2$.

16.3 The loop and junction rules

Kirchoff's junction rule is a statement of conservation of charge. It says that the sum of the currents flowing into any junction in a circuit must be zero (if the junction has no way to store charge).

Kirchoff's loop rule is a statement of conservation of energy. For any loop in a circuit, the sum of the voltage drops must be zero.

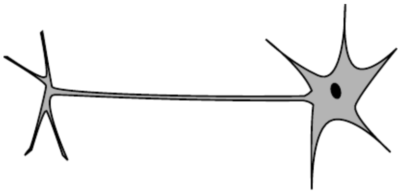
Problems

16-a1 In a wire carrying a current of 1.0 pA, how long do you have to wait, on the average, for the next electron to pass a given point? Express your answer in units of microseconds.

▷ Solution, p. 357

16-a2 Referring back to our old friend the neuron from problem 14-a1 on page 158, let's now consider what happens when the nerve is stimulated to transmit information. When the blob at the top (the cell body) is stimulated, it causes Na$^+$ ions to rush into the top of the tail (axon). This electrical pulse will then travel down the axon, like a flame burning down from the end of a fuse, with the Na$^+$ ions at each point first going out and then coming back in. If 10^{10} Na$^+$ ions cross the cell membrane in 0.5 ms, what amount of current is created?

√



Problem 16-a2.

16-a3 Lightning discharges a cloud during an electrical storm. Suppose that the current in the lightning bolt varies with time as $I = bt$, where b is a constant. Find the cloud's charge as a function of time.

√

16-a4 (a) You take an LP record out of its sleeve, and it acquires a static charge of 1 nC. You play it at the normal speed of $33\frac{1}{3}$ r.p.m., and the charge moving in a circle creates an electric current. What is the current, in amperes?

√

(b) Although the planetary model of the atom can be made to work with any value for the radius of the electrons' orbits, more advanced models that we will study later in this course predict definite radii. If the electron is imagined as circling around the proton at a speed of 2.2×10^6 m/s, in an orbit with a radius of 0.05 nm, what electric current is created?

√

Problem 16-a4.

16-a5 A silk thread is uniformly charged by rubbing it with llama fur. The thread is then dangled vertically above a metal plate and released. As each part of the thread makes contact with the conducting plate, its charge is deposited onto the plate. Since the thread is accelerating due to gravity, the rate of charge deposition increases with time, and by time t the cumulative amount of charge is $q = ct^2$, where c is a constant. (a) Find the current flowing onto the plate. √

(b) Suppose that the charge is immediately carried away through a resistance R. Find the power dissipated as heat. √

16-a6 In AM (amplitude-modulated) radio, an audio signal $f(t)$ is multiplied by a sine wave $\sin \omega t$ in the megahertz frequency range. For simplicity, let's imagine that the transmitting antenna is a whip, and that charge goes back and forth between the top and bottom. Suppose that, during a certain time interval, the audio signal varies linearly with time, giving a charge $q = (a + bt) \sin \omega t$ at the top of the whip and $-q$ at the bottom. Find the current as a function of time.

√

16-d1 If a typical light bulb draws about 900 mA from a 110 V household circuit, what is its resistance? (Don't worry about the fact that it's alternating current.)

✓

16-d2 (a) Express the power dissipated by a resistor in terms of R and ΔV only, eliminating I. ✓

(b) Electrical receptacles in your home are mostly 110 V, but circuits for electric stoves, air conditioners, and washers and driers are usually 220 V. The two types of circuits have differently shaped receptacles. Suppose you rewire the plug of a drier so that it can be plugged in to a 110 V receptacle. The resistor that forms the heating element of the drier would normally draw 200 W. How much power does it actually draw now?

✓

16-d3 A resistor has a voltage difference ΔV across it, causing a current I to flow.

(a) Find an equation for the power it dissipates as heat in terms of the variables I and R only, eliminating ΔV. ✓

(b) If an electrical line coming to your house is to carry a given amount of current, interpret your equation from part a to explain whether the wire's resistance should be small, or large.

16-d4 We have referred to resistors *dissipating* heat, i.e., we have assumed that $P = I\Delta V$ is always greater than zero. Could $I\Delta V$ come out to be negative for a resistor? If so, could one make a refrigerator by hooking up a resistor in such a way that it absorbed heat instead of dissipating it?

16-d5 What resistance values can be created by combining a 1 kΩ resistor and a 10 kΩ resistor?

▷ Solution, p. 357

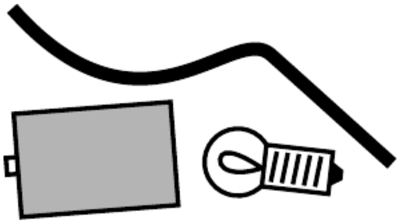
16-d6 Wire is sold in a series of standard diameters, called "gauges." The difference in diameter between one gauge and the next in the series is about 20%. How would the resistance of a given length of wire compare with the resistance of the same length of wire in the next gauge in the series?

✓

16-g1 You are given a battery, a flashlight bulb, and a single piece of wire. Draw at least two configurations of these items that would result in lighting up the bulb, and at least two that would not light it. (Don't draw schematics.) Note that the bulb has two electrical contacts: one is the threaded metal jacket, and the other is the tip (at the bottom in the figure).

If you're not sure what's going on, there are a couple of ways to check. The best is to try it in real life by either borrowing the materials from your instructor or scrounging the materials from around the house. (If you have a flashlight with this type of bulb, you can remove the bulb.) Another method is to use the simulation at `phet.colorado.edu/en/simulation/circuit-construction-kit-dc`.

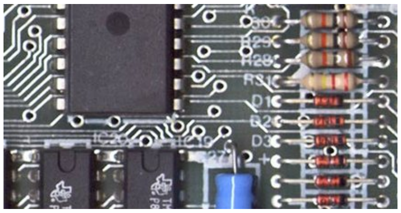
[Problem by Arnold Arons.]



Problem 16-g1.

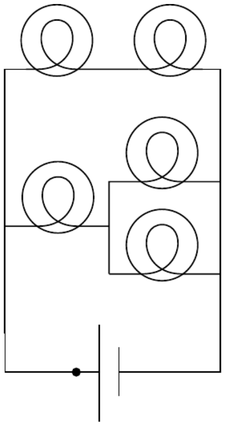
16-g2 You have to do different things with a circuit to measure current than to measure a voltage difference. Which would be more practical for a printed circuit board, in which the wires are actually strips of metal embedded inside the board?

▷ Solution, p. 358



Problem 16-g2.

16-g3 The figure shows a circuit containing five lightbulbs connected to a battery. Suppose you're going to connect one probe of a voltmeter to the circuit at the point marked with a dot. How many unique, nonzero voltage differences could you measure by connecting the other probe to other wires in the circuit?



Problems 16-g3 and 16-g4.

16-g4 The lightbulbs in the figure are all identical. If you were inserting an ammeter at various places in the circuit, how many unique currents could you measure? If you know that the current measurement will give the same number in more than one place, only count that as one unique current.

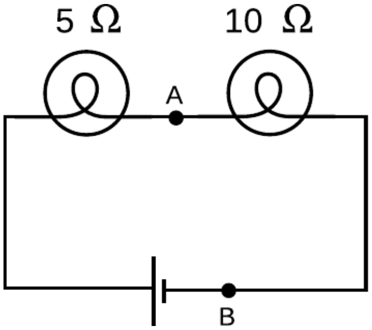
16-g5 In the figure, the battery is 9 V.

(a) What are the voltage differences across each light bulb? ✓

(b) What current flows through each of the three components of the circuit? ✓

(c) If a new wire is added to connect points A and B, how will the appearances of the bulbs change? What will be the new voltages and currents?

(d) Suppose no wire is connected from A to B, but the two bulbs are switched. How will the results compare with the results from the original setup as drawn?



Problem 16-g5.

16-g6 A student in a biology lab is given the following instructions: "Connect the cerebral eraser (C.E.) and the neural depolarizer (N.D.) in parallel with the power supply (P.S.). (Under no circumstances should you ever allow the cerebral eraser to come within 20 cm of your head.) Connect a voltmeter to measure the voltage across the cerebral eraser, and also insert an ammeter in the circuit so that you can make sure you don't put more than 100 mA through the neural depolarizer." The diagrams show two lab groups' attempts to follow the instructions. (a) Translate diagram a into a standard-style schematic. What is correct and incorrect about this group's setup? (b) Do the same for diagram b.

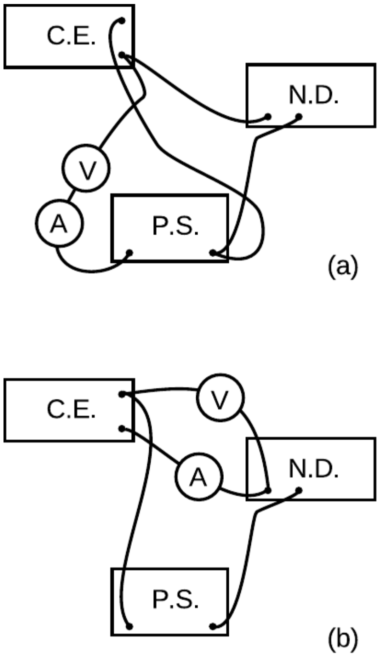
16-g7 A 1.0 Ω toaster and a 2.0 Ω lamp are connected in parallel with the 110-V supply of your house. (Ignore the fact that the voltage is AC rather than DC.)

(a) Draw a schematic of the circuit.

(b) For each of the three components in the circuit, find the current passing through it and the voltage drop across it. ✓

(c) Suppose they were instead hooked up in series. Draw a schematic and calculate the same things. ✓

16-g8 The heating element of an electric stove is connected in series with a switch that opens and closes many times per second. When you



Problem 16-g6.

turn the knob up for more power, the fraction of the time that the switch is closed increases. Suppose someone suggests a simpler alternative for controlling the power by putting the heating element in series with a variable resistor controlled by the knob. (With the knob turned all the way clockwise, the variable resistor's resistance is nearly zero, and when it's all the way counterclockwise, its resistance is essentially infinite.) (a) Draw schematics. (b) Why would the simpler design be undesirable?

16-g9 You have a circuit consisting of two unknown resistors in series, and a second circuit consisting of two unknown resistors in parallel.
(a) What, if anything, would you learn about the resistors in the series circuit by finding that the currents through them were equal?
(b) What if you found out the voltage differences across the resistors in the series circuit were equal?
(c) What would you learn about the resistors in the parallel circuit from knowing that the currents were equal?
(d) What if the voltages in the parallel circuit were equal?

16-g10 How many different resistance values can be created by combining three unequal resistors? (Don't count possibilities in which not all the resistors are used, i.e., ones in which there is zero current in one or more of them.)

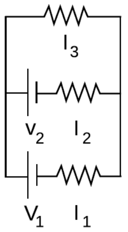
16-g11 Suppose six identical resistors, each with resistance R, are connected so that they form the edges of a tetrahedron (a pyramid with three sides in addition to the base, i.e., one less side than an Egyptian pyramid). What resistance value or values can be obtained by making connections onto any two points on this arrangement?

▷ Solution, p. 360

16-g12 A person in a rural area who has no electricity runs an extremely long extension cord to a friend's house down the road so she can run an electric light. The cord is so long that its resistance, x, is not negligible. Show that the lamp's brightness is greatest if its resistance, y, is equal to x. Explain physically why the lamp is dim for values of y that are too small or too large.

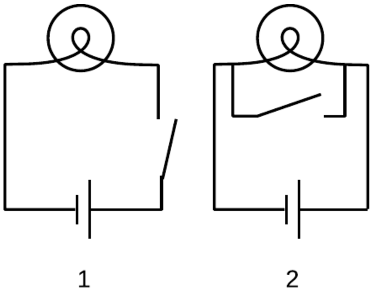
16-g13 All three resistors have the same resistance, R. Find the three unknown currents in terms of V_1, V_2, and R.

√



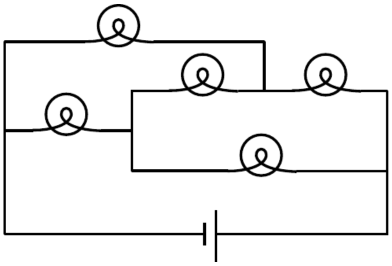
Problem 16-g13.

16-g14 The figure shows two possible ways of wiring a flashlight with a switch. Both will serve to turn the bulb on and off, although the switch functions in the opposite sense. Why is method 1 preferable?



Problem 16-g14.

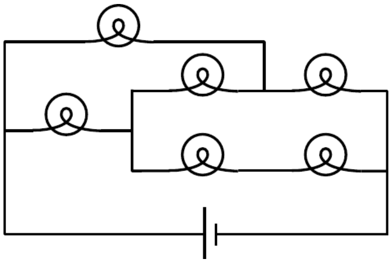
16-g15 The bulbs are all identical. Which one doesn't light up?



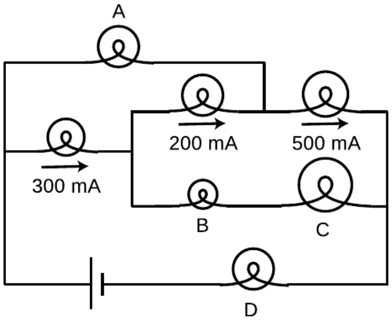
Problem 16-g15.

16-g16 Each bulb has a resistance of one ohm. How much power is drawn from the one-volt battery? √

16-g17 The bulbs all have unequal resistances. Given the three currents shown in the figure, find the currents through bulbs A, B, C, and D. √



Problem 16-g16.



Problem 16-g17.

16-g18 It's fairly common in electrical circuits for additional, undesirable resistances to occur because of factors such as dirty, corroded, or loose connections. Suppose that a device with resistance R normally dissipates power P, but due to an additional series resistance r the *total* power is reduced to P'. We might, for example, detect this change because the battery powering our device ran down more slowly than normal.
(a) Find the unknown resistance r. √
(b) Check that the units of your result make sense.
(c) Check that your result makes sense in the special cases $P' = P$ and $P' = 0$.
(d) Suppose we redefine P' as the useful power dissipated in R. For example, this would be the change we would notice because a flashlight was dimmer. Find r. √

16-j1 Suppose a parallel-plate capacitor is built so that a slab of dielectric material can be slid in or out. (This is similar to the way a stud finder works.) We insert the dielectric, hook the capacitor up to a battery to charge it, and then use an ammeter and a voltmeter to observe what happens when the dielectric is withdrawn. Predict the changes observed on the meters, and correlate them with the expected change in capacitance. Discuss the energy transformations involved, and determine whether positive or negative work is done in removing the dielectric.

Problem 16-j1.

16-j2 Repeat problem 16-j1, but with one change in the procedure: after we charge the capacitor, we open the circuit, and then continue with the observations.

17 Basics of relativity

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

17.1 The Lorentz transformation

There is a saying among biologists that without evolution, nothing in biology makes sense. Similarly, it is impossible to make sense out of electricity and magnetism, beyond simple electrostatics and DC circuits, without understanding a few basic ideas about Einstein's theory of special relativity.

According to Galileo and Newton, motion is relative but time is absolute. This theory of time and space is called Galilean relativity. According to Galilean relativity, observers in different states of motion will have position and time coordinates that relate to one another in the manner shown in figure 17.1.

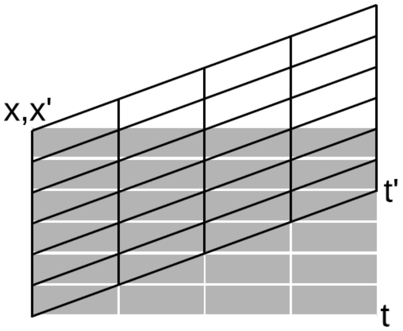


Figure 17.1: The relationship between time and space coordinates in two different frames of reference, according to Galilean relativity.

Experiments show that this absoluteness of time is only an approximation, valid at low speeds. At high speeds, or with sufficiently precise experiments, we find that time is relative. Although this idea dates back to a 1905 paper by Einstein, and certain types of indirect experimental evidence go back as far as the 19th century, modern technology has made it easier to demonstrate this in more direct and conceptually simple experiments. In 2010, for example, Chou *et al.* succeeded in building an atomic clock accurate enough to detect an effect at speeds as low as 10 m/s. The correct relationship between time and space in different frames of reference, proposed mathematically by Lorentz and interpreted correctly by Einstein, is called the Lorentz transformation, figure 17.2.

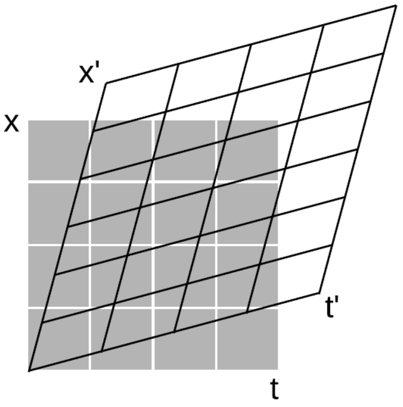


Figure 17.2: The Lorentz transformation.

The Lorentz transformation shown in the figure has a simple symmetry with respect to a flip across the diagonal. This symmetry is present only when we use units specially adapted to relativity. In such units, the slope of the 45-degree diagonal is a special speed having the value 1, and time and space are measured in the same units. In SI units, this special speed is notated

c, and it has a defined value, equal to approximately 3.0×10^8 m/s. One of the predictions of relativity is that anything without mass must move at this speed.

Since light was historically the first example encountered, c is often referred to as the speed of light, but relativity tells us that it is better to think of c as a kind of conversion factor between space and time.

The speed c has the following fundamentally important properties. It is the only speed that observers in different states of motion agree on. It is the speed at which massless objects always travel, and it is an ultimate speed limit for massive objects. It is the maximum speed of propagations for signals or for any mechanism of cause and effect.

The Lorentz transformation can be expressed algebraically, although it would be a distraction to do so here. Its form is determined entirely by the facts that (1) the slope of the t' axis is the velocity of one observer relative to the other, (2) the main diagonal keeps the same slope, and (3) the area of the boxes is preserved in the transformation.

17.2 Length contraction and time dilation

From these facts about the Lorentz transformation, it can be shown that different observers disagree about lengths and times in the following way. Let $\gamma = 1/\sqrt{1-v^2}$, where v is the velocity of one observer relative to another. (In SI units, substitute v/c for v.)

A clock appears to run fastest to an observer at rest relative to the clock. According to other observers, the clock's rate is lower by a factor of γ.

A meter stick appears longest to an observer at rest relative to the stick. An observer in motion parallel to the stick measures the stick to have been shortened by a factor of γ.

Problems

17-a1 The figure illustrates a Lorentz transformation using the conventions described in sec. 17.1, p. 181. For simplicity, the transformation chosen is one that lengthens one diagonal by a factor of 2. Since Lorentz transformations preserve area, the other diagonal is shortened by a factor of 2. Let the original frame of reference, depicted with the square, be A, and the new one B. (a) By measuring with a ruler on the figure, show that the velocity of frame B relative to frame A is $0.6c$. (b) Print out a copy of the page. With a ruler, draw a third parallelogram that represents a second successive Lorentz transformation, one that lengthens the long diagonal by another factor of 2. Call this third frame C. Use measurements with a ruler to determine frame C's velocity relative to frame A. Does it equal double the velocity found in part a? Explain why it should be expected to turn out the way it does.

$\checkmark$

17-a2 Astronauts in three different spaceships are communicating with each other. Those aboard ships A and B agree on the rate at which time is passing, but they disagree with the ones on ship C.
(a) Alice is aboard ship A. How does she describe the motion of her own ship, in its frame of reference?
(b) Describe the motion of the other two ships according to Alice.
(c) Give the description according to Betty, whose frame of reference is ship B.
(d) Do the same for Cathy, aboard ship C.

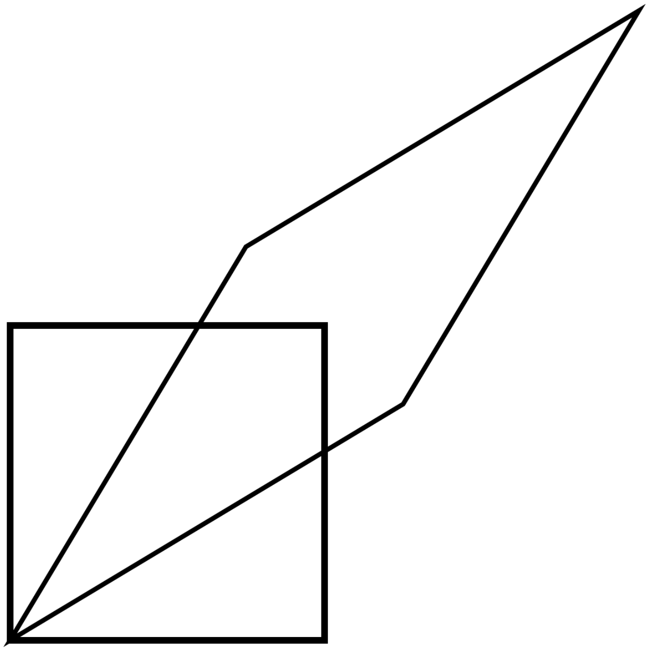
17-a3 What happens in the equation for γ when you put in a negative number for v? Explain what this means physically, and why it makes sense.

17-a4 The Voyager 1 space probe, launched in 1977, is moving faster relative to the earth than any other human-made object, at 17,000 meters per second.
(a) Calculate the probe's γ.
(b) Over the course of one year on earth, slightly less than one year passes on the probe. How much less? (There are 31 million seconds in a year.)

$\checkmark$

17-a5 The earth is orbiting the sun, and therefore is contracted relativistically in the direction of its motion. Compute the amount by which its diameter shrinks in this direction. $\checkmark$

17-a6 (a) Show that for $v = (3/5)c$, γ comes out to be a simple fraction.
(b) Find another value of v for which γ is a simple fraction.



Problem 17-a1.

18 Electromagnetism

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

18.1 Electromagnetism

The top panel of figure 18.1 shows a charged particle moving to the right, parallel to a current formed by two countermoving lines of opposite charge, moving at velocities $\pm u$. The two lines of charge are drawn offset from each other to make them easy to distinguish, but we think of them as coinciding, so that the line is electrically neutral over all, much like a current-carrying copper wire. Based on our knowledge of electrostatics, we would expect the lone charge to feel zero force, since the neutral "wire" has no electric field.

The bottom panel of the figure shows the same situation in the rest frame of the lone charge. Although velocities do not exactly add and subtract in special relativity as they would in Galilean relativity (problem 17-a1, p. 183), they approximately do if the velocities are not too big, so that the velocities of the two lines of charge are approximately $u - v$ and $-u - v$. Since the magnitudes of these velocities are unequal, the length contractions are unequal, and the "wire" is charged, according to an observer in this frame. Therefore the lone charge feels an attractive (downward) electrical force.

The descriptions in the two frames of reference is inconsistent, so we introduce a force in the original frame. A moving charge always interacts with other moving charges through such a force, called a magnetic force. Thus if we had only known about electrical interactions, relativity would have compelled us to introduce magnetic interactions as well. Relativity *unifies* the electrical and magnetic interactions as two sides of the same coin. The unified theory of electricity and magnetism is called electromagnetism.

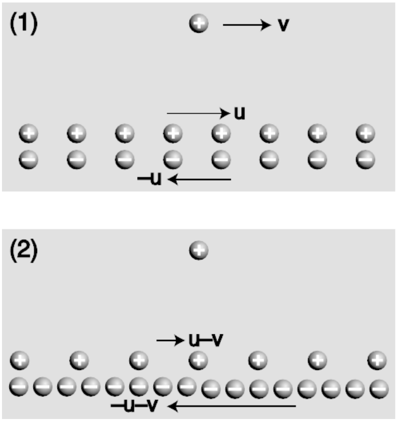


Figure 18.1: A charged particle and a current, seen in two different frames of reference. The second frame is moving at velocity v with respect to the first frame, so all the velocities have v subtracted from them (approximately).

18.2 The magnetic field

The magnetic force acting on a charged particle is $q\mathbf{v} \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic field. This is partly a definition of $\mathbf{B}$ and partly a prediction about how the force depends on $\mathbf{v}$. The units of the electric field are N·s/C·m, which can be abbreviated as tesla, 1 T = 1 N·s/C·m.

Empirically, we find that the magnetic field has no sources or sinks. Gauss' law for magnetism is

$$\Phi_B = 0.$$

In other words, there are no magnetic monopoles. There are, however, magnetic dipoles. Subatomic particles such as electrons and neutrons have magnetic dipole moments, as do some molecules. As a standard of comparison, the magnetic dipole moment $\mathbf{m}$ of a loop of current has magnitude $m = IA$, and is in the (right-handed) direction perpendicular to the loop. The energy of a magnetic dipole in an external magnetic field is $-\mathbf{m} \cdot \mathbf{B}$, and the torque acting on it is $\mathbf{m} \times \mathbf{B}$.

The energy of the magnetic field is

$$\mathrm{d}U_m = \frac{c^2}{8\pi k} B^2 \,\mathrm{d}v.$$

When a static magnetic field is caused by a current loop, the *Biot-Savart law*,

$$\mathrm{d}\mathbf{B} = \frac{kI \,\mathrm{d}\boldsymbol{\ell} \times \mathbf{r}}{c^2 r^3},$$

gives the field when we integrate over the loop.

Ampère's law is another way of relating static magnetic fields to the static currents that created them, and it is more easily extended to nonstatic fields than is the Biot-Savart law. Ampère's law states that the *circulation* of the magnetic field,

$$\Gamma_B = \int \mathbf{B} \cdot \mathrm{d}\mathbf{s},$$

around the edge of a surface is related to the current $I_{through}$ passing through the surface,

$$\Gamma = \frac{4\pi k}{c^2} I_{through}.$$

Problems

18-a1 A particle with a charge of 1.0 C and a mass of 1.0 kg is observed moving past point P with a velocity (1.0 m/s)$\hat{\mathbf{x}}$. The electric field at point P is (1.0 V/m)$\hat{\mathbf{y}}$, and the magnetic field is (2.0 T)$\hat{\mathbf{y}}$. Find the force experienced by the particle. $\surd$

18-a2 For a positively charged particle moving through a magnetic field, the directions of the $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{F}$ vectors are related by a right-hand rule:

$\mathbf{v}$	along the fingers, with the hand flat
$\mathbf{B}$	along the fingers, with the knuckles bent
$\mathbf{F}$	along the thumb

Make a three-dimensional model of the three vectors using pencils or rolled-up pieces of paper to represent the vectors assembled with their tails together. Make all three vectors perpendicular to each other. Now write down every possible way in which the rule could be rewritten by scrambling up the three symbols $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{F}$. Referring to your model, which are correct and which are incorrect?

18-a3 A charged particle is released from rest. We see it start to move, and as it gets going, we notice that its path starts to curve. Can we tell whether this region of space has $\mathbf{E} \neq 0$, or $\mathbf{B} \neq 0$, or both? Assume that no other forces are present besides the possible electrical and magnetic ones, and that the fields, if they are present, are uniform.

18-a4 A charged particle is in a region of space in which there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. There is no electric field, and no other forces act on the particle. In each case, describe the future motion of the particle, given its initial velocity.
(a) $\mathbf{v}_\text{o} = 0$
(b) $\mathbf{v}_\text{o} = (1\text{ m/s})\hat{\mathbf{z}}$
(c) $\mathbf{v}_\text{o} = (1\text{ m/s})\hat{\mathbf{y}}$

18-a5 (a) A line charge, with charge per unit length λ, moves at velocity v along its own length. How much charge passes a given point in time dt? What is the resulting current?
(b) Show that the units of your answer in part a work out correctly.

Remark: This constitutes a physical model of an electric current, and it would be a physically realistic model of a beam of particles moving in a vacuum, such as the electron beam in a television tube. It is not a physically realistic model of the motion of the electrons in a current-carrying wire, or of the ions in your nervous system; the motion of the charge carriers in these systems is much more complicated and chaotic, and there are charges of both signs, so that the total charge is zero. But even when the model is physically unrealistic, it still gives the right answers when you use it to compute magnetic effects. This is a remarkable fact, which we will not prove. The interested reader is referred to E.M. Purcell, *Electricity and Magnetism*, McGraw Hill, 1963.

18-a6 Two parallel wires of length L carry currents I_1 and I_2. They are separated by a distance R, and we assume R is much less than L, so that our results for long, straight wires are accurate. The goal of this problem is to compute the magnetic forces acting between the wires.
(a) Neither wire can make a force on *itself.* Therefore, our first step in computing wire 1's force on wire 2 is to find the magnetic field made only by wire 1, in the space *occupied* by wire 2. Express this field in terms of the given quantities. $\surd$
(b) Let's model the current in wire 2 by pretending that there is a line charge inside it, possessing density per unit length λ_2 and moving at velocity v_2. Relate λ_2 and v_2 to the current I_2, using the result of problem 18-a5a. Now find the magnetic force wire 1 makes on wire 2, in terms of I_1, I_2, L, and R.
(c) Show that the units of the answer to part b work out to be newtons.

18-a7 Suppose a charged particle is moving through a region of space in which there is an electric field perpendicular to its velocity vector, and also a magnetic field perpendicular to both the particle's velocity vector and the electric field. Show that there will be one particular velocity at which the particle can be moving that results in a total force of zero on it; this requires that you analyze both the magnitudes and the directions of the forces compared to one another. Relate this velocity to the magnitudes of the electric and magnetic fields. (Such an arrangement, called a velocity filter, is one way of determining the speed of an unknown particle.)

18-a8 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$\begin{aligned} q_1 &= 1\ \mu\text{C}\,, & q_2 &= -2\ \mu\text{C}\,,\\ \mathbf{v}_1 &= (1\ \text{m/s})\hat{\mathbf{x}}\,, & \mathbf{v}_2 &= (-1\ \text{m/s})\hat{\mathbf{x}}\,,\\ \mathbf{F}_1 &= (-1\ \text{mN})\hat{\mathbf{y}} & \mathbf{F}_2 &= (-2\ \text{mN})\hat{\mathbf{y}} \end{aligned}$$

The data are insufficient to determine the magnetic field vector; demonstrate this by giving two different magnetic field vectors, both of which are consistent with the data.

18-a9 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$\begin{aligned} q_1 &= 1\ \text{nC}\,, & q_2 &= 1\ \text{nC}\,,\\ \mathbf{v}_1 &= (1\ \text{m/s})\hat{\mathbf{z}}\,, & \mathbf{v}_2 &= (3\ \text{m/s})\hat{\mathbf{z}}\,,\\ \mathbf{F}_1 &= (5\ \text{pN})\hat{\mathbf{x}} & \mathbf{F}_2 &= (10\ \text{pN})\hat{\mathbf{x}}\\ &\quad +(2\ \text{pN})\hat{\mathbf{y}} & &\quad +(4\ \text{pN})\hat{\mathbf{y}} \end{aligned}$$

Is there a nonzero electric field at this point? A nonzero magnetic field?

18-a10 This problem is a continuation of problem 18-a6. Note that the answer to problem 18-a6b is given on page **??**.

(a) Interchanging the 1's and 2's in the answer to problem 18-a6b, what is the magnitude of the magnetic force from wire 2 acting on wire 1? Is this consistent with Newton's third law?

(b) Suppose the currents are in the same direction. Make a sketch, and use the right-hand rule to determine whether wire 1 pulls wire 2 towards it, or pushes it away.

(c) Apply the right-hand rule again to find the direction of wire 2's force on wire 1. Does this agree with Newton's third law?

(d) What would happen if wire 1's current was in the opposite direction compared to wire 2's?

18-a11 (a) In the photo, magnetic forces cause a beam of electrons to move in a circle. The beam is created in a vacuum tube, in which a small amount of hydrogen gas has been left. A few of the electrons strike hydrogen molecules, creating light and letting us see the beam. A magnetic field is produced by passing a current (meter) through the circular coils of wire in front of and behind the tube. In the bottom figure, with the magnetic field turned on, the force perpendicular to the electrons' direction of motion causes them to move in a circle. infer the direction of the magnetic field from the motion of the electron beam. (The answer is given in the answer to the self-check on that page.)

(b) Based on your answer to part a, find the direction of the currents in the coils.

(c) What direction are the electrons in the coils going?

(d) Are the currents in the coils repelling the currents consisting of the beam inside the tube, or attracting them? Check your answer by comparing with the result of problem 18-a10.

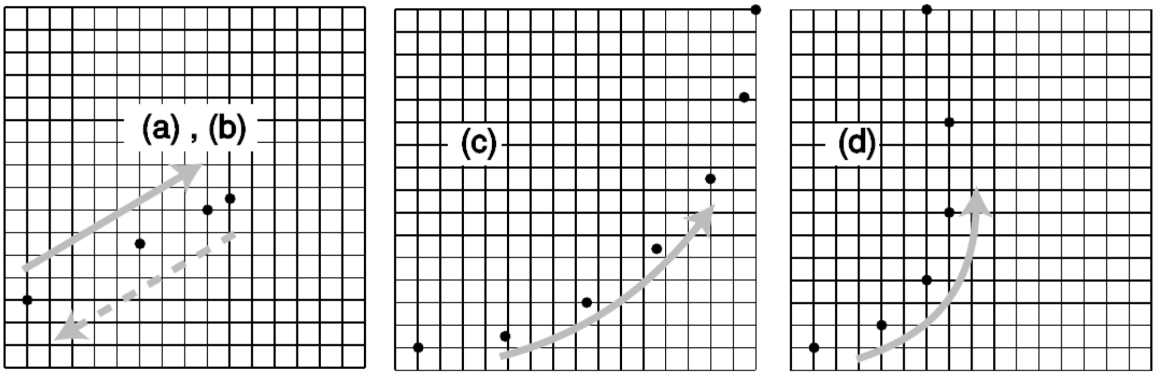
18-a12 A charged particle of mass m and charge q moves in a circle due to a uniform magnetic field of magnitude B, which points perpendicular to the plane of the circle.

(a) Assume the particle is positively charged. Make a sketch showing the direction of motion and the direction of the field, and show that the resulting force is in the right direction to produce circular motion.

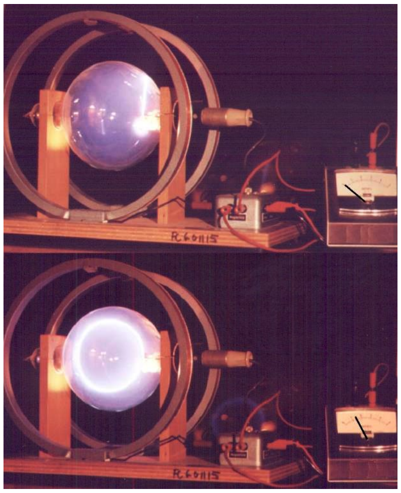
(b) Find the radius, r, of the circle, in terms of m, q, v, and B. √

(c) Show that your result from part b has the right units.

(d) Discuss all four variables occurring on the right-hand side of your answer from part b. Do they make sense? For instance, what should happen to the radius when the magnetic field is made stronger? Does your equation behave this



Problem 18-a13.



Problem 18-a11.

way?

(e) Restate your result so that it gives the particle's angular frequency, ω, in terms of the other variables, and show that v drops out. $\surd$

Remark: A charged particle can be accelerated in a circular device called a cyclotron, in which a magnetic field is what keeps them from going off straight. This frequency is therefore known as the cyclotron frequency. The particles are accelerated by other forces (electric forces), which are AC. As long as the electric field is operated at the correct cyclotron frequency for the type of particles being manipulated, it will stay in sync with the particles, giving them a shove in the right direction each time they pass by. The particles are speeding up, so this only works because the cyclotron frequency is independent of velocity.

18-a13 Each figure represents the motion of a positively charged particle. The dots give the particles' positions at equal time intervals. In each case, determine whether the motion was caused by an electric force, a magnetic force, or a frictional force, and explain your reasoning. If possible, determine the direction of the magnetic or electric field. All fields are uniform. In (a), the particle stops for an instant at the upper right, but then comes back down and to the left, retracing the same dots. In (b), it stops on the upper right and stays there.

18-a14 One model of the hydrogen atom has the electron circling around the proton at a speed of 2.2×10^6 m/s, in an orbit with a radius of 0.05 nm. (Although the electron and proton really orbit around their common center of mass, the center of mass is very close to the proton, since it is 2000 times more massive. For this problem, assume the proton is stationary.) In homework problem 16-a4, p. 174, you calculated the electric current created.

(a) Now estimate the magnetic field created at the center of the atom by the electron. We are treating the circling electron as a current loop, even though it's only a single particle. $\surd$

(b) Does the proton experience a nonzero force

from the electron's magnetic field? Explain.
(c) Does the electron experience a magnetic field from the proton? Explain.
(d) Does the electron experience a magnetic field created by its own current? Explain.
(e) Is there an electric force acting between the proton and electron? If so, calculate it. √
(f) Is there a gravitational force acting between the proton and electron? If so, calculate it.
(g) An inward force is required to keep the electron in its orbit – otherwise it would obey Newton's first law and go straight, leaving the atom. Based on your answers to the previous parts, which force or forces (electric, magnetic and gravitational) contributes significantly to this inward force?
[Based on a problem by Arnold Arons.]

18-a15 The following data give the results of three experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$\begin{aligned} q_1 &= 1\ \text{C}\,, & \mathbf{v}_1 &= 0\,, & \mathbf{F}_1 &= (1\ \text{N})\hat{\mathbf{y}} \\ q_2 &= 1\ \text{C}\,, & \mathbf{v}_2 &= (1\ \text{m/s})\hat{\mathbf{x}}\,, & \mathbf{F}_2 &= (1\ \text{N})\hat{\mathbf{y}} \\ q_3 &= 1\ \text{C}\,, & \mathbf{v}_3 &= (1\ \text{m/s})\hat{\mathbf{z}}\,, & \mathbf{F}_3 &= 0 \end{aligned}$$

Determine the electric and magnetic fields. √

18-a16 If you put four times more current through a solenoid, how many times more energy is stored in its magnetic field? √

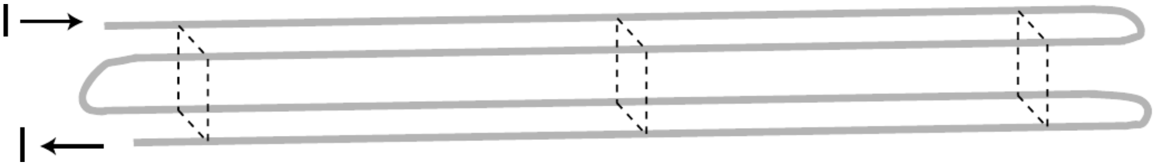
18-a17 A Helmholtz coil is defined as a pair of identical circular coils lying in parallel planes and separated by a distance, h, equal to their radius, b. (Each coil may have more than one turn of wire.) Current circulates in the same direction in each coil, so the fields tend to reinforce each other in the interior region. This configuration has the advantage of being fairly open, so that other apparatus can be easily placed inside and subjected to the field while remaining visible from the outside. The choice of $h = b$ results in the most uniform possible field near the center. A photograph of a Helmholtz coil is shown in example **??** on page **??**.
(a) Find the percentage drop in the field at the center of one coil, compared to the full strength at the center of the whole apparatus. √
(b) What value of h (not equal to b) would make this difference equal to zero? √

18-a18 The figure shows a nested pair of circular wire loops used to create magnetic fields. (The twisting of the leads is a practical trick for reducing the magnetic fields they contribute, so the fields are very nearly what we would expect for an ideal circular current loop.) The coordinate system below is to make it easier to discuss directions in space. One loop is in the $y-z$ plane, the other in the $x-y$ plane. Each of the loops has a radius of 1.0 cm, and carries 1.0 A in the direction indicated by the arrow.
(a) Calculate the magnetic field that would be produced by *one* such loop, at its center. √
(b) Describe the direction of the magnetic field that would be produced, at its center, by the loop in the $x-y$ plane alone.
(c) Do the same for the other loop.
(d) Calculate the magnitude of the magnetic field produced by the two loops in combination, at their common center. Describe its direction. √

18-a19 Four long wires are arranged, as shown, so that their cross-section forms a square, with connections at the ends so that current flows through all four before exiting. Note that the current is to the right in the two back wires, but to the left in the front wires. If the dimensions of the cross-sectional square (height and front-to-back) are b, find the magnetic field (magnitude and direction) along the long central axis. √

18-a20 In problem 18-a15, the three experiments gave enough information to determine both fields. Is it possible to design a procedure so that, using only two such experiments, we can always find **E** and **B**? If so, design it. If not, why not?

18-a21 Consider two solenoids, one of which is smaller so that it can be put inside the other.



Problem 18-a19.

Assume they are long enough so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is inside the big one with their currents circulating in the same direction, and a second configuration in which the currents circulate in opposite directions. Compare the energies of these configurations with the energy when the solenoids are far apart. Based on this reasoning, which configuration is stable, and in which configuration will the little solenoid tend to get twisted around or spit out?

18-a22 Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough to act like ideal solenoids, so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is partly inside and partly hanging out of the big one, with their currents circulating in the same direction. Their axes are constrained to coincide.
(a) Find the difference in the magnetic energy between the configuration where the solenoids are separate and the configuration where the small one is inserted into the big one. Your equation will include the length x of the part of the small solenoid that is inside the big one, as well as other relevant variables describing the two solenoids. $\surd$
(b) Based on your answer to part a, find the force acting

18-a23 Two long, parallel strips of thin metal foil form a configuration like a long, narrow sandwich. The air gap between them has height h, the width of each strip is w, and their length is ℓ. Each strip carries current I, and we assume for concreteness that the currents are in opposite directions, so that the magnetic force, F, between the strips is repulsive.
(a) Find the force in the limit of $w \gg h$. $\surd$
(b) Find the force in the limit of $w \ll h$, which is like two ordinary wires.
(c) Discuss the relationship between the two results.

18-a24 Suppose we are given a permanent magnet with a complicated, asymmetric shape. Describe how a series of measurements with a magnetic compass could be used to determine the strength and direction of its magnetic field at some point of interest. Assume that you are only able to see the direction to which the compass needle settles; you cannot measure the torque acting on it.

18-d1 This problem will lead you through the steps of applying the Biot-Savart law to prove that the magnetic field of a long, straight wire has magnitude

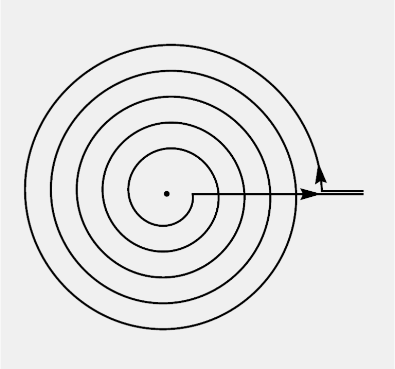
$$B = \frac{2kI}{c^2R}.$$

Almost everything in this equation has to be the way it is because of units, the only exception being the unitless factor of 2, so this problem amounts to proving that it really does come out to be 2[.] (a) Set up the integral prescribed by the Biot-Savart law, and simplify it so that it involves only scalar variables rather than a vector cross product, but do not evaluate it yet.
(b) Your integral will contain several different

variables, each of which is changing as we integrate along the wire. These will probably include a position on the wire, a distance from the point on the wire to the point at which the field is to be found, and an angle between the wire and this point-to-point line. In order to evaluate the integral, it is necessary to express the integral in terms of only one of these variables. It's not obvious, but the integral turns out to be easiest to evaluate if you express it in terms of the angle and eliminate the other variables. Do so. Note that the d... part of the integral has to be reexpressed in the same way we would do any time we attacked an integral by substitution ("u-substitution").
(c) Pulling out all constant factors now gives a definite integral. Evaluate this integral, which you should find is a trivial one, and show that it equals 2.

18-d2 A regular polygon with n sides can be inscribed within a circle of radius R and can have a circle inscribed inside it with radius h. Let $\rho = \sqrt{hR}$ be the geometric mean of these two radii. A current loop is constructed in the shape of a regular n-gon. Show that the magnetic field at the center can be calculated in a simple way from the perimeter and ρ, and make sense of the result in the extreme cases $n = 2$ (a degenerate polygon enclosing no area) and $n \to \infty$.

18-d3 Magnet coils are often wrapped in multiple layers. The figure shows the special case where the layers are all confined to a single plane, forming a spiral. Since the thickness of the wires (plus their insulation) is fixed, the spiral that results is a mathematical type known as an Archimedean spiral, in which the turns are evenly spaced. The equation of the spiral is $r = w\theta$, where w is a constant. For a spiral that starts from $r = a$ and ends at $r = b$, show that the field at the center is given by $(kI/c^2w)\ln b/a$.
▷ Solution, p. 370

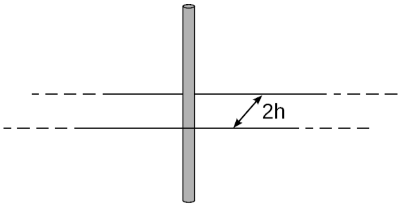


Problem 18-d3.

18-d4 Perform a calculation similar to the one in problem 18-d3, but for a logarithmic spiral, defined by $r = we^{u\theta}$, and show that the field is $B = (kI/c^2u)(1/a - 1/b)$. Note that the solution to problem 18-d3 is given in the back of the book.

18-g1 A certain region of space has a magnetic field given by $\mathbf{B} = bx\hat{\mathbf{y}}$. Find the electric current flowing through the square defined by $z = 0$, $0 \le x \le a$, and $0 \le y \le a$.
√

18-g2 Verify Ampère's law in the case shown in the figure, assuming the known equation for the field of a wire. A wire carrying current I passes perpendicularly through the center of the rectangular Ampèrian surface. The length of the rectangle is infinite, so it's not necessary to compute the contributions of the ends.



Problem 18-g2.

19 Maxwell's equations and electromagnetic waves

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

19.1 Maxwell's equations

The fundamental laws of physics governing electric and magnetic fields are Maxwell's equations, which state that for any closed surface, the fluxes through the surface are

$$\begin{aligned} \Phi_E &= 4\pi k q_{in} \qquad \text{and} \\ \Phi_B &= 0. \end{aligned}$$

For any surface that is not closed, the circulations around the edges of the surface are given by

$$\begin{aligned} \Gamma_E &= -\frac{\partial \Phi_B}{\partial t} \qquad \text{and} \\ c^2\Gamma_B &= \frac{\partial \Phi_E}{\partial t} + 4\pi k I_{through}. \end{aligned}$$

19.2 Electromagnetic waves

The most important result of Maxwell's equations is the existence of electromagnetic waves which propagate at the velocity of light — that's what light is. The waves are transverse, and the electric and magnetic fields are perpendicular to each other. There are no purely electric or purely magnetic waves; their amplitudes are always related to one another by $B = E/c$. They propagate in the right-handed direction given by the cross product $\mathbf{E} \times \mathbf{B}$, and carry momentum $p = U/c$.

19.3 Maxwell's equations in matter

A complete statement of Maxwell's equations in the presence of electric and magnetic materials is as follows:

$$\begin{aligned} \Phi_D &= q_{\text{free}} \\ \Phi_B &= 0 \\ \Gamma_E &= -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\ \Gamma_H &= \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} + I_{\text{free}}, \end{aligned}$$

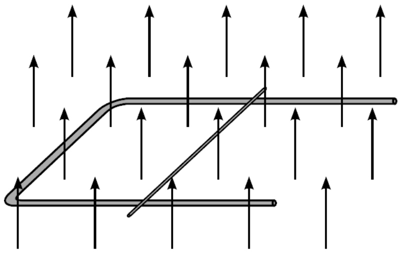
where the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ are defined as

$$\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} \qquad \text{and} \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu}, \end{aligned}$$

and ϵ and μ are the permittivity and permeability of the substance.

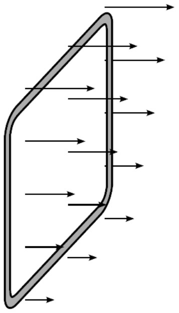
Problems

19-a1 A U-shaped wire makes electrical contact with a second, straight wire, which rolls along it to the right, as shown in the figure. The whole thing is immersed in a uniform magnetic field, which is perpendicular to the plane of the circuit. The resistance of the rolling wire is much greater than that of the U.
(a) Find the direction of the force on the wire based on conservation of energy.
(b) Verify the direction of the force using right-hand rules.
(c) Find the magnitude of the force acting on the wire. There is more than one way to do this, but please do it using Faraday's law (which works even though it's the Ampèrian surface itself that is changing, rather than the field). ✓
(d) Consider how the answer to part a would have changed if the direction of the field had been reversed, and also do the case where the direction of the rolling wire's motion is reversed. Verify that this is in agreement with your answer to part c.



Problem 19-a1.

19-a2 A wire loop of resistance R and area A, lying in the $y-z$ plane, falls through a nonuniform magnetic field $\mathbf{B} = kz\hat{\mathbf{x}}$, where k is a constant. The z axis is vertical.
(a) Find the direction of the force on the wire based on conservation of energy.
(b) Verify the direction of the force using right-hand rules.
(c) Find the magnetic force on the wire.



Problem 19-a2.

✓

19-a3 (a) For each term appearing on the right side of Maxwell's equations, give an example of an everyday situation it describes.
(b) Most people doing calculations in the SI system of units don't use k and k/c^2. Instead, they express everything in terms of the constants

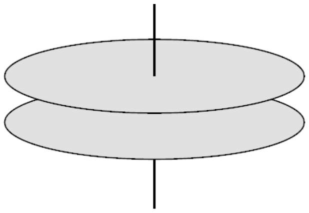
$$\epsilon_o = \frac{1}{4\pi k} \qquad \text{and}$$
$$\mu_o = \frac{4\pi k}{c^2}.$$

Rewrite Maxwell's equations in terms of these constants, eliminating k and c everywhere.

19-a4 The circular parallel-plate capacitor shown in the figure is being charged up over time, with the voltage difference across the plates varying as $V = st$, where s is a constant. The plates have radius b, and the distance between them is d. We assume $d \ll b$, so that the electric field between the plates is uniform, and parallel to the axis. Find the induced magnetic field at a point between the plates, at a distance R from the axis.

✓

19-d1 A charged particle is in motion at speed v, in a region of vacuum through which an electromagnetic wave is passing. In what direction should the particle be moving in order to minimize the total force acting on it? Consider both possibilities for the sign of the charge. (Based on a problem by David J. Raymond.)



Problem 19-a4.

19-d2 (a) Prove that in an electromagnetic plane wave, half the energy is in the electric field and half in the magnetic field.
(b) Based on your result from part a, find the proportionality constant in the relation $\mathrm{d}\mathbf{p} \propto \mathbf{E}\times\mathbf{B}\,\mathrm{d}v$, where $\mathrm{d}\mathbf{p}$ is the momentum of the part of a plane light wave contained in the volume $\mathrm{d}v$. The vector $\mathbf{E}\times\mathbf{B}$, multiplied by the appropriate constant, is known as the Poynting vector, and even outside the context of an electromagnetic plane wave, it can be interpreted as a momentum density or rate of energy flow. (To do this problem, you need to know the relativistic relationship between the energy and momentum of a beam of light from problem **??** on p. **??**.) √

19-d3 (a) A beam of light has cross-sectional area A and power P, i.e., P is the number of joules per second that enter a window through which the beam passes. Find the energy density U/v in terms of P, A, and universal constants.
(b) Find $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, the amplitudes of the electric and magnetic fields, in terms of P, A, and universal constants (i.e., your answer should *not* include U or v). You will need the result of problem 19-d2a. A real beam of light usually consists of many short wavetrains, not one big sine wave, but don't worry about that. √
(c) A beam of sunlight has an intensity of $P/A = 1.35\times10^3$ W/m^2, assuming no clouds or atmospheric absorption. This is known as the solar constant. Compute $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, and compare with the strengths of static fields you experience in everyday life: $E \sim 10^6$ V/m in a thunderstorm, and $B \sim 10^{-3}$ T for the Earth's magnetic field.

√

19-d4 Electromagnetic waves are supposed to have their electric and magnetic fields perpendicular to each other. (Throughout this problem, assume we're talking about waves traveling through a vacuum, and that there is only a single sine wave traveling in a single direction, not a superposition of sine waves passing through each other.) Suppose someone claims they can make an electromagnetic wave in which the electric and magnetic fields lie in the same plane. Prove that this is impossible based on Maxwell's equations.

19-d5 A positively charged particle is released from rest at the origin at $t = 0$, in a region of vacuum through which an electromagnetic wave is passing. The particle accelerates in response to the wave. In this region of space, the wave varies as $\mathbf{E} = \hat{\mathbf{x}}\tilde{E}\sin\omega t$, $\mathbf{B} = \hat{\mathbf{y}}\tilde{B}\sin\omega t$, and we assume that the particle has a relatively large value of m/q, so that its response to the wave is sluggish, and it never ends up moving at any speed comparable to the speed of light. Therefore we don't have to worry about the spatial variation of the wave; we can just imagine that these are uniform fields imposed by some external mechanism on this region of space.
(a) Find the particle's coordinates as functions of time. √
(b) Show that the motion is confined to $-z_{max} \le z \le z_{max}$, where $z_{max} = 1.101\left(q^2\tilde{E}\tilde{B}/m^2\omega^3\right)$.

19-d6 If you watch a movie played backwards, some vectors reverse their direction. For instance, people walk backwards, with their velocity vectors flipped around. Other vectors, such as forces, keep the same direction, e.g., gravity still pulls down. An electric field is another example of a vector that doesn't turn around: positive charges are still positive in the time-reversed universe, so they still make diverging electric fields, and likewise for the converging fields around negative charges.
(a) How does the momentum of a material object behave under time-reversal?

(b) The laws of physics are still valid in the time-reversed universe. For example, show that if two material objects are interacting, and momentum is conserved, then momentum is still conserved in the time-reversed universe.
(c) Discuss how currents and magnetic fields would behave under time reversal.
(d) Similarly, show that the equation $\mathrm{d}\mathbf{p} \propto \mathbf{E}\times\mathbf{B}$ is still valid under time reversal.

▷ Solution, p. 375

19-d7 This problem is a more advanced exploration of the time-reversal ideas introduced in problem 19-d6.
(a) In that problem, we assumed that charge did not flip its sign under time reversal. Suppose we make the opposite assumption, that charge *does* change its sign. This is an idea introduced by Richard Feynman: that antimatter is really matter traveling backward in time! Determine the time-reversal properties of $\mathbf{E}$ and $\mathbf{B}$ under this new assumption, and show that $\mathrm{d}\mathbf{p} \propto \mathbf{E}\times\mathbf{B}$ is still valid under time-reversal.
(b) Show that Maxwell's equations are time-reversal symmetric, i.e., that if the fields $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ satisfy Maxwell's equations, then so do $\mathbf{E}(x, y, z, -t)$ and $\mathbf{B}(x, y, z, -t)$. Demonstrate this under both possible assumptions about charge, $q \rightarrow q$ and $q \rightarrow -q$.

19-g1 (a) Figure **??** on page **??** shows a hollow sphere with $\mu/\mu_\mathrm{o} = x$, inner radius a, and outer radius b, which has been subjected to an external field $\mathbf{B}_\mathrm{o}$. Finding the fields on the exterior, in the shell, and on the interior requires finding a set of fields that satisfies five boundary conditions: (1) far from the sphere, the field must approach the constant $\mathbf{B}_\mathrm{o}$; (2) at the outer surface of the sphere, the field must have $\mathbf{H}_{\parallel,1} = \mathbf{H}_{\parallel,2}$, as discussed on page **??**; (3) the same constraint applies at the inner surface of the sphere; (4) and (5) there is an additional constraint on the fields at the inner and outer surfaces, as found in problem **??**. The goal of this problem is to find the solution for the fields, and from it, to prove that the interior field is uniform, and given by

$$\mathbf{B} = \left[\frac{9x}{(2x+1)(x+2) - 2\frac{a^3}{b^3}(x-1)^2}\right]\mathbf{B}_\mathrm{o}.$$

This is a very difficult problem to solve from first principles, because it's not obvious what form the fields should have, and if you hadn't been told, you probably wouldn't have guessed that the interior field would be uniform. We could, however, guess that once the sphere becomes polarized by the external field, it would become a dipole, and at $r \gg b$, the field would be a uniform field superimposed on the field of a dipole. It turns out that even close to the sphere, the solution has exactly this form. In order to complete the solution, we need to find the field in the shell ($a < r < b$), but the only way this field could match up with the detailed angular variation of the interior and exterior fields would be if it was also a superposition of a uniform field with a dipole field. The final result is that we have four unknowns: the strength of the dipole component of the external field, the strength of the uniform and dipole components of the field within the shell, and the strength of the uniform interior field. These four unknowns are to be determined by imposing constraints (2) through (5) above.
(b) Show that the expression from part a has physically reasonable behavior in its dependence on x and a/b.

$\star\star$

20 LRC circuits

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

20.1 Complex numbers

For a more detailed treatment of complex numbers, see ch. 3 of James Nearing's free book at physics.miami.edu/nearing/mathmethods/.

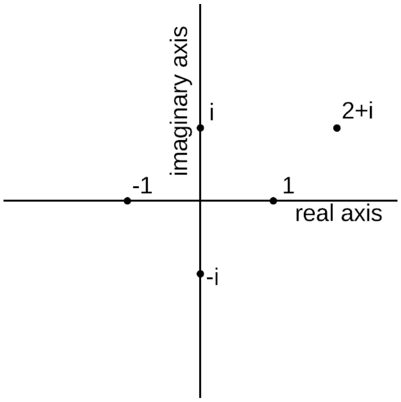


Figure 20.1: Visualizing complex numbers as points in a plane.

We assume there is a number, i, such that $i^2 = -1$. The square roots of -1 are then i and $-i$. (In electrical engineering work, where i stands for current, j is sometimes used instead.) This gives rise to a number system, called the complex numbers, containing the real numbers as a subset. Any complex number z can be written in the form $z = a+bi$, where a and b are real, and a and b are then referred to as the real and imaginary parts of z. A number with a zero real part is called an imaginary number. The complex numbers can be visualized as a plane, with the real number line placed horizontally like the x axis of the familiar $x-y$ plane, and the imaginary numbers running along the y axis. The complex numbers are complete in a way that the real numbers aren't: every nonzero complex number has two square roots. For example, 1 is a real number, so it is also a member of the complex numbers, and its square roots are -1 and 1. Likewise, -1 has square roots i and $-i$, and the number i has square roots $1/\sqrt{2} + i/\sqrt{2}$ and $-1/\sqrt{2} - i/\sqrt{2}$.

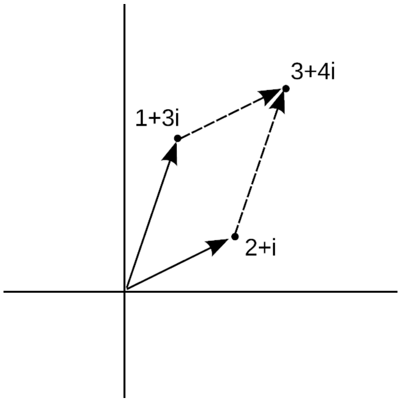


Figure 20.2: Addition of complex numbers is just like addition of vectors, although the real and imaginary axes don't actually represent directions in space.

Complex numbers can be added and subtracted by adding or subtracting their real and imaginary parts. Geometrically, this is the same as vector addition.

The complex numbers $a + bi$ and $a - bi$, lying at equal distances above and below the real axis, are called complex conjugates. The results of the quadratic formula are either both real, or complex conjugates of each other. The complex conjugate of a number z is notated as $\bar{z}$ or z^*.

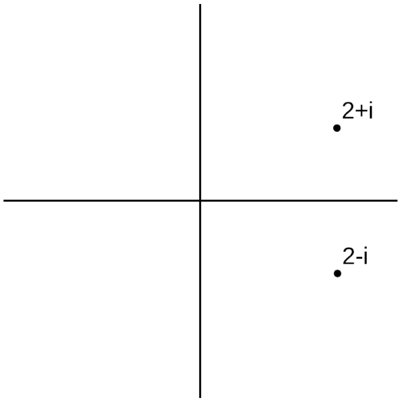


Figure 20.3: A complex number and its conjugate.

The complex numbers obey all the same rules of arithmetic as the reals, except that they can't be ordered along a single line. That is, it's not possible to say whether one complex number is greater than another. We can compare them in terms of their magnitudes (their distances from the origin), but two distinct complex numbers may have the same magnitude, so, for example, we can't say whether 1 is greater than i or i is greater than 1.

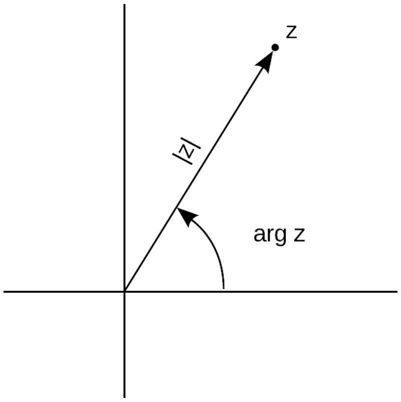


Figure 20.4: A complex number can be described in terms of its magnitude and argument.

There is a nice interpretation of complex multiplication. We define the argument of a complex number as its angle in the complex plane, measured counterclockwise from the positive real axis. Multiplying two complex numbers then corresponds to multiplying their magnitudes, and adding their arguments.

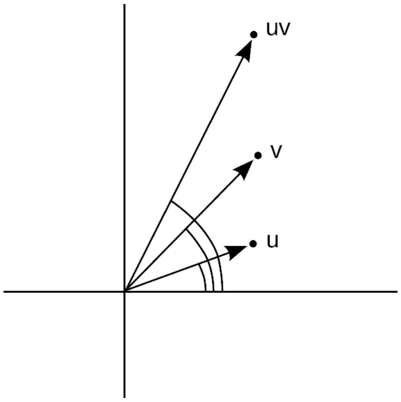


Figure 20.5: The argument of uv is the sum of the arguments of u and v.

Having expanded our horizons to include the complex numbers, it's natural to want to extend functions we knew and loved from the world of real numbers so that they can also operate on complex numbers. The only really natural way to do this in general is to use Taylor series. A particularly beautiful thing happens with the functions e^x, $\sin x$, and $\cos x$:

$$\begin{aligned} e^x &= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots \end{aligned}$$

If $x = i\phi$ is an imaginary number, we have

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

a result known as Euler's formula. The geometrical interpretation in the complex plane is shown in figure 20.6.

Although the result may seem like something out of a freak show at first, applying the definition of the exponential function makes it clear how natural it is:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n .$$

When $x = i\phi$ is imaginary, the quantity $(1 + i\phi/n)$ represents a number lying just above 1 in the complex plane. For large n, $(1 + i\phi/n)$ becomes very close to the unit circle, and its argument is the small angle ϕ/n. Raising this number to the nth power multiplies its argument by n, giving a number with an argument of ϕ.

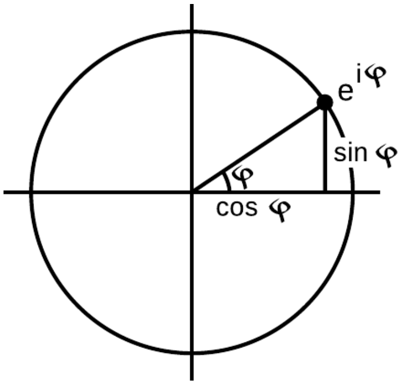


Figure 20.6: The complex number $e^{i\phi}$ lies on the unit circle.

Sinusoidal functions have a remarkable property, which is that if you add two different sinusoidal functions having the same frequency, the result is also a sinusoid with that frequency. For example, $\cos\omega t + \sin\omega t = \sqrt{2}\sin(\omega t + \pi/4)$, which can be proved using trig identities. The trig identities can get very cumbersome, however, and there is a much easier technique involving complex numbers.

Figure 20.7 shows a useful way to visualize what's going on. When a circuit is oscillating at a frequency ω, we use points in the plane to represent sinusoidal functions with various phases and amplitudes.

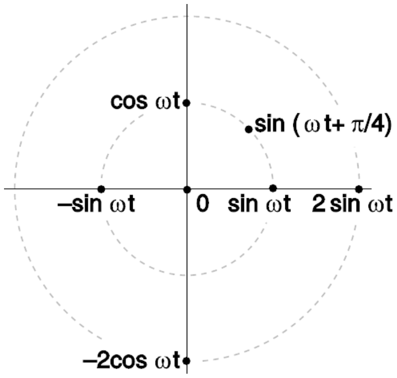


Figure 20.7: Representing functions with points in polar coordinates.

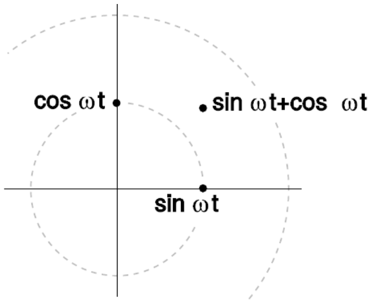


Figure 20.8: Adding two sinusoidal functions.

The simplest examples of how to visualize this in polar coordinates are ones like $\cos\omega t + \cos\omega t = 2\cos\omega t$, where everything has the same phase, so all the points lie along a single line in the polar plot, and addition is just like adding numbers on the number line. The less trivial example $\cos\omega t + \sin\omega t = \sqrt{2}\sin(\omega t + \pi/4)$, can be visualized as in figure 20.8.

Figure 20.8 suggests that all of this can be tied together nicely if we identify our plane with the plane of complex numbers. For example, the complex numbers 1 and i represent the functions $\sin\omega t$ and $\cos\omega t$.

20.2 LRC circuits

As discussed previously, capacitance, C, is defined as

$$U_C = \frac{1}{2C} q^2 .$$

Similarly, inductance, L, is defined as

$$U_L = \frac{L}{2} I^2 ,$$

and measured in units of henries. The magnitude of the voltage difference across a capacitor or inductor is given by

$$V_C = \frac{q}{C}$$

or

$$V_L = L \frac{\mathrm{d}I}{\mathrm{d}t} .$$

In the equation for the inductor, the direction of the voltage drop (plus or minus sign) is such that the inductor resists the change in current.

A series LRC circuit exhibits *oscillation*, and, if driven by an external voltage, resonates. The Q of the circuit relates to the resistance value. For large Q, the resonant frequency is

$$\omega \approx \frac{1}{\sqrt{LC}} .$$

A series RC or RL circuit exhibits exponential *decay*,

$$q = q_\mathrm{o} \exp\left(-\frac{t}{RC}\right)$$

or

$$I = I_\mathrm{o} \exp\left(-\frac{R}{L} t\right) ,$$

and the quantity RC or L/R is known as the time constant.

When driven by a sinusoidal AC voltage with amplitude $\tilde{V}$, a capacitor, resistor, or inductor responds with a current having amplitude

$$\tilde{I} = \frac{\tilde{V}}{Z} ,$$

where the *impedance*, Z, is a frequency-dependent quantity having units of ohms. In a capacitor, the current has a phase that is 90° ahead of the voltage, while in an inductor the current is 90° behind. We can represent these phase relationships by defining the impedances as complex numbers:

$$\begin{aligned} Z_C &= -\frac{i}{\omega C} \\ Z_R &= R \\ Z_L &= i\omega L \end{aligned}$$

The arguments of the complex impedances are to be interpreted as phase relationships between the oscillating voltages and currents. The complex impedances defined in this way combine in series and parallel according to the same rules as resistances.

When a voltage source is driving a load through a transmission line, the maximum power is delivered to the load when the impedances of the line and the load are matched.

Problems

20-a1 Find $\arg i$, $\arg(-i)$, and $\arg 37$, where $\arg z$ denotes the argument of the complex number z.

▷ Solution, p. 375

20-a2 Visualize the following multiplications in the complex plane using the interpretation of multiplication in terms of multiplying magnitudes and adding arguments: $(i)(i) = -1$, $(i)(-i) = 1$, $(-i)(-i) = -1$.

▷ Solution, p. 375

20-a3 If we visualize z as a point in the complex plane, how should we visualize $-z$? What does this mean in terms of arguments? Give similar interpretations for z^2 and $\sqrt{z}$.

▷ Solution, p. 375

20-a4 Find four different complex numbers z such that $z^4 = 1$.

▷ Solution, p. 375

20-a5 Compute the following. For the final two, use the magnitude and argument, not the real and imaginary parts.

$$|1+i| \quad , \quad \arg(1+i) \quad , \quad \left|\frac{1}{1+i}\right| \quad , \quad \arg\left(\frac{1}{1+i}\right)$$

From these, find the real and imaginary parts of $1/(1+i)$.

▷ Solution, p. 375

20-d1 (a) Use complex number techniques to rewrite the function $f(t) = 4\sin\omega t + 3\cos\omega t$ in the form $A\sin(\omega t + \delta)$. ✓

(b) Verify the result using the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

20-d2 This problem deals with the cubes and cube roots of complex numbers, but the principles involved apply more generally to other exponents besides 3 and 1/3. These examples are designed to be much easier to do using the magnitude-argument representation of complex numbers than with the cartesian representation. If done by the easiest technique, none of these requires more than two or three lines of *simple* math. In the following, the symbols θ, a, and b represent real numbers, and all angles are to be expressed in radians. As often happens with fractional exponents, the cube root of a complex number will typically have more than one possible value. (Cf. $4^{1/2}$, which can be 2 or -2.) In parts c and d, this ambiguity is resolved explicitly in the instructions, in a way that is meant to make the calculation as easy as possible.

(a) Calculate $\arg\left[(e^{i\theta})^3\right]$. ✓

(b) Of the points u, v, w, and x shown in the figure, which could be a cube root of z?

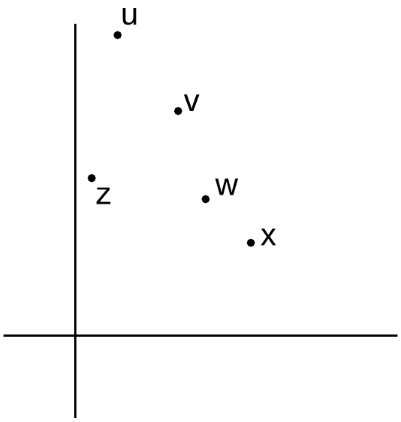
(c) Calculate $\arg\left[\sqrt[3]{a+bi}\right]$. For simplicity, assume that $a + bi$ is in the first quadrant of the complex plane, and compute the answer for a root that also lies in the first quadrant. ✓

(d) Compute

$$\frac{1+i}{(-2+2i)^{1/3}}.$$

Because there is more than one possible root to use in the denominator, multiple answers are possible in this problem. Use the root that results in the final answer that lies closest to the real line. (This is also the easiest one to find by using the magnitude-argument techniques introduced in the text.)

✓



Problem 20-d2.

20-d3 Calculate the quantity i^i (i.e., find its real and imaginary parts).

√

20-g1 (a) Show that the equation $V_L = L\,\mathrm{d}I/\,\mathrm{d}t$ has the right units.
(b) Verify that RC has units of time.
(c) Verify that L/R has units of time.

20-g2 Find the inductance of two identical inductors in parallel.

20-g3 The wires themselves in a circuit can have resistance, inductance, and capacitance. Would "stray" inductance and capacitance be most important for low-frequency or for high-frequency circuits? For simplicity, assume that the wires act like they're in *series* with an inductor or capacitor.

20-g4 Starting from the relation $V = L\,\mathrm{d}I/\,\mathrm{d}t$ for the voltage difference across an inductor, show that an inductor has an impedance equal to $L\omega$.

20-g5 Suppose that an FM radio tuner for the US commercial broadcast band (88-108 MHz) consists of a series LRC circuit. If the inductor is 1.0 μH, what range of capacitances should the variable capacitor be able to provide? √

20-g6 (a) Find the parallel impedance of a 37 kΩ resistor and a 1.0 nF capacitor at $f = 1.0 \times 10^4$ Hz. √
(b) A voltage with an amplitude of 1.0 mV drives this impedance at this frequency. What is the amplitude of the current drawn from the voltage source, what is the current's phase angle with respect to the voltage, and does it lead the voltage, or lag behind it? √

20-g7 A series LRC circuit consists of a 1.000 Ω resistor, a 1.000 F capacitor, and a 1.000 H inductor. (These are not particularly easy values to find on the shelf!)
(a) Plot its impedance as a point in the complex plane for each of the following frequencies: ω=0.250, 0.500, 1.000, 2.000, and 4.000 Hz.
(b) What is the resonant angular frequency, ω_{res}, and how does this relate to your plot? √
(c) What is the resonant frequency f_{res} corresponding to your answer in part b? √

20-g8 At a frequency of 53.1 kHz, a certain series LR circuit has an impedance of $1.6\ \mathrm{k}\Omega + (1.2\ \mathrm{k}\Omega)i$. Suppose that instead we want to achieve the same impedance using two circuit elements in parallel. What must the elements be? As a check on your answer, you should find that both values are round numbers when rounded off to the correct number of significant figures.

20-g9 (a) In a series LC circuit driven by a DC voltage ($\omega = 0$), compare the energy stored in the inductor to the energy stored in the capacitor.
(b) Carry out the same comparison for an LC circuit that is oscillating freely (without any driving voltage).
(c) Now consider the general case of a series LC circuit driven by an oscillating voltage at an arbitrary frequency. Let $\overline{U_L}$ and be the average energy stored in the inductor, and similarly for $\overline{U_C}$. Define a quantity $u = \overline{U_C}/(\overline{U_L} + \overline{U_C})$, which can be interpreted as the capacitor's average share of the energy, while $1 - u$ is the inductor's average share. Find u in terms of L, C, and ω, and sketch a graph of u and $1 - u$ versus ω. What happens at resonance? Make sure your result is consistent with your answer to part a. √

21 Thermodynamics

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

21.1 Pressure, temperature, and heat

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 5.1
- OpenStax University Physics, openstax.org, v. 2, ch. 1

21.2 Kinetic theory

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 5.2
- OpenStax University Physics, openstax.org, v. 2, ch. 2-3

21.3 Entropy

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 5.3-4
- OpenStax University Physics, openstax.org, v. 2, ch. 4

Problems

21-a1 (a) Show that under conditions of standard pressure and temperature, the volume of a sample of an ideal gas depends only on the number of molecules in it.
(b) One mole is defined as 6.0×10^{23} atoms. Find the volume of one mole of an ideal gas, in units of liters, at standard temperature and pressure (0°C and 101 kPa). $\surd$

21-a2 A gas in a cylinder expands its volume by an amount $\mathrm{d}V$, pushing out a piston. Show that the work done by the gas on the piston is given by $\mathrm{d}W = P\,\mathrm{d}V$.

21-a3 (a) Atmospheric pressure at sea level is 101 kPa. The deepest spot in the world's oceans is a valley called the Challenger Deep, in the Marianas Trench, with a depth of 11.0 km. Find the pressure at this depth, in units of atmospheres. Although water under this amount of pressure does compress by a few percent, assume for the purposes of this problem that it is incompressible.
(b) Suppose that an air bubble is formed at this depth and then rises to the surface. Estimate the change in its volume and radius.
▷ Solution, p. 379

21-a4 The ideal gas law is $PV = nkT$, and if everything is expressed in SI units, then the Boltzmann constant k has the numerical value $k = 1.38\times10^{-23}$ J/K. You may also have seen a version of the equation in which n is expressed in units of moles rather than molecules, and the constant of proportionality then has a different value. The following are three other ways in which we could imagine making a different version of the equation. In each case, explain whether this makes sense, and if so, find the value of the constant of proportionality that would replace the SI value of k.
(a) Temperature is measured in degrees Celsius.
(b) The amount of gas is measured in kilograms.
(c) The pressure and volume are measured in the centimeter-gram-second (cgs) system, where, for example, the unit of force is 1 $\mathrm{g\cdot cm/s^2}$, and all other units are based on the centimeter, gram, and second, rather than the SI base units of meters, kilograms, and seconds.

21-d1 (a) A helium atom contains 2 protons, 2 electrons, and 2 neutrons. Find the mass of a helium atom. $\surd$
(b) Find the number of atoms in 1.0 kg of helium. $\surd$
(c) Helium gas is monoatomic. Find the amount of heat needed to raise the temperature of 1.0 kg of helium by 1.0 degree C. (This is known as helium's heat capacity at constant volume.) $\surd$

21-d2 A sample of gas is enclosed in a sealed chamber. The gas consists of molecules, which are then split in half through some process such as exposure to ultraviolet light, or passing an electric spark through the gas. The gas returns to the same temperature as the surrounding room, but the molecules remain split apart, at least for some amount of time. (To achieve these conditions, we would need an extremely dilute gas. Otherwise the recombination of the molecules would be faster than the cooling down to the same temperature as the room.) How does its pressure now compare with its pressure before the molecules were split?

21-d3 Most of the atoms in the universe are in the form of gas that is not part of any star or galaxy: the intergalactic medium (IGM). The IGM consists of about 10^{-5} atoms per cubic centimeter, with a typical temperature of about 10^3 K. These are, in some sense, the density and temperature of the universe (not counting light, or the exotic particles known as "dark matter"). Calculate the pressure of the universe (or, speaking more carefully, the typical pressure due to the IGM). $\surd$

21-d4 (a) Determine the ratio between the escape velocities from the surfaces of the earth and the moon. $\surd$
(b) The temperature during the lunar daytime

gets up to about 130°C. In the extremely thin (almost nonexistent) lunar atmosphere, estimate how the typical velocity of a molecule would compare with that of the same type of molecule in the earth's atmosphere. Assume that the earth's atmosphere has a temperature of 0°C. √

(c) Suppose you were to go to the moon and release some fluorocarbon gas, with molecular formula C_nF_{2n+2}. Estimate what is the smallest fluorocarbon molecule (lowest n) whose typical velocity would be lower than that of an N_2 molecule on earth in proportion to the moon's lower escape velocity. The moon would be able to retain an atmosphere made of these molecules. √

21-d5 Our sun is powered by nuclear fusion reactions, and as a first step in these reactions, one proton must approach another proton to within a short enough range r. This is difficult to achieve, because the protons have electric charge $+e$ and therefore repel one another electrically. (It's a good thing that it's so difficult, because otherwise the sun would use up all of its fuel very rapidly and explode.) To make fusion possible, the protons must be moving fast enough to come within the required range. Even at the high temperatures present in the core of our sun, almost none of the protons are moving fast enough.

(a) For comparison, the early universe, soon after the Big Bang, had extremely high temperatures. Estimate the temperature T that would have been required so that protons with average energies could fuse. State your result in terms of r, the mass m of the proton, and universal constants.

(b) Show that the units of your answer to part a make sense.

(c) Evaluate your result from part a numerically, using $r = 10^{-15}$ m and $m = 1.7 \times 10^{-27}$ kg. As a check, you should find that this is much hotter than the sun's core temperature of $\sim 10^7$ K.

▷ Solution, p. 380

21-d6 The sun is mainly a mixture of hydrogen and helium, some of which is ionized. As a simplified model, let's pretend that it's made purely out of neutral, monatomic hydrogen, and that the whole mass of the sun is in thermal equilibrium. Given its mass, it would then contain 1.2×10^{57} atoms. It generates energy from nuclear reactions at a rate of 3.8×10^{26} W, and it is in a state of equilibrium in which this amount of energy is radiated off into space as light. Suppose that its ability to radiate light were somehow blocked. Find the rate at which its temperature would increase. √

21-d7 In metals, some electrons, called conduction electrons, are free to move around, rather than being bound to one atom. Classical physics gives an adequate description of many of their properties. Consider a metal at temperature T, and let m be the mass of the electron. Find expressions for (a) the average kinetic energy of a conduction electron, and (b) the average square of its velocity, $\overline{v^2}$. (It would not be of much interest to find $\bar{v}$, which is just zero.) Numerically, $\sqrt{\overline{v^2}}$, called the root-mean-square velocity, comes out to be surprisingly large — about two orders of magnitude greater than the normal thermal velocities we find for atoms in a gas. Why?

Remark: From this analysis, one would think that the conduction electrons would contribute greatly to the heat capacities of metals. In fact they do not contribute very much in most cases; if they did, Dulong and Petit's observations would not have come out as described in the text. The resolution of this contradiction was only eventually worked out by Sommerfeld in 1933, and involves the fact that electrons obey the Pauli exclusion principle. √

21-d8 Most of the mass of an atom comes from its protons and neutrons. The mass of a proton is approximately the same as the mass of a neutron. The planet Jupiter is made mostly of hydrogen molecules. A normal hydrogen molecule (H_2) contains two protons (one in each atom), and no neutrons. A small percentage of the hydrogen in the universe is in a form in which the nucleus contains both a proton and a neutron; this is called deuterium, often notated D. Since deuterium isn't very abundant, the most common thing to happen to a deuterium atom in Jupiter's atmosphere is that it would find itself in a molecule whose other atom was a normal hydrogen atom. The resulting molecule could be described as a DH. A DH molecule contains a total of two protons and one neutron. Compare the typical speed of a DH molecule in Jupiter's atmosphere with the typical speed of an H_2. Give a quantitative comparison, and notate it so that it's clear which is the higher speed.

21-g1 Object A is a brick. Object B is half of a similar brick. If A is heated, we have $\Delta S = Q/T$. Show that if this equation is valid for A, then it is also valid for B.

▷ Solution, p. 380

21-g2 You use a spoon at room temperature, 22°C, to mix your coffee, which is at 80°C. During this brief period of thermal contact, 1.3 J of heat is transferred from the coffee to the spoon. Find the total change in the entropy of the universe.

√

21-g3 Refrigerators, air conditioners, and heat pumps are heat engines that work in reverse. You put in mechanical work, and the effect is to take heat out of a cooler reservoir and deposit heat in a warmer one: $Q_L + W = Q_H$. As with the heat engines discussed previously, the efficiency is defined as the energy transfer you want (Q_L for a refrigerator or air conditioner, Q_H for a heat pump) divided by the energy transfer you pay for (W).

Efficiencies are supposed to be unitless, but the efficiency of an air conditioner is normally given in terms of an EER rating (or a more complex version called an SEER). The EER is defined as Q_L/W, but expressed in the barbaric units of of Btu/watt-hour. A typical EER rating for a residential air conditioner is about 10 Btu/watt-hour, corresponding to an efficiency of about 3. The standard temperatures used for testing an air conditioner's efficiency are 80°F (27°C) inside and 95°F (35°C) outside.

(a) What would be the EER rating of a reversed Carnot engine used as an air conditioner? √

(b) If you ran a 3-kW residential air conditioner, with an efficiency of 3, for one hour, what would be the effect on the total entropy of the universe? Is your answer consistent with the second law of thermodynamics?

√

21-g4 Even when resting, the human body needs to do a certain amount of mechanical work to keep the heart beating. This quantity is difficult to define and measure with high precision, and also depends on the individual and her level of activity, but it's estimated to be about 1 to 5 watts. Suppose we consider the human body as nothing more than a pump. A person who is just lying in bed all day needs about 1000 kcal/day worth of food to stay alive. (a) Estimate the person's thermodynamic efficiency as a pump, and (b) compare with the maximum possible efficiency imposed by the laws of thermodynamics for a heat engine operating across the difference between a body temperature of 37°C and an ambient temperature of 22°C. (c) Interpret your answer.

21-g5 (a) Consider a *one-dimensional* ideal gas consisting of n material particles, at temperature T. Trace back through the logic of the equipartition theorem on p. **??** to determine the total energy.

(b) Explain why it should matter how many dimensions there are.

(c) Gases that we encounter in everyday life are made of atoms, but there are gases made out of other things. For example, soon after the big bang, there was a period when the universe was very hot and dominated by light rather than matter. A particle of light is called a photon, so the early universe was a "photon gas." For simplicity, consider a photon gas in one dimension. Photons are massless, and we will see in ch. **??** on relativity that for a massless particle, the energy is related to the momentum by $E = pc$, where c is the speed of light. (Note that $p = mv$ does *not* hold for a photon.) Again, trace back through the logic of equipartition on p. **??**. Does the photon gas have the same heat capacity as the one you found in part a?

21-g6 Example **??** on page **??** suggests analyzing the resonance of a violin at 300 Hz as a Helmholtz resonance. However, we might expect the equation for the frequency of a Helmholtz resonator to be a rather crude approximation here, since the f-holes are not long tubes, but slits cut through the face of the instrument, which is only about 2.5 mm thick. (a) Estimate the frequency that way anyway, for a violin with a volume of about 1.6 liters, and f-holes with a total area of 10 cm^2. (b) A common rule of thumb is that at an open end of an air column, such as the neck of a real Helmholtz resonator, some air beyond the mouth also vibrates as if it was inside the tube, and that this effect can be taken into account by adding 0.4 times the diameter of the tube for each open end (i.e., 0.8 times the diameter when both ends are open). Applying this to the violin's f-holes results in a huge change in L, since the ~ 7 mm width of the f-hole is considerably greater than the thickness of the wood. Try it, and see if the result is a better approximation to the observed frequency of the resonance.

22 Optics

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

22.1 Geometric optics

- Crowell, *Simple Nature*, lightandmatter.com, sec. 12.1-4
- OpenStax University Physics, openstax.org, v. 3, ch. 1-2

22.2 Wave optics

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 12.5
- OpenStax University Physics, openstax.org, v. 3, ch. 3-4

Problems

22-a1 Draw a ray diagram showing why a small light source (a candle, say) produces sharper shadows than a large one (e.g., a long fluorescent bulb).

22-a2 A Global Positioning System (GPS) receiver is a device that lets you figure out where you are by receiving timed radio signals from satellites. It works by measuring the travel time for the signals, which is related to the distance between you and the satellite. By finding the ranges to several different satellites in this way, it can pin down your location in three dimensions to within a few meters. How accurate does the measurement of the time delay have to be to determine your position to this accuracy?

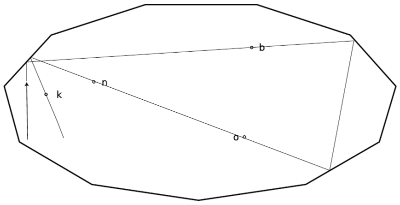
22-a3 Estimate the frequency of an electromagnetic wave whose wavelength is similar in size to an atom (about a nm). In what part of the electromagnetic spectrum would such a wave lie (infrared, gamma-rays, . . .)?

22-a4 The Stealth Bomber is designed with flat, smooth surfaces. Why would this make it difficult to detect using radar?

▷ Solution, p. 381

22-a5 The natives of planet Wumpus play pool using light rays on an eleven-sided table with mirrors for bumpers, shown in the figure on the next page. Trace this shot accurately with a ruler to reveal the hidden message. To get good enough accuracy, you'll need to photocopy the page (or download the book and print the page) and construct each reflection using a protractor.

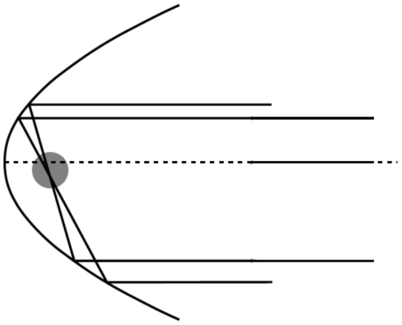
▷ Solution, p. 382



Problem 22-a5.

22-a6 The figure on the next page shows a curved (parabolic) mirror, with three parallel light rays coming toward it. One ray is approaching along the mirror's center line. (a) Continue the light rays until they are about to undergo their second reflection. To get good enough accuracy, you'll need to photocopy the page (or download the book and print the page) and draw in the normal at each place where a ray is reflected. What do you notice? (b) Make up an example of a practical use for this device. (c) How could you use this mirror with a small lightbulb to produce a parallel beam of light rays going off to the right?

▷ Solution, p. 382



Problem 22-a6.

22-a7 Suppose we have a polygonal room whose walls are mirrors, and there a pointlike light source in the room. In most such examples, every point in the room ends up being illuminated by the light source after some finite number of reflections. A difficult mathematical question, first posed in the middle of the last century, is whether it is ever possible to have an example in which the whole room is *not* illuminated. (Rays are assumed to be absorbed if they strike exactly at a vertex of the polygon, or if they pass exactly through the plane of a mirror.)

The problem was finally solved in 1995 by G.W. Tokarsky, who found an example of a room that was not illuminable from a certain point. Figure 22-a7 shows a slightly simpler example found two years later by D. Castro. If a light source is placed at either of the locations shown with dots, the other dot remains unilluminated, although every other point is lit up. It is not straightforward to prove rigorously that Castro's solution has this property. However, the plausibility of the solution can be demonstrated as follows.

Suppose the light source is placed at the right-hand dot. Locate all the images formed by single reflections. Note that they form a regular pattern. Convince yourself that none of these images illuminates the left-hand dot. Because of the regular pattern, it becomes plausible that even if we form images of images, images of images of images, etc., none of them will ever illuminate the other dot.

There are various other versions of the problem, some of which remain unsolved. The book by Klee and Wagon gives a good introduction to the topic, although it predates Tokarsky and Castro's work.

References:

G.W. Tokarsky, "Polygonal Rooms Not Illuminable from Every Point." Amer. Math. Monthly 102, 867-879, 1995.

D. Castro, "Corrections." Quantum 7, 42, Jan. 1997.

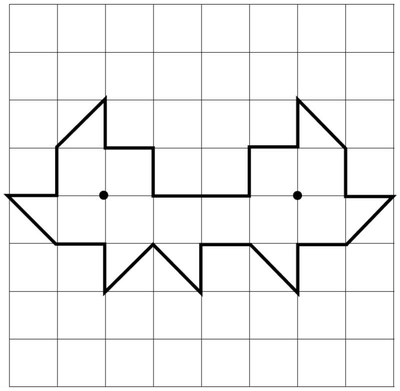
V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory.* Mathematical Association of America, 1991.

22-d1 A man is walking at 1.0 m/s directly towards a flat mirror. At what speed is his separation from his image decreasing?

√

22-d2 If a mirror on a wall is only big enough for you to see yourself from your head down to your waist, can you see your entire body by backing up? Test this experimentally and come up with an explanation for your observations, including a ray diagram.

Note that when you do the experiment, it's easy to confuse yourself if the mirror is even a tiny bit off of vertical. One way to check yourself is to artificially lower the top of the mirror by putting a piece of tape or a post-it note where it blocks your view of the top of your head. You can then check whether you are able to see more of yourself both above *and* below by backing up.



Problem 22-a7.

22-g1 Draw a ray diagram for the formation of an image by a convex mirror. (a) How does the image's distance from the mirror compare with the actual object's distance from the mirror? From this comparison, determine whether the magnification is greater than or less than one. (b) Is the image real, or virtual? Could this mirror ever make the other type of image?

22-g2 As discussed in question 22-g1, there are two types of curved mirrors, concave and convex. Make a list of all the possible combinations of types of images (virtual or real) with types of mirrors (concave and convex). (Not all of the four combinations are physically possible.) Now for each one, use ray diagrams to determine whether increasing the distance of the object from the mirror leads to an increase or a decrease in the distance of the image from the mirror.

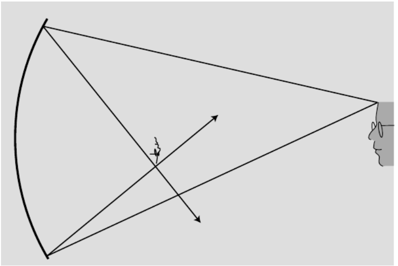
Draw BIG ray diagrams! Each diagram should use up about half a page of paper.

Some tips: To draw a ray diagram, you need two rays. For one of these, pick the ray that comes straight along the mirror's axis, since its reflection is easy to draw. After you draw the two rays and locate the image for the original object position, pick a new object position that results in the same type of image, and start a new ray diagram, in a different color of pen, right on top of the first one. For the two new rays, pick the ones that just happen to hit the mirror at the same two places; this makes it much easier to get the result right without depending on extreme accuracy in your ability to draw the reflected rays.

22-g3 If the user of an astronomical telescope moves her head closer to or farther away from the image she is looking at, does the magnification change? Does the angular magnification change? Explain. (For simplicity, assume that no eyepiece is being used.)

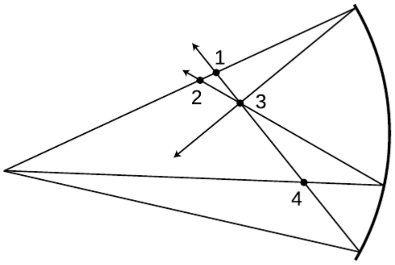
▷ Solution, p. 384

22-g4 In the figure, only the image of my forehead was located by drawing rays. Either photocopy the figure or download the book and print out the relevant page. On this copy of the figure, make a new set of rays coming from my chin, and locate its image. To make it easier to judge the angles accurately, draw rays from the chin that happen to hit the mirror at the same points where the two rays from the forehead were shown hitting it. By comparing the locations of the chin's image and the forehead's image, verify that the image is actually upside-down, as shown in the original figure.



Problem 22-g4.

22-g5 The figure shows four points where rays cross. Of these, which are image points? Explain.



Problem 22-g5.

22-g6 Here's a game my kids like to play. I sit next to a sunny window, and the sun reflects from the glass on my watch, making a disk of light on the wall or floor, which they pretend to chase as I move it around. Is the spot a disk because that's the shape of the sun, or because it's the shape of my watch? In other words, would a square watch make a square spot, or do we just have a circular image of the circular sun, which will be circular no matter what?

22-g7 Apply the equation $M = d_i/d_o$ to the case of a flat mirror.

▷ Solution, p. 384

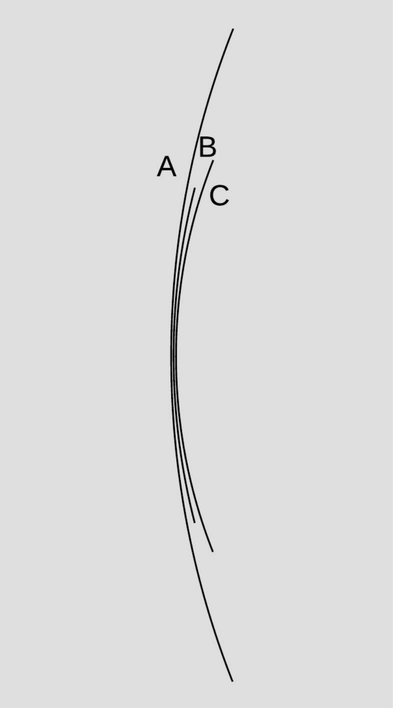
22-g8 Find the focal length of the mirror in problem 22-a6 .

✓

22-g9 Rank the focal lengths of the mirrors in the figure, from shortest to longest. Explain.

22-g10 (a) A converging mirror with a focal length of 20 cm is used to create an image, using an object at a distance of 10 cm. Is the image real, or is it virtual? (b) How about $f = 20$ cm and $d_o = 30$ cm? (c) What if it was a *diverging* mirror with $f = 20$ cm and $d_o = 10$ cm? (d) A diverging mirror with $f = 20$ cm and $d_o = 30$ cm?

▷ Solution, p. 384



Problem 22-g9.

22-g11 (a) Make up a numerical example of a virtual image formed by a converging mirror with a certain focal length, and determine the magnification. (You will need the result of problem **??**.) Make sure to choose values of d_o and f that would actually produce a virtual image, not a real one. Now change the location of the object *a little bit* and redetermine the magnification, showing that it changes. At my local department store, the cosmetics department sells hand mirrors advertised as giving a magnification of 5 times. How would you interpret this?

(b) Suppose a Newtonian telescope is being used for astronomical observing. Assume for simplicity that no eyepiece is used, and assume a value for the focal length of the mirror that would be reasonable for an amateur instrument that is to fit in a closet. Is the angular magnification different for objects at different distances? For example, you could consider two planets, one of which is twice as far as the other.

▷ Solution, p. 384

22-g12 (a) Find a case where the magnification of a curved mirror is infinite. Is the *angular* magnification infinite from any realistic viewing position? (b) Explain why an arbitrarily large magnification can't be achieved by having a sufficiently small value of d_o.

▷ Solution, p. 385

22-g13 A concave surface that reflects sound waves can act just like a converging mirror. Suppose that, standing near such a surface, you are able to find a point where you can place your head so that your own whispers are focused back on your head, so that they sound loud to you. Given your distance to the surface, what is the surface's focal length?

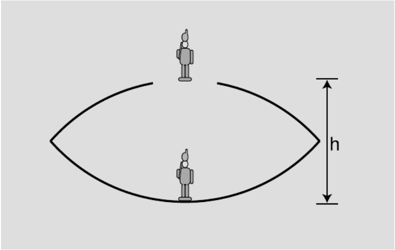
✓

22-g14 The figure shows a device for constructing a realistic optical illusion. Two mirrors of equal focal length are put against each other with their silvered surfaces facing inward. A small object placed in the bottom of the cavity will have its image projected in the air above. The way it works is that the top mirror produces a virtual image, and the bottom mirror then creates a real image of the virtual image. (a) Show that if the image is to be positioned as shown, at the mouth of the cavity, then the focal length of the mirrors is related to the dimension h via the equation

$$\frac{1}{f}=\frac{1}{h}+\frac{1}{h+\left(\frac{1}{h}-\frac{1}{f}\right)^{-1}}.$$

(b) Restate the equation in terms of a single variable $x=h/f$, and show that there are two solutions for x. Which solution is physically consistent with the assumptions of the calculation?

22-g15 (a) A converging mirror is being used to create a virtual image. What is the range of possible magnifications? (b) Do the same for the other types of images that can be formed by curved mirrors (both converging and diverging).



Problem 22-g14.

22-g16 A diverging mirror of focal length f is fixed, and faces down. An object is dropped from the surface of the mirror, and falls away from it with acceleration g. The goal of the problem is to find the maximum velocity of the image.
(a) Describe the motion of the image verbally, and explain why we should expect there to be a maximum velocity.
(b) Use arguments based on units to determine the form of the solution, up to an unknown unitless multiplicative constant.
(c) Complete the solution by determining the unitless constant.
$\checkmark$

22-g17 A converging mirror has focal length f. An object is located at a distance $(1+\epsilon)f$ from the mirror, where ϵ is small. Find the distance of the image from the mirror, simplifying your result as much as possible by using the assumption that ϵ is small.

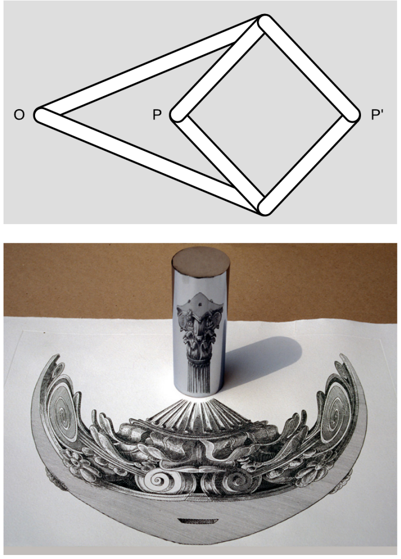
22-g18 A mechanical linkage is a device that changes one type of motion into another. The most familiar example occurs in a gasoline car's engine, where a connecting rod changes the linear motion of the piston into circular motion of the crankshaft. The top panel of the figure shows a mechanical linkage invented by Peaucellier in 1864, and independently by Lipkin around the same time. It consists of six rods joined by hinges, the four short ones forming a rhombus. Point O is fixed in space, but the apparatus is free to rotate about O. Motion at P is transformed into a different motion at P$'$ (or vice versa).

Geometrically, the linkage is a mechanical implementation of the ancient problem of inversion in a circle. Considering the case in which the rhombus is folded flat, let the k be the distance from O to the point where P and P$'$ coincide. Form the circle of radius k with its center at O. As P and P$'$ move in and out, points on the inside of the circle are always mapped to points on its outside, such that $rr'=k^2$. That is, the linkage is a type of analog computer that exactly solves the problem of finding the inverse of a number r. Inversion in a circle has many remarkable geometrical properties, discussed in H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1961. If a pen is inserted through a hole at P, and P$'$ is traced over a geometrical figure, the Peaucellier linkage can be used to draw a kind of image of the figure.

A related problem is the construction of pictures, like the one in the bottom panel of the figure, called anamorphs. The drawing of the column on the paper is highly distorted, but when the reflecting cylinder is placed in the correct spot on top of the page, an undistorted image is produced inside the cylinder. (Wide-format movie technologies such as Cinemascope are based on similar principles.)

Show that the Peaucellier linkage does *not* convert correctly between an image and its anamorph, and design a modified version of the linkage that does. Some knowledge of analytic geometry will be helpful.

22-j1 Diamond has an index of refraction of 2.42, and part of the reason diamonds sparkle is that this encourages a light ray to undergo many total internal reflections before it emerges. (a) Calculate the critical angle at which total internal reflection occurs in diamond. (b) Explain the interpretation of your result: Is it measured from the normal, or from the surface? Is it a minimum angle for total internal reflection, or is it a maximum? How would the critical angle have been different for a substance such as glass or plastic, with a lower index of refraction?
$\checkmark$



Problem 22-g18.

22-j2 Under ordinary conditions, gases have indices of refraction only a little greater than that of vacuum, i.e., $n = 1 + \epsilon$, where ϵ is some small number. Suppose that a ray crosses a boundary between a region of vacuum and a region in which the index of refraction is $1 + \epsilon$. Find the maximum angle by which such a ray can ever be deflected, in the limit of small ϵ. $\surd$

22-j3 In an experiment to measure the unknown index of refraction n of a liquid, you send a laser beam from air into a tank filled with the liquid. Let ϕ be the angle of the beam relative to the normal while in the air, and let θ be the angle in the liquid. You can set ϕ to any value you like by aiming the laser from an appropriate direction, and you measure θ as a result. We wish to plan such an experiment so as to minimize the error dn in the result of the experiment, for a fixed error dθ in the measurement of the angle in the liquid. We assume that there is no significant contribution to the error from uncertainty in the index of refraction of air (which is very close to 1) or from the angle ϕ. Find dn in terms of dθ, and determine the optimal conditions.

▷ Solution, p. 387

22-j4 The intensity of a beam of light is defined as the power per unit area incident on a perpendicular surface. Suppose that a beam of light in a medium with index of refraction n reaches the surface of the medium, with air on the outside. Its incident angle with respect to the normal is θ. (All angles are in radians.) Only a fraction f of the energy is transmitted, the rest being reflected. Because of this, we might expect that the transmitted ray would always be less intense than the incident one. But because the transmitted ray is refracted, it becomes narrower, causing an additional change in intensity by a factor $g > 1$. The product of these factors $I = fg$ can be greater than one. The purpose of this problem is to estimate the maximum amount of intensification.

We will use the small-angle approximation $\theta \ll 1$ freely, in order to make the math tractable. In our previous studies of waves, we have only studied the factor f in the one-dimensional case where $\theta = 0$. The generalization to $\theta \neq 0$ is rather complicated and depends on the polarization, but for unpolarized light, we can use Schlick's approximation,

$$f(\theta) = f(0)(1 - \cos\theta)^5,$$

where the value of f at $\theta = 0$ is found as in problem 13-d7 on p. 152.

(a) Using small-angle approximations, obtain an expression for g of the form $g \approx 1 + P\theta^2$, and find the constant P.

(b) Find an expression for I that includes the two leading-order terms in θ. We will call this expression I_2. Obtain a simple expression for the angle at which I_2 is maximized. As a check on your work, you should find that for $n = 1.3$, $\theta = 63°$. (Trial-and-error maximization of I gives 60°.)

(c) Find an expression for the maximum value of I_2. You should find that for $n = 1.3$, the maximum intensification is 31%.

22-j5 Prove that the principle of least time leads to Snell's law.

22-m1 When you take pictures with a camera, the distance between the lens and the film or chip has to be adjusted, depending on the distance at which you want to focus. This is done by moving the lens. If you want to change your focus so that you can take a picture of something farther away, which way do you have to move the lens? Explain using ray diagrams. [Based on a problem by Eric Mazur.]

22-m2 A diverging lens with $f = 30$ cm is forming an image of an object. The object is 60 cm from the lens. (a) Draw a ray diagram, and label the image on the diagram. (b) Is the image real, or virtual? Explain. (c) Calculate the position of the image.

22-m3 Nearsighted people wear glasses whose lenses are diverging. (a) Draw a ray diagram. For simplicity pretend that there is no eye behind the glasses. (b) If the focal length of the lens is 50.0 cm, and the person is looking at an object at a distance of 80.0 cm, locate the image. ✓

22-m4 Suppose a converging lens is constructed of a type of plastic whose index of refraction is less than that of water. How will the lens's behavior be different if it is placed underwater?

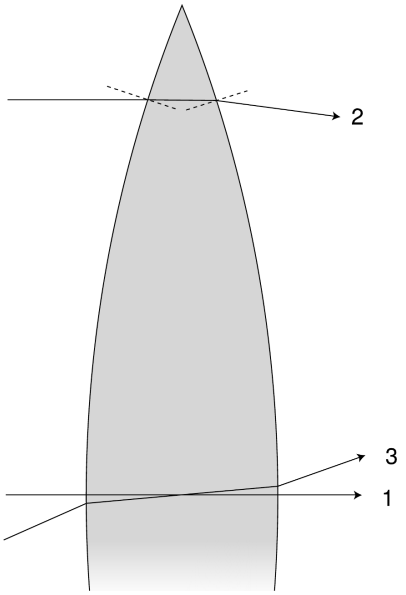
▷ Solution, p. 388

22-m5 There are two main types of telescopes, refracting (using a lens) and reflecting (using a mirror). (Some telescopes use a mixture of the two types of elements: the light first encounters a large curved mirror, and then goes through an eyepiece that is a lens. To keep things simple, assume no eyepiece is used.) What implications would the color-dependence of focal length have for the relative merits of the two types of telescopes? Describe the case where an image is formed of a white star. You may find it helpful to draw a ray diagram.

22-m6 Based on Snell's law, explain why rays of light passing through the edges of a converging lens are bent more than rays passing through parts closer to the center. It might seem like it should be the other way around, since the rays at the edge pass through less glass — shouldn't they be affected less? In your answer:

- Include a ray diagram showing a huge, full-page, close-up view of the relevant part of the lens.
- Make use of the fact that the front and back surfaces aren't always parallel; a lens in which the front and back surfaces *are* always parallel doesn't focus light at all, so if your explanation doesn't make use of this fact, your argument must be incorrect.
- Make sure your argument still works even if the rays don't come in parallel to the axis or from a point on the axis.

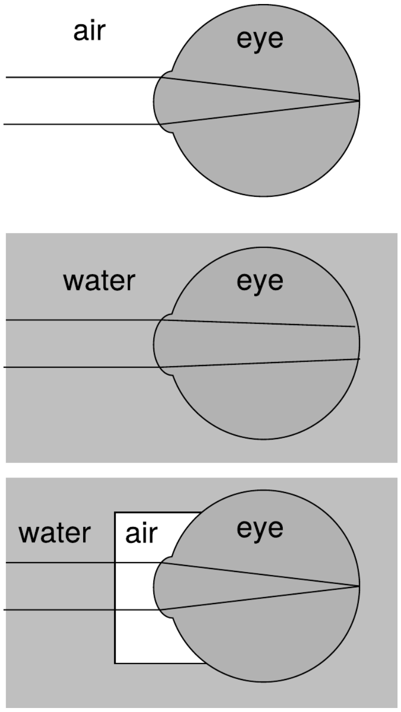
▷ Solution, p. 388



Problem 22-m6.

22-m7 When swimming underwater, why is your vision made much clearer by wearing goggles with flat pieces of glass that trap air behind them? [Hint: You can simplify your reasoning by considering the special case where you are looking at an object far away, and along the optic axis of the eye.]

▷ Solution, p. 389



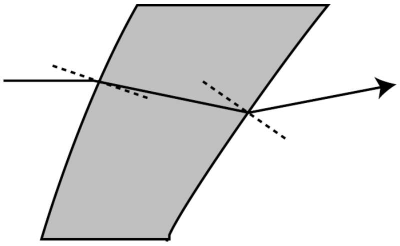
Problem 22-m7.

22-m8 The figure shows four lenses. Lens 1 has two spherical surfaces. Lens 2 is the same as lens 1 but turned around. Lens 3 is made by cutting through lens 1 and turning the bottom around. Lens 4 is made by cutting a central circle out of lens 1 and recessing it.

(a) A parallel beam of light enters lens 1 from the left, parallel to its axis. Reasoning based on Snell's law, will the beam emerging from the lens be bent inward, or outward, or will it remain parallel to the axis? Explain your reasoning. As part of your answer, make a huge drawing of one small part of the lens, and apply Snell's law at both interfaces. Recall that rays are bent more if they come to the interface at a larger angle with respect to the normal.

(b) What will happen with lenses 2, 3, and 4? Explain. Drawings are not necessary.

▷ Solution, p. 389



Problem 22-m8.

22-m9 The drawing shows the anatomy of the human eye, at twice life size. Find the radius of curvature of the outer surface of the cornea by measurements on the figure, and then derive the focal length of the air-cornea interface, where almost all the focusing of light occurs. You will need to use physical reasoning to modify the lensmaker's equation for the case where there is only a single refracting surface. Assume that the index of refraction of the cornea is essentially that of water.

√

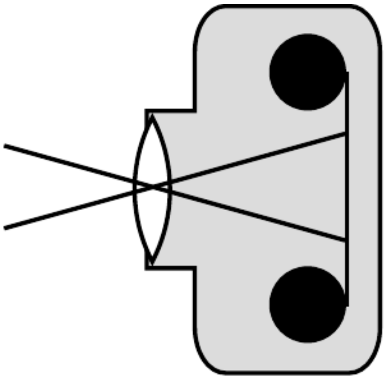
22-m10 (a) Light is being reflected diffusely from an object 1.000 m underwater. The light that comes up to the surface is refracted at the water-air interface. If the refracted rays all appear to come from the same point, then there will be a virtual image of the object in the water, above the object's actual position, which will be visible to an observer above the water. Consider three rays, A, B and C, whose angles in the water with respect to the normal are $\theta_i = 0.000°$, $1.000°$ and $20.000°$ respectively. Find the depth of the point at which the refracted parts of A and B appear to have intersected, and do the same for A and C. Show that the intersections are at nearly the same depth, but not quite. [Check: The difference in depth should be about 4 cm.]

(b) Since all the refracted rays do not quite appear to have come from the same point, this is technically not a virtual image. In practical terms, what effect would this have on what you see?

(c) In the case where the angles are all small, use algebra and trig to show that the refracted rays do appear to come from the same point, and find an equation for the depth of the virtual image. Do not put in any numerical values for the angles or for the indices of refraction — just keep them as symbols. You will need the approximation $\sin\theta \approx \tan\theta \approx \theta$, which is valid for small angles measured in radians.

22-m11 Two standard focal lengths for camera lenses are 50 mm (standard) and 28 mm (wide-angle). To see how the focal lengths relate to the angular size of the field of view, it is helpful to visualize things as represented in the figure. Instead of showing many rays coming from the same point on the same object, as we normally do, the figure shows two rays from two different objects. Although the lens will intercept infinitely many rays from each of these points, we have shown only the ones that pass through the center of the lens, so that they suffer no angular deflection. (Any angular deflection at the front surface of the lens is canceled by an opposite deflection at the back, since the front and back surfaces are parallel at the lens's center.) What is special about these two rays is that they are aimed at the edges of one 35-mm-wide frame of film; that is, they show the limits of the field of view. Throughout this problem, we assume that d_o is much greater than d_i. (a) Compute the angular width of the camera's field of view when these two lenses are used. (b) Use small-angle approximations to find a simplified equation for the angular width of the field of view, θ, in terms of the focal length, f, and the width of the film, w. Your equation should not have any trig functions in it. Compare the results of this approximation with your answers from part a. (c) Suppose that we are holding constant the aperture (amount of surface area of the lens being used to collect light). When switching from a 50-mm lens to a 28-mm lens, how many times longer or shorter must the exposure be in order to make a properly developed picture, i.e., one that is not under- or overexposed? [Based on a problem by Arnold Arons.]

▷ Solution, p. 390



Problem 22-m11.

22-m12 A nearsighted person is one whose eyes focus light too strongly, and who is therefore unable to relax the lens inside her eye sufficiently to form an image on her retina of an object that is too far away.

(a) Draw a ray diagram showing what happens when the person tries, with uncorrected vision, to focus at infinity.

(b) What type of lenses do her glasses have? Explain.

(c) Draw a ray diagram showing what happens when she wears glasses. Locate both the image formed by the glasses and the final image.

(d) Suppose she sometimes uses contact lenses instead of her glasses. Does the focal length of her contacts have to be less than, equal to, or greater than that of her glasses? Explain.

22-m13 Fred's eyes are able to focus on things as close as 5.0 cm. Fred holds a magnifying glass with a focal length of 3.0 cm at a height of 2.0 cm above a flatworm. (a) Locate the image, and find the magnification. (b) Without the magnifying glass, from what distance would Fred want to view the flatworm to see its details as well as possible? With the magnifying glass? (c) Compute the angular magnification.

22-m14 The figure shows a lens with surfaces that are curved, but whose thickness is constant along any horizontal line. Use the lensmaker's equation to prove that this "lens" is not really a lens at all.

▷ Solution, p. 391

22-m15 Zahra likes to play practical jokes on the friends she goes hiking with. One night, by a blazing camp fire, she stealthily uses a lens of focal length f to gather light from the fire and make a hot spot on Becky's neck. (a) Draw a ray diagram and set up the equation for the image location, inferring the correct plus and minus signs from the diagram. (b) Let A be the distance from the lens to the campfire, and B the distance from the lens to Becky's neck. Consider the following nine possibilities:

		B		
		$< f$	$= f$	$> f$
A	$< f$			
	$= f$			
	$> f$			

By reasoning about your equation from part a, determine which of these are possible and which are not.

▷ Solution, p. 391

22-m16 It would be annoying if your eyeglasses produced a magnified or reduced image. Prove that when the eye is very close to a lens, and the lens produces a virtual image, the angular magnification is always approximately equal to 1 (regardless of whether the lens is diverging or converging).

22-q1 The figure shows a diffraction pattern made by a double slit, along with an image of a meter stick to show the scale. Sketch the diffraction pattern from the figure on your paper. Now consider the four variables in the equation $\lambda/d = \sin\theta/m$. Which of these are the same for all five fringes, and which are different for each fringe? Which variable would you naturally use in order to label which fringe was which? Label the fringes on your sketch using the values of that variable.

22-q2 Match gratings A-C with the diffraction patterns 1-3 that they produce. Explain.

22-q3 The figure below shows two diffraction patterns. The top one was made with yellow light, and the bottom one with red. Could the slits used to make the two patterns have been the same?

22-q4 The figure on p. **??** shows a diffraction pattern made by a double slit, along with an image of a meter stick to show the scale. The slits were 146 cm away from the screen on which the diffraction pattern was projected. The spacing of the slits was 0.050 mm. What was the wavelength of the light?

√

22-q5 Why would blue or violet light be the best for microscopy?

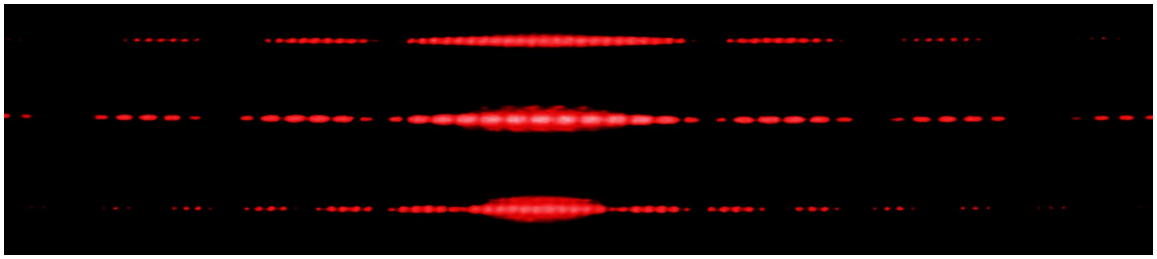
▷ Solution, p. 392

22-q6 The figure below shows two diffraction patterns, both made with the same wavelength of red light. (a) What type of slits made the patterns? Is it a single slit, double slits, or something else? Explain. (b) Compare the dimensions of the slits used to make the top and bottom pattern. Give a numerical ratio, and state which way the ratio is, i.e., which slit pattern was the larger one. Explain.

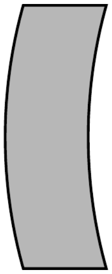
▷ Solution, p. 392

22-q7 When white light passes through a diffraction grating, what is the smallest value of m for which the visible spectrum of order m overlaps the next one, of order $m + 1$? (The visible spectrum runs from about 400 nm to about 700 nm.)

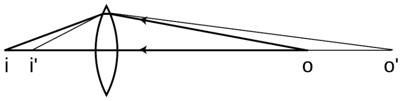
22-q8 For star images such as the ones in figure **??**, estimate the angular width of the diffraction spot due to diffraction at the mouth of the telescope. Assume a telescope with a diameter of 10 meters (the largest currently in existence),



Problem 22-q9.



Problem 22-m14.



Problem 22-m15.

and light with a wavelength in the middle of the visible range. Compare with the actual angular size of a star of diameter 10^9 m seen from a distance of 10^{17} m. What does this tell you?

▷ Solution, p. 392

22-q9 The figure below shows three diffraction patterns. All were made under identical conditions, except that a different set of double slits was used for each one. The slits used to make the top pattern had a center-to-center separation $d = 0.50$ mm, and each slit was $w = 0.04$ mm wide. (a) Determine d and w for the slits used to make the pattern in the middle. (b) Do the same for the slits used to make the bottom pattern.

▷ Solution, p. 392

22-q10 The beam of a laser passes through a diffraction grating, fans out, and illuminates a wall that is perpendicular to the original beam, lying at a distance of 2.0 m from the grating. The beam is produced by a helium-neon laser, and has a wavelength of 694.3 nm. The grating has 2000 lines per centimeter. (a) What is the distance on the wall between the central maximum and the maxima immediately to its right and left? (b) How much does your answer change when you use the small-angle approximations $\theta \approx \sin\theta \approx \tan\theta$?

√

22-q11 Ultrasound, i.e., sound waves with frequencies too high to be audible, can be used for imaging fetuses in the womb or for breaking up kidney stones so that they can be eliminated by the body. Consider the latter application. Lenses can be built to focus sound waves, but because the wavelength of the sound is not all that small compared to the diameter of the lens, the sound will not be concentrated exactly at the geometrical focal point. Instead, a diffraction pattern will be created with an intense central spot surrounded by fainter rings. About 85% of the power is concentrated within the central spot. The angle of the first minimum (surrounding the central spot) is given by $\sin\theta = \lambda/b$, where b is the diameter of the lens. This is similar to the corresponding equation for a single slit, but with a factor of 1.22 in front which arises from the circular shape of the aperture. Let the distance from the lens to the patient's kidney stone be $L = 20$ cm. You will want $f > 20$ kHz, so that the sound is inaudible. Find values of b and f that would result in a usable design, where the central spot is small enough to lie within a kidney stone 1 cm in diameter.

22-q12 Under what circumstances could one get a mathematically undefined result by solving the double-slit diffraction equation for θ? Give a physical interpretation of what would actually be observed.

▷ Solution, p. 393

22-q13 When ultrasound is used for medical imaging, the frequency may be as high as 5-20 MHz. Another medical application of ultrasound is for therapeutic heating of tissues inside the body; here, the frequency is typically 1-3 MHz. What fundamental physical reasons could you suggest for the use of higher frequencies for imaging?

23 More about relativity

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

23.1 More about relativity

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 7.3
- OpenStax University Physics, openstax.org, v. 3, sec. 5.8-9

Problems

23-a1 (a) A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and undergoes beta decay (which you may want to review). The masses of the particles involved are as follows:

neutron	1.67495×10^{-27} kg
proton	1.67265×10^{-27} kg
electron	0.00091×10^{-27} kg
antineutrino	$< 10^{-35}$ kg

Find the energy released in the decay of a free neutron. √

(b) Neutrons and protons make up essentially all of the mass of the ordinary matter around us. We observe that the universe around us has no free neutrons, but lots of free protons (the nuclei of hydrogen, which is the element that 90% of the universe is made of). We find neutrons only inside nuclei along with other neutrons and protons, not on their own.

If there are processes that can convert neutrons into protons, we might imagine that there could also be proton-to-neutron conversions, and indeed such a process does occur sometimes in nuclei that contain both neutrons and protons: a proton can decay into a neutron, a positron, and a neutrino. A positron is a particle with the same properties as an electron, except that its electrical charge is positive. A neutrino, like an antineutrino, has negligible mass.

Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive, and you wouldn't exist!)

23-a2 (a) Find a relativistic equation for the velocity of an object in terms of its mass and momentum (eliminating γ). Use natural units (i.e., discard factors of c) throughout. √

(b) Show that your result is approximately the same as the nonrelativistic value, p/m, at low velocities.

(c) Show that very large momenta result in speeds close to the speed of light.

(d) Insert factors of c to make your result from part a usable in SI units. √

23-a3 (a) A charged particle is surrounded by a uniform electric field. Starting from rest, it is accelerated by the field to speed v after traveling a distance d. Now it is allowed to continue for a further distance $3d$, for a total displacement from the start of $4d$. What speed will it reach, assuming newtonian physics?

(b) Find the relativistic result for the case of $v = c/2$.

23-a4 Expand the equation $K = m(\gamma - 1)$ in a Taylor series, and find the first two nonvanishing terms. Explain why the vanishing terms are the ones that should vanish physically. Show that the first term is the nonrelativistic expression for kinetic energy.

23-a5 Consider the relativistic relation for momentum as a function of velocity (for a particle with nonzero mass). Expand this in a Taylor series, and find the first two nonvanishing terms. Explain why the vanishing terms are the ones that should vanish physically. Show that the first term is the newtonian expression.

23-a6 Expand the relativistic equation for the longitudinal Doppler shift of light $D(v)$ in a Taylor series, and find the first two nonvanishing terms. Show that these two terms agree with the nonrelativistic expression, so that any relativistic effect is of higher order in v.

23-a7 (a) Let L be the diameter of our galaxy. Suppose that a person in a spaceship of mass m wants to travel across the galaxy at constant speed, taking proper time τ. Find the kinetic energy of the spaceship. (b) Your friend is impatient, and wants to make the voyage in an hour. For $L = 10^5$ light years, estimate the energy in units of megatons of TNT (1 megaton=4×10^9 J).

24 Quantum physics

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

24.1 The nucleus, half-life, and probability

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 8.2, 13.1
- OpenStax University Physics, openstax.org, v. 3, ch. 10

24.2 Wave-particle duality

References:

- Crowell, *Simple Nature*, lightandmatter.com, sec. 13.2-3
- OpenStax University Physics, openstax.org, v. 3, ch. 6

Problems

24-a1 ^{241}Pu decays either by electron decay or by alpha decay. (A given ^{241}Pu nucleus may do either one; it's random.) What are the isotopes created as products of these two modes of decay?

24-a2 The nuclear process of beta decay by electron capture is $\mathrm{p} + \mathrm{e}^- \rightarrow \mathrm{n} + \nu$. One of the protons in the nucleus absorbs a proton.
(a) Show that charge is conserved in this reaction.
(b) Explain why electron capture doesn't occur in hydrogen atoms. (If it did, matter wouldn't exist!)

24-d1 If a radioactive substance has a half-life of one year, does this mean that it will be completely decayed after two years? Explain.

24-d2 (a) A nuclear physicist is studying a nuclear reaction caused in an accelerator experiment, with a beam of ions from the accelerator striking a thin metal foil and causing nuclear reactions when a nucleus from one of the beam ions happens to hit one of the nuclei in the target. After the experiment has been running for a few hours, a few billion radioactive atoms have been produced, embedded in the target. She does not know what nuclei are being produced, but she suspects they are an isotope of some heavy element such as Pb, Bi, Fr or U. Following one such experiment, she takes the target foil out of the accelerator, sticks it in front of a detector, measures the activity every 5 min, and makes a graph (figure). The isotopes she thinks may have been produced are:

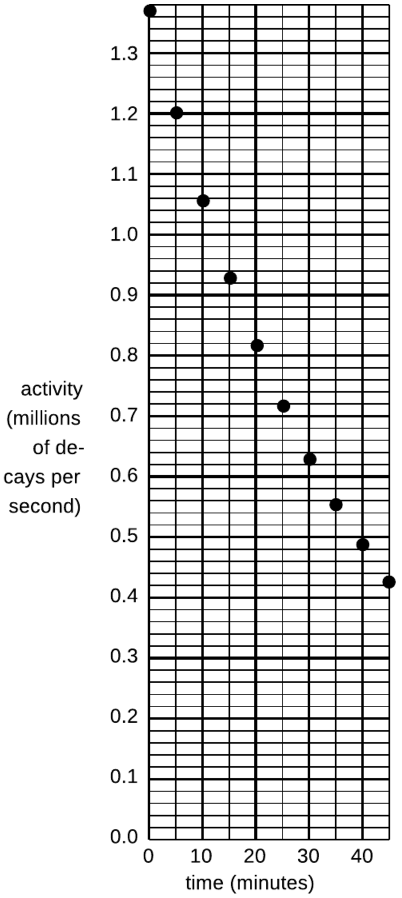
isotope	half-life (minutes)
^{211}Pb	36.1
^{214}Pb	26.8
^{214}Bi	19.7
^{223}Fr	21.8
^{239}U	23.5

Which one is it?
(b) Having decided that the original experimental conditions produced one specific isotope, she now tries using beams of ions traveling at several different speeds, which may cause different reactions. The following table gives the activity of the target 10, 20 and 30 minutes after the end of the experiment, for three different ion speeds.

	activity (millions of decays/s) after...		
	10 min	20 min	30 min
first ion speed	1.933	0.832	0.382
second ion speed	1.200	0.545	0.248
third ion speed	7.211	1.296	0.248

Since such a large number of decays is being counted, assume that the data are only inaccurate due to rounding off when writing down the table. Which are consistent with the production of a single isotope, and which imply that more than one isotope was being created?

24-d3 All helium on earth is from the decay of naturally occurring heavy radioactive elements such as uranium. Each alpha particle that is emitted ends up claiming two electrons, which makes it a helium atom. If the original ^{238}U atom is in solid rock (as opposed to the earth's molten regions), the He atoms are unable to diffuse out of the rock. This problem involves dating a rock using the known decay properties of uranium 238. Suppose a geologist finds a sample of hardened lava, melts it in a furnace, and finds that it contains 1230 mg of uranium and 2.3 mg of helium. ^{238}U decays by alpha emission, with a half-life of 4.5×10^9 years. The subsequent chain of alpha and electron (beta) decays involves much shorter half-lives, and terminates in the stable nucleus ^{206}Pb. Almost all natural uranium is ^{238}U, and the chemical composition of this rock indicates that there were no decay chains involved other than that of ^{238}U.
(a) How many alphas are emitted per decay chain? [Hint: Use conservation of mass.]
(b) How many electrons are emitted per decay chain? [Hint: Use conservation of charge.]
(c) How long has it been since the lava originally hardened?



Problem 24-d2.

√

24-g1 What is the probability of rolling a pair of dice and getting "snake eyes," i.e., both dice come up with ones?

24-g2 Devise a method for testing experimentally the hypothesis that a gambler's chance of winning at craps is independent of her previous record of wins and losses. If you don't invoke the mathematical definition of statistical independence, then you haven't proposed a test. This has nothing to do with the details of the rules of craps, or with the fact that it's a game played using dice.

24-g3 A blindfolded person fires a gun at a circular target of radius b, and is allowed to continue firing until a shot actually hits it. Any part of the target is equally likely to get hit. We measure the random distance r from the center of the circle to where the bullet went in.
(a) Show that the probability distribution of r must be of the form $D(r) = kr$, where k is some constant. (Of course we have $D(r) = 0$ for $r > b$.)
(b) Determine k by requiring D to be properly normalized. √
(c) Find the average value of r. √
(d) Interpreting your result from part c, how does it compare with $b/2$? Does this make sense? Explain.

24-g4 We are given some atoms of a certain radioactive isotope, with half-life $t_{1/2}$. We pick one atom at random, and observe it for one half-life, starting at time zero. If it decays during that one-half-life period, we record the time t at which the decay occurred. If it doesn't, we reset our clock to zero and keep trying until we get an atom that cooperates. The final result is a time $0 \le t \le t_{1/2}$, with a distribution that looks like the usual exponential decay curve, but with its tail chopped off.
(a) Find the distribution $D(t)$, with the proper normalization. √
(b) Find the average value of t. √
(c) Interpreting your result from part b, how does it compare with $t_{1/2}/2$? Does this make sense? Explain.

24-g5 The speed, v, of an atom in an ideal gas has a probability distribution of the form $D(v) = bve^{-cv^2}$, where $0 \le v < \infty$, c relates to the temperature, and b is determined by normalization.
(a) Sketch the distribution.
(b) Find b in terms of c. √
(c) Find the average speed in terms of c, eliminating b. (Don't try to do the indefinite integral,

because it can't be done in closed form. The relevant definite integral can be found in tables or done with computer software.)

$\surd$

24-j1 Suppose we want to build an electronic light sensor using an apparatus in which the light strikes a metal plate and generates a current through the photoelectric effect. How would its ability to detect different parts of the spectrum depend on the type of metal used in the capacitor plates?

24-j2 The photoelectric effect can occur not just for metal cathodes but for any substance, including living tissue. Ionization of DNA molecules can cause cancer or birth defects. If the energy required to ionize DNA is on the same order of magnitude as the energy required to produce the photoelectric effect in a metal, which of the following types of electromagnetic waves might pose such a hazard? Explain.

60 Hz waves from power lines
100 MHz FM radio
microwaves from a microwave oven
visible light
ultraviolet light
x-rays

24-j3 When light is reflected from a mirror, perhaps only 80% of the energy comes back. One could try to explain this in two different ways: (1) 80% of the photons are reflected, or (2) all the photons are reflected, but each loses 20% of its energy. Based on your everyday knowledge about mirrors, how can you tell which interpretation is correct? [Based on a problem from PSSC Physics.]

24-j4 Give a numerical comparison of the number of photons per second emitted by a hundred-watt FM radio transmitter and a hundred-watt lightbulb.

$\surd$

24-j5 The beam of a 100 W overhead projector covers an area of 1 m × 1 m when it hits the screen 3 m away. Estimate the number of photons that are in flight at any given time. (Since this is only an estimate, we can ignore the fact that the beam is not parallel.)

$\surd$

24-j6 In the photoelectric effect, electrons are observed with virtually no time delay (~ 10 ns), even when the light source is very weak. (A weak light source does however only produce a small number of ejected electrons.) The purpose of this problem is to show that the lack of a significant time delay contradicted the classical wave theory of light, so throughout this problem you should put yourself in the shoes of a classical physicist and pretend you don't know about photons at all. At that time, it was thought that the electron might have a radius on the order of 10^{-15} m. (Recent experiments have shown that if the electron has any finite size at all, it is far smaller.)
(a) Estimate the power that would be soaked up by a single electron in a beam of light with an intensity of 1 mW/m^2. $\surd$
(b) The energy, E_s, required for the electron to escape through the surface of the cathode is on the order of 10^{-19} J. Find how long it would take the electron to absorb this amount of energy, and explain why your result constitutes strong evidence that there is something wrong with the classical theory.

$\surd$

24-j7 (a) A radio transmitter radiates power P in all directions, so that the energy spreads out spherically. Find the energy density at a distance r. $\surd$
(b) Let the wavelength be λ. As described in example **??** on p. **??**, find the number of photons in a volume λ^3 at this distance r. $\surd$
(c) For a 1000 kHz AM radio transmitting station, assuming reasonable values of P and r, verify, as claimed in the example, that the result from part b is very large.

24-m1 In a television, suppose the electrons are accelerated from rest through a voltage difference of 10^4 V. What is their final wavelength?

$\surd$

24-m2 Use the Heisenberg uncertainty principle to estimate the minimum velocity of a proton or neutron in a ^{208}Pb nucleus, which has a diameter of about 13 fm (1 fm=10^{-15} m). Assume that the speed is nonrelativistic, and then check at the end whether this assumption was warranted.

$\checkmark$

24-m3 Find the energy of a nonrelativistic particle in a one-dimensional box of length L, expressing your result in terms of L, the particle's mass m, the number of peaks and valleys n in the wavefunction, and fundamental constants. $\checkmark$

24-m4 A free electron that contributes to the current in an ohmic material typically has a speed of 10^5 m/s (much greater than the drift velocity).

(a) Estimate its de Broglie wavelength, in nm. $\checkmark$

(b) If a computer memory chip contains 10^8 electric circuits in a 1 cm^2 area, estimate the linear size, in nm, of one such circuit. $\checkmark$

(c) Based on your answers from parts a and b, does an electrical engineer designing such a chip need to worry about wave effects such as diffraction?

(d) Estimate the maximum number of electric circuits that can fit on a 1 cm^2 computer chip before quantum-mechanical effects become important.

24-m5 In classical mechanics, an interaction energy of the form $U(x) = \frac{1}{2}kx^2$ gives a harmonic oscillator: the particle moves back and forth at a frequency $\omega = \sqrt{k/m}$. This form for $U(x)$ is often a good approximation for an individual atom in a solid, which can vibrate around its equilibrium position at $x = 0$. (For simplicity, we restrict our treatment to one dimension, and we treat the atom as a single particle rather than as a nucleus surrounded by electrons). The atom, however, should be treated quantum-mechanically, not clasically. It will have a wave function. We expect this wave function to have one or more peaks in the classically allowed region, and we expect it to tail off in the classically forbidden regions to the right and left. Since the shape of $U(x)$ is a parabola, not a series of flat steps as in figure **??** on page **??**, the wavy part in the middle will not be a sine wave, and the tails will not be exponentials.

(a) Show that there is a solution to the Schrödinger equation of the form

$$\Psi(x) = e^{-bx^2},$$

and relate b to k, m, and $\hbar$. To do this, calculate the second derivative, plug the result into the Schrödinger equation, and then find what value of b would make the equation valid for *all* values of x. This wavefunction turns out to be the ground state. Note that this wavefunction is not properly normalized — don't worry about that.

(b) Sketch a graph showing what this wavefunction looks like.

(c) Let's interpret b. If you changed b, how would the wavefunction look different? Demonstrate by sketching two graphs, one for a smaller value of b, and one for a larger value.

(d) Making k greater means making the atom more tightly bound. Mathematically, what happens to the value of b in your result from part a if you make k greater? Does this make sense physically when you compare with part c?

$\checkmark$

24-m6 Estimate the angular momentum of a spinning basketball, in units of $\hbar$. Explain how this result relates to the correspondence principle.

24-m7 Before the quantum theory, experimentalists noted that in many cases, they would find three lines in the spectrum of the same atom that satisfied the following mysterious rule: $1/\lambda_1 = 1/\lambda_2 + 1/\lambda_3$. Explain why this would occur. Do not use reasoning that only works for hydrogen — such combinations occur in the spectra of all elements. [Hint: Restate the equation in terms of the energies of photons.]

24-m8 The wavefunction of the electron in the ground state of a hydrogen atom, shown in the top left of figure **??** on p. **??**, is

$$\Psi = \pi^{-1/2} a^{-3/2} e^{-r/a},$$

where r is the distance from the proton, and $a = \hbar^2/kme^2 = 5.3 \times 10^{-11}$ m is a constant that sets the size of the wave. The figure doesn't show the proton; let's take the proton to be a sphere with a radius of $b = 0.5$ fm.
(a) Reproduce figure **??** in a rough sketch, and indicate, relative to the size of your sketch, some idea of how big a and b are.
(b) Calculate symbolically, without plugging in numbers, the probability that at any moment, the electron is inside the proton. [Hint: Does it matter if you plug in $r = 0$ or $r = b$ in the equation for the wavefunction?] ✓
(c) Calculate the probability numerically. ✓
(d) Based on the equation for the wavefunction, is it valid to think of a hydrogen atom as having a finite size? Can a be interpreted as the size of the atom, beyond which there is nothing? Or is there any limit on how far the electron can be from the proton?

24-m9 Show that the wavefunction given in problem 24-m8 is properly normalized.

24-m10 *This problem generalizes the one-dimensional result from problem 24-m3.*
Find the energy levels of a particle in a three-dimensional rectangular box with sides of length a, b, and c.
✓

24-m11 Hydrogen is the only element whose energy levels can be expressed exactly in an equation. Calculate the ratio λ_E/λ_F of the wavelengths of the transitions labeled E and F in problem **??** on p. **??**. Express your answer as an exact fraction, not a decimal approximation. In an experiment in which atomic wavelengths are being measured, this ratio provides a natural, stringent check on the precision of the results.
✓

24-m12 The electron, proton, and neutron were discovered, respectively, in 1897, 1919, and 1932. The neutron was late to the party, and some physicists felt that it was unnecessary to consider it as fundamental. Maybe it could be explained as simply a proton with an electron trapped inside it. The charges would cancel out, giving the composite particle the correct neutral charge, and the masses at least approximately made sense (a neutron is heavier than a proton).
(a) Given that the diameter of a proton is on the order of 10^{-15} m, use the Heisenberg uncertainty principle to estimate the trapped electron's minimum momentum. ✓
(b) Find the electron's minimum kinetic energy. ✓
(c) Show via $E = mc^2$ that the proposed explanation may have a problem, because the contribution to the neutron's mass from the electron's kinetic energy would be comparable to the neutron's entire mass.

24-m13 Suppose that an electron, in one dimension, is confined to a certain region of space so that its wavefunction is given by

$$\Psi = \begin{cases} 0 & \text{if } x < 0 \\ A\sin(2\pi x/L) & \text{if } 0 \le x \le L \\ 0 & \text{if } x > L \end{cases}$$

Determine the constant A from normalization.
✓

24-m14 Show that a wavefunction of the form $\Psi = e^{by}\sin ax$ is a possible solution of the Schrödinger equation in two dimensions, with a constant potential U. Can we tell whether it would apply to a classically allowed region, or a classically forbidden one?

Answers

1-a1

$$134\ \mathrm{mg} \times \frac{10^{-3}\ \mathrm{g}}{1\ \mathrm{mg}} \times \frac{10^{-3}\ \mathrm{kg}}{1\ \mathrm{g}} = 1.34 \times 10^{-4}\ \mathrm{kg}$$

1-a2 (a) $10^4\ \mu$g; (b) $10^{10}\ \mu$g; (c) $10^{10}\ \mu$g; (d) $10^{11}\ \mu$g; (e) 1 μg

1-a3 (a)

$$\begin{aligned}\frac{0.5\ \mathrm{mg}}{0.2\ \mathrm{ng/hamburger}} &= \frac{0.5 \times 10^{-3}\ \mathrm{g}}{0.2 \times 10^{-9}\ \mathrm{g}}\ \mathrm{hamburgers} \\ &= 2\ \text{million hamburgers!}\end{aligned}$$

(Note how the fake units of "hamburgers" help to reassure us that dividing the first number by the second one gives the right result.)

(b) (0.5 mg)/(300 ng/serving)=2000 servings

1-d1 (a) Let's do 10.0 g and 1000 g. The arithmetic mean is 505 grams. It comes out to be 0.505 kg, which is consistent. (b) The geometric mean comes out to be 100 g or 0.1 kg, which is consistent. (c) If we multiply meters by meters, we get square meters. Multiplying grams by grams should give square grams! This sounds strange, but it makes sense. Taking the square root of square grams (g^2) gives grams again. (d) No. The superduper mean of two quantities with units of grams wouldn't even be something with units of grams! Related to this shortcoming is the fact that the superduper mean would fail the kind of consistency test carried out in the first two parts of the problem.

1-d2 (a) They're all defined in terms of the ratio of side of a triangle to another. For instance, the tangent is the length of the opposite side over the length of the adjacent side. Dividing meters by meters gives a unitless result, so the tangent, as well as the other trig functions, is unitless. (b) The tangent function gives a unitless result, so the units on the right-hand side had better cancel out. They do, because the top of the fraction has units of meters squared, and so does the bottom.

1-d3 The final line is supposed to be an equation for the height, so the units of the expression on the right-hand side had better equal meters. The pi and the 3 are unitless, so we can ignore them. In terms of units, the final becomes

$$\mathrm{m} = \frac{\mathrm{m}^2}{\mathrm{m}^3} = \frac{1}{\mathrm{m}}.$$

This is false, so there must be a mistake in the algebra. The units of lines 1, 2, and 3 check out, so the mistake must be in the step from line 3 to line 4. In fact the result should have been

$$h = \frac{3V}{\pi r^2}.$$

Now the units check: $\mathrm{m} = \mathrm{m}^3/\mathrm{m}^2$.

1-d4 Both r and h have units of meters. The product rh has units of m^2, and taking its square root gives units of meters again. That makes sense, because the result is supposed to be a distance.

1-g1 The range of uncertainty amounts to more than a factor of 2. That means that each of the two figures has been quoted with some completely meaningless sig figs. A better way to write the numbers would be 6 days and 14 days.

1-g2 The weight is given in U.S. units with not much more than one sig fig of precision, but the metric conversion is given with three sig figs. It's silly to try to give tenths of a gram here; there must be an order of magnitude more variation than that from one bag to the next, since one pretzel is probably several grams.

1-j1 The proportionality $V \propto L^3$ can be restated in terms of ratios as $V_1/V_2 = (L_1/L_2)^3 = (1/10)^3 = 1/1000$, so scaling down the linear dimensions by a factor of 1/10 reduces the volume by 1/1000, to a milliliter.

1-j2

$$1\ \mathrm{mm}^2 \times \left(\frac{1\ \mathrm{cm}}{10\ \mathrm{mm}}\right)^2 = 10^{-2}\ \mathrm{cm}^2$$

1-j3 The bigger scope has a diameter that's ten times greater. Area scales as the square of the linear dimensions, so $A \propto d^2$, or in the language of ratios $A_1/A_2 = (d_1/d_2)^2 = 100$. Its light-gathering power is a hundred times greater.

1-j4 The cone of mixed gin and vermouth is the same shape as the cone of vermouth, but its linear dimensions are doubled. Translating the proportionality $V \propto L^3$ into an equation about ratios, we have $V_1/V_2 = (L_1/L_2)^3 = 8$. Since the ratio of the whole thing to the vermouth is 8, the ratio of gin to vermouth is 7.

1-j5

$$(1\ \text{ft}^3) \times (12\ \text{in}/1\ \text{ft})^3 = 1728\ \text{in}^3$$

1-j6 Volume scales like the third power of length, so the dog has 8 times more brain cells.

1-k1 Since they differ by two steps on the Richter scale, the energy of the bigger quake is 10^4 times greater. The wave forms a hemisphere, and the surface area of the hemisphere over which the energy is spread is proportional to the square of its radius, $A \propto r^2$, or $r \propto \sqrt{A}$, which means $r_1/r_2 = \sqrt{A_1/A_2}$. If the amount of vibration was the same, then the surface areas must be in the ratio $A_1/A_2 = 10^4$, which means that the ratio of the radii is 10^2.

1-k2 Area scales like the square of the linear dimensions, so 0.35 of the linear dimensions corresponds to $0.35^2 = 0.11$, or 11% of the area.

1-k3 Area scales like the square of the linear dimensions, so Earth's total surface area is four times that of Mars. Since only 30 percent of Earth's area is land, the ratio is 1.2 to 1 (Earth to Mars).

1-k4 The relationship between volume and linear dimensions is $V \propto L^3$. We're given information about V and asked to find out about L, so we solve this for $L \propto V^{1/3}$. Converting this into a statement about ratios, we have $L_1/L_2 = (V_1/V_2)^{1/3}$. The result is that the balls could have a diameter as big as 4.48 cm.

1-k5 In the language of proportionalities, we have $L \propto t$, and $V \propto L^3$, so $t \propto V^{1/3}$. Translating this into a statement about ratios, we have $t_2/t_1 = (V_2/V_1)^{1/3} = 100^{1/3} = 4.64$. The signals have already been propagating outward for $t_1 =$ 90 years, so to fill 100 times the volume, they would have to propagate for (90 years)(4.64), or 418 years. Adding this to 1920, we get 2338 AD.

1-k6 Humans and ants are both mostly water, so the ratio of their body masses is probably about the same as the ratio of their volumes. We therefore have $m \propto V \propto L^3$, or $L \propto m^{1/3}$. Since the ratio of the body masses is $m_1/m_2 \sim 7 \times 10^6$, the ratio of the thicknesses is $L_1/L_2 \sim (m_1/m_2)^{1/3} \sim 200$.

1-m1 Check: The actual number of species of lupine occurring in the San Gabriels is 22. You should find that your answer comes out in the same ballpark as this figure, but not exactly the same, of course, because the scaling rule is only a generalization.

1-m2 Solving the proportionality $A \propto L^2$ for L, we have $L \propto \sqrt{A}$. Converting this into the language of ratios gives $L_{LA}/L_{SF} = \sqrt{A_{LA}/A_{SF}} = 1.2$.

1-m3 (a) You can get two A5 sheets by cutting an A4 in half, so the area of an A5 is half that of an A4. The two types are the same shape, so the area and linear dimensions are related by $A \propto L^2$, or $L \propto \sqrt{A}$. Cutting the area in half means reducing all the linear dimensions by a factor of $1/\sqrt{2}$.

(b) Let the dimensions of the B0 box, in units of meters, be a, b, and c ($a > b > c$). If you can pack two B1 boxes inside a B0 box, two of the dimensions of a B1 must be the same as the two smallest dimensions of the B0, b and c, while the third is $a/2$. Since $L \propto V^{1/3}$, the dimensions of the B1 must be smaller than those of the B0 by a factor of $1/2^{1/3}$. If the series is to continue with B2, B3, ..., the three sides of a particular box must be related by this same factor:

$$b = \frac{a}{2^{1/3}}$$

$$c = \frac{b}{2^{1/3}} = \frac{a}{2^{2/3}}$$

Since the volume is one cubic meter, we have

$$\begin{aligned} 1\text{ m}^3 &= abc \\ &= (a)\left(\frac{a}{2^{1/3}}\right)\left(\frac{a}{2^{2/3}}\right) \\ &= \frac{a^3}{2} \\ a &= 2^{1/3}\text{ m} \\ b &= 1\text{ m} \\ c &= 1/2^{1/3}\text{ m} \end{aligned}$$

1-p1 Directly guessing the number of jelly beans would be like guessing volume directly. That would be a mistake. Instead, we start by estimating the linear dimensions, in units of beans. The contents of the jar look like they're about 10 beans deep. Although the jar is a cylinder, its exact geometrical shape doesn't really matter for the purposes of our order-of-magnitude estimate. Let's pretend it's a rectangular jar. The horizontal dimensions are also something like 10 beans, so it looks like the jar has about $10 \times 10 \times 10$ or $\sim 10^3$ beans inside.

1-p2 First let's estimate the surface area of the earth's oceans. One of the rules of thumb for order-of-magnitude estimates is never to guess area or volume directly, but to do it indirectly by estimating linear dimensions first. We're only attempting an order-of-magnitude estimate, so let's just say that the diameter of the earth is on the order of 10,000 km, or 10^7 m. Since we're not aiming to take into account details such as the proportion of land to ocean, there is no point in worrying about numerical factors such as π, so the surface area of the earth is on the order of 10^{14} m^2. Multiplying by 1 mm gives 10^{11} m^3, which converts to 10^{14} liters.

1-p3 Let's estimate the linear dimensions of a small classroom at 12 m by 14 m by 2 m, for a total volume of about 300 m^3. A penny is roughly 1 cm by 1 cm by 1 mm, which makes 0.1 cm^3 = 10^{-7} m^3. Dividing the two volumes gives something like 3×10^9 pennies, or 30 million dollars.

1-p4 We never try to estimate mass or volume directly; we estimate linear dimensions first. Let's say the hair has a diameter of 0.1 mm, and a length of 1 cm. There's no real point in trying to compute its volume as a cylinder, since this is just an estimate. Let's just consider it to be a long, skinny box, with dimensions 0.1 mm × 0.1 mm × 1 cm. We'll assume it has the same density as water, 1 g/cm^3, so let's get all the dimensions in terms of centimeters before computing the volume and mass: 0.01 cm × 0.01 cm × 1 cm = 10^{-4} cm^3. Using the density, we get a mass of 10^{-4} g = 10^{-7} kg.

1-p5 I don't even know the exact width of a football field, but let's guess the dimensions are 100 yd × 30 yd, which is close enough to 100 m × 30 m, for a total area of 3000 m^2. If we assume each blade of grass takes up on the order of 1 mm^2, then the number of blades equals 3000 m^2/1 mm^2 = 3×10^9 mm^2/1 mm^2 = 3×10^9 . Call it 10^9 blades.

1-p6 The world's population is about 5 billion, or call it 10^{10}. Suppose each person gets an apartment that's 5 m x 5 m x 2 m, for a volume of 50 m^3, or round off to 100 m^3. The total volume of the building needs to be about 10^{12} m^3. Its area at the base is 10^8 m^2, so its height needs to be 10^{12} m^3/10^8 m^2 = 10^4 m = 10 km.

1-p7 (a) Living things typically have about the same density as water, 1g/cm^3 = 1000 kg/m^3. The linear dimensions are about 1 μm, so the volume is about 10^{-18} m^3, giving a mass of 10^{-15} kg.

(b) Estimating the mass of the feces to be 1 kg (some of us are more full of it than others), we get about 10^{15} bacteria in the body. This is two orders of magnitude greater than the number of human cells.

(c) Since the human cells are much less numerous, but take up much more space, they must be many orders of magnitude bigger than bacterial cells.

1-q1 Let's estimate the Great Wall's volume, and then figure out how many bricks that would represent. The wall is famous because it covers pretty much all of China's northern border, so let's say it's 1000 km long. From pictures, it looks like it's about 10 m high and 10 m wide, so the total volume would be $10^6\ \mathrm{m}\times 10\ \mathrm{m}\times 10\ \mathrm{m} = 10^8\ \mathrm{m}^3$. If a single brick has a volume of 1 liter, or $10^{-3}\ \mathrm{m}^3$, then this represents about 10^{11} bricks. If one person can lay 10 bricks in an hour (taking into account all the preparation, etc.), then this would be 10^{10} man-hours.

1-q2 A thousand metric tons is 10^9 g of plutonium. To estimate the volume of the earth's oceans, let r be the earth's radius and d the average depth of the ocean. Looking up r, estimating d to be 1 km, and taking the earth to be mostly ocean, we get a volume of $4\pi r^2 d$, or about $10^9\ \mathrm{km}^3 = 10^{24}\ \mathrm{cm}^3$. The concentration comes out to be $\sim 10^{-15}\ \mathrm{g/cm}^3$, which is fifty times lower than the government standard. This method of disposal would therefore have no measurable human health effects (and probably none on any part of the biosphere).

2-a1 Velocity is relative, so having to lean tells you nothing about the current value of the train's velocity. Fullerton is moving at a huge speed relative to Beijing, but that doesn't produce any noticeable effect in either city. The fact that you have to lean tells you that the train is *changing* its speed, but it doesn't tell you what the train's current speed is.

2-a2 We write v_{AB} for the velocity of A relative to B, meaning the velocity that A has if B is considered to be at rest. Similarly v_{BA} is the velocity of B if A is taken as the point of reference. The rule $v_{AB} = -v_{BA}$ tells us that if A is moving in a certain direction at a certain speed relative to B, then B is moving in the opposite direction relative to A, at the same speed.

2-a3 Let W be Wa-Chuen, E the escalator, and M the mall. Then $v_{WM} = 0$, which happens because $v_{WE} + v_{EM} = 0$.

2-a4 Take a frame defined by an observer in another car who is in the lane next to yours, who is initially moving at the same speed as you, and who continues at the speed even while you slow down. In that observer's frame, you started out with zero speed and then gained some (backward) speed when you stepped on the brakes.

2-b1 (a) Let the subscripts A, E, and G refer to the arrow, Eowyn, and the ground. Let the positive x axis point forward.

$$\begin{aligned} v_{EG} &= 11\ \mathrm{m/s} \\ v_{AE} &= -25\ \mathrm{m/s} \end{aligned}$$

(b)

$$\begin{aligned} v_{AG} &= v_{AE} + v_{EG} \\ &= -14\ \mathrm{m/s} \end{aligned}$$

The answer makes sense: the arrow flies backward, at less than normal speed.

2-b2 Let north be positive, and let Y=you, W=walkway, F=friend, and A=airport. The given information is that $v_{WA} = 3.0$ m/s, $v_{YW} = 2.0$ m/s, and $v_{FA} = -1.5$ m/s.
(a) $v_{YW} = 2.0$ m/s
(b) $v_{YA} = v_{YW} + v_{WA} = 5.0$ m/s
(c) $v_{YF} = v_{YA} + v_{AF} = v_{YA} - v_{FA} = 6.5$ m/s
(d) Your speed is greater than his by a factor of 10/3, so it takes him (45 s)(10/3) = 150 s.

2-b3 To avoid writing lots of numbers, which is hard to read or debug, let's convert the given information into symbols. Let $L = 10$ km, let $u = 8$ km/hour, let $V = 12$ km/hour, and let v be the the velocity for the second half of the ride.

We want the total time taken for the ride to be $2L/V$. The time it will actually take is $L/u + L/v$. Equating these, we find $2L/V = L/u + L/v$, or $2/V = 1/u + 1/v$. Solving for v gives $v = 1/(2/V - 1/u) = 24$ km/hour.

2-b4 (a) Let the length of the race be 2ℓ. The time required for the first half is ℓ/u, and for the second ℓ/v, so the total time is $\ell/u + \ell/v$. Dividing the total distance by the total time gives an over-all speed of $2/(1/u + 1/v)$.
(b) The units of $1/u$ are seconds per meter, and so are the units of $1/v$. Since they have the same units, it makes sense to add them. The final result has units that are the inverse of these, meters per second, and that makes sense, because what we're trying to calculate is a velocity.
(c) When $u = v$, we have $2/(1/u + 1/u) = u$, which makes sense.
(d) Mathematically, making u bigger makes $1/u$ smaller, but $1/u$ is in the denominator, so the

final result gets bigger. This makes sense physically: if you do the first half of the race faster, your over-all speed is higher. The analysis for v is identical.

2-b5 (a) In the equation $x = At^5 - Bt^2$, all three terms must have the same units, so the units of A have to be m/s^5, and B should be in m/s^2 (acceleration).
(b) $v = \mathrm{d}x/\,\mathrm{d}t = 5At^4 - 2Bt$
(c)

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{x-0}{t-0} = At^4 - Bt$$

(d) Setting the answers for parts b and c equal to each other and solving for t, we have $t = (B/4A)^{1/3}$. Note that everything about this answer except for the unitless factor of 4 is determinable from units.

2-b6 (a) The circumference of a circle is 2π times the radius, so we have $v = 2\pi R/T$.
(b) The factor of 2π is unitless. The radius R has units of meters, and T has units of seconds. Therefore the right-hand side has units of m/s, which makes sense, because we're supposed to be calculating a velocity.
(c) When T is very large, we'd be dividing by a very large number in our expression from part a. Dividing by a very large number gives a very small result. This makes sense, because if the earth stopped rotating, the velocity should equal zero.
(d) Plugging in numbers gives 460 m/s. This is faster than the speed of sound, over a thousand miles an hour! (I rounded the result to 2 sig figs, since a really precise answer would have to take into account other factors, such as how far Nairobi is from the equator, how much the earth deviates from being a perfect sphere, and the distinction between a solar day and a sideral day.)

2-c1 (a) Split the isoceles triangle into two right triangles, and let the long sides of one of the right triangles be L_p and L, where the subscript p refers to the runner who is passing. Applying the definition of the cosine gives $L/L_P = \cos\theta$. Since the runners cover these distances in equal times, we have $v_p = L_P/t$ and $v = L/t$, so that $v_p = (L_P/L)v = v/\cos\theta$. Therefore $\Delta v = v_p - v = (1/\cos\theta - 1)v$.
(b) The cosine is unitless, so the entire factor $(1/\cos\theta - 1)$ in parentheses is also unitless. The velocity v has units of m/s, so multiplying by the quantity in parentheses still gives m/s. This makes sense, because the thing we're calculating, on the left side of the equals sign, is supposed to be a velocity, with units of m/s.
(c) Since the cosine of 0 is 1, the result in the case of $\theta = 0$ is $\Delta v = 0$. This makes sense. If there is a hundred-mile straightaway available in which to pass, the extra speed required is going to be nearly zero.
(d) Plugging in numbers gives $\Delta v = 1.1 \times 10^{-3}$ m/s. This is not just small (as expected from part c) but insanely small — in fact too small to matter — basically because L_p does not differ from L by as much as we would intuitively expect. It appears that in reality, the extra speed required by the passing runner is not the speed required to run the extra distance, but simply the same extra speed that would have been required if the runners had been in parallel lanes. That is, our geometrical model turned out to include the factor that was unimportant, and ignore the one that would really be important.

2-c2 The speed of light is constant, so the equation $v = \Delta x/\Delta t$ holds. The distance traveled by the ray of light is $\Delta x = 2L$. The spacing between teeth is $1/n$ of a revolution, so during the time Δt the wheel moves through $1/2n$ of a revolution. Setting up a ratio between a whole revolution and this small part of a revolution, we have

$$\frac{\Delta t}{\text{time for one revolution}} = \frac{1}{2n}$$

The time for one revolution is $1/f$, so $\Delta t = 1/2nf$. Substituting into the equation for the velocity, we have $c = \Delta x/\Delta t = (2L)/(1/2nf) = 4Lnf$.
(b) The factor of $4n$ on the right is unitless. The units of Lf are (m)(s^{-1}) = m/s, which are the units we expect for a velocity.
(c) All three numbers are being multiplied on the right, so an increase in any one of these corre-

sponds to an *increase* in c. Since c is big, Fizeau wants all three numbers to be big: a long distance to the mirror, a large number of teeth, and a fast rotation of the wheel.

(d) The result is 3.13×10^8 m/s.

2-e1 Zero. $\Delta v = 0$, because the velocity didn't change.

2-e2 They both have accelerations equal to g. Let's say that the bullet leaves the gun with a velocity of 100 m/s, and for convenience, let's just say g is approximately 10 m/s^2. Then the rock's speed at one-second intervals goes like 0, 10, 20, … m/s, while the bullet goes 100, 110, 120, … m/s.

2-e3 No. The acceleration is greatest at the ends, where $v = 0$. Just because the velocity is passing through zero on a graph doesn't mean the acceleration is zero. The velocity is changing rapidly (large Δv), even though $v = 0$.

2-e4 For example, suppose you accelerate away from a stop sign to 10 km/hr, then 20 km/hr, then 30 km/hr. The equation is telling you that the distance you cover increases faster than your speed. The distance from the stop sign to where your speedometer reads 20 km/hr is four times the distance from the stop sign to where it read 10 km/hr, and likewise the total distance from the stop sign to where it reads 30 km/hr is nine times the distance at 10 km/hr. This makes sense, because later in the motion, you're covering ground more quickly.

2-f1 Taking g to be 10 m/s^2, the bullet loses 10 m/s of speed every second, so it will take 10 s to come to a stop, and then another 10 s to come back down, for a total of 20 s.

2-f2 The units don't work out: $\text{m/s}^2 \times \text{s} = \text{m/s}$, not m.

2-f3 $s = \frac{1}{2}at^2$, and $a = g = 10$ m/s^2, so $s = 45$ m, or call it 40 m (1 sig fig).

2-f4 We're given distance and acceleration, and we want to find time. Of the constant acceleration equations, the one that has the right variables in it is $\Delta x = v_o\Delta t + \frac{1}{2}a\Delta t^2$. The initial velocity is zero, and solving for Δt, we have $\Delta t = \sqrt{2\Delta x/a}$. Converting the given distance to SI (meter-kilogram-second) units, we have 4000 mi $= 6.4 \times 10^6$ m, which results in $\Delta t = 1100$ s, or about 20 minutes (1 sig fig).

2-f5 The constant-acceleration acceleration that has the right variables in it is $v_f^2 = v_0^2 + 2a\Delta x$. The final velocity is zero, so we have $a = -v_0^2/2\Delta x$. If we wanted to worry about the sign, then we would have to pick a coordinate so that we could define a sign for Δx, and then we would get some sign for the acceleration, expressed in those coordinates. However, we're only trying to find the magnitude of the deceleration, so ignoring signs, the result is 1.5 m/s^2.

2-f6 The constant-acceleration equation with the right variables in it is $\Delta x = v_0\Delta t + (1/2)a\Delta t^2$. The initial velocity is zero. Solving for the acceleration gives $a = 2\Delta x/\Delta t^2 = 16.3$ m/s^2.

2-i1 (a) Let positive be up. We have $v_f^2 = v_0^2 + 2a\Delta x$, with $v_f = 0$, $a = -g$, and $\Delta x = 2.0$ m, so $v_0 = \sqrt{2g\Delta x} = 6.3$ m/s.

(b) The equation $\Delta x = v_0t + (1/2)at^2$ would allow us to find t by plugging in the result of part a for v_0 and then solving a quadratic. A simpler approach is to calculate the fall time, which is the same as the rise time. For the purpose of computing the fall time, we have $v_0 = 0$ and $\Delta x < 0$, so $t = \sqrt{2\Delta x/a} = 0.64$ s.

(c) Since the rise time and fall time are equal, this is double the result from part b, or 1.3 s.

2-i2 (a) We have $v_f^2 = v_0^2 + 2a\Delta x$, and $v_0 = 0$, so $a = v_f^2/2\Delta x = 680$ m/s^2. This is a remarkable acceleration, equal to about 70 times the acceleration of gravity!

(b) The tricky and simple way to find this is to recognize that the ball's average velocity $\Delta x/t$ is half its final velocity. Therefore $t = 2\Delta x/v_f = 0.067$ s. (This could also have been done without the trick simply by taking a constant acceleration equation and using the acceleration found in part a.)

(c) The time on the fly is (18.0 m)/(45.0 m/s) = 0.400 s, so the total time in the air is 0.467 s.

2-i3 Of the time t he spent aloft, he spent $t/2$ rising and $t/2$ falling. The acceleration is $g = (9.8\text{ m/s}^2)/6$. $\Delta x = (1/2)g(t/2)^2 = 0.43$ m.

2-i4 The calculation of $\Delta v = a\Delta t$ is correct, because the equation it's based on, $a = \Delta v/\Delta t$, is valid for constant acceleration. The calculation of $\Delta x = v\Delta t$ is wrong, because the equation it's based on, $v = \Delta x/\Delta t$, is only valid for constant velocity. We know the velocity is increasing steadily, not staying constant. The speed of 10 m/s is only correct for the very end of the time period, not the whole time.

2-i5

$$H = v_0T + \frac{1}{2}gT^2$$
$$v_0 = H/T - \frac{1}{2}gT$$

2-i6 Let v_A be the acceleration of object A, $v_{B,0}$ the initial velocity of object B, and a the acceleration of object B. Setting their positions equal gives $v_At = v_{B,0}t+(1/2)at^2$. This equation is satisfied at $t = 0$, but that isn't the solution we're interested in, so we can divide both sides by t to give $v_A = v_{B,0} + (1/2)at$. Solving this for the time gives $t = (2/a)(v_A - v_{B,0}) = 4$ s. A nice way of interpreting this result is that at this time, the average velocity of B will be 2 m/s, the same as for A.

2-i7 (a) This is easiest to calculate in the frame of the elevator. Let positive be down, so that Δx and $a = +g$ are both positive. $\Delta x = (1/2)gt^2$, so $t = \sqrt{2\Delta x/g} = 0.78$ s (2 sig figs, because we don't know g to more than 2 sig figs for an unstated point on the earth's surface).

(b) Again this is easiest in the frame of the elevator. $v_f^2 = 2gx$, so $v_f = \sqrt{2gx} = 7.6$ m/s, where we pick the positive square root because positive is down.

(c) Let B be the bolt, E the elevator, and L the landing. What we found in part B was v_{BE}. We then have $v_{BL} = v_{BE} + v_{EL} = 5.1$ m/s.

2-i8 (a) Plugging our symbols into the standard equation for motion with constant acceleration, we have

$$x_F = L + ut/2.$$

(b) Similarly, for the Honda,

$$x_H = ut - \frac{1}{2}at^2.$$

(c) Subtraction gives

$$d = L - ut/2 + \frac{1}{2}at^2.$$

(d) Setting $d = 0$ gives a quadratic equation. We're interested in whether this has any real solutions, so all we need to do is convert the equation into the standard form $At^2 + Bt + C = 0$ and check whether the discriminant $B^2 - 4AC$ is positive. The coefficients are $A = a/2$, $B = -u/2$, and $C = L$, so the discriminant equals $u^2/4-2aL$. The critical value of the acceleration occurs when this discriminant equals zero, and solving this for the acceleration gives $a = u^2/8L$.

(e) The units on the right-hand side are $(\text{m/s})^2/\text{m}$, which do work out to be m/s^2.

(f) If u is greater, the acceleration required becomes greater. That makes sense, because if both cars are moving at higher speeds, it gets harder for the Honda to decelerate in time. If L is greater, the acceleration needed is smaller. That also makes sense, because if there is more room to stop, the Honda won't have to accelerate as violently.

2-i9 Defining some notation, we have:

$$x_{0F} = 45 \text{ m}$$
$$v_{0F} = 30.0 \text{ m/s}$$
$$a_F = -5.0 \text{ m/s}^2$$
$$v_{0H} = 40.0 \text{ m/s}$$
$$a_H = -10.0 \text{ m/s}^2$$
$$t_R = 1.0 \text{ s}$$

(a) Plugging our symbols into the standard equation for motion with constant acceleration, we have

$$x_F = x_{0F} + v_{0F}t + \frac{1}{2}a_Ft^2.$$

(b) Up until $t = t_R = 1.0$ s, we have $x_H = v_{0H}t$. After that time, the deceleration kicks in, so

$$x_H = v_{0H}t + \frac{1}{2}a_H(t - t_R)^2.$$

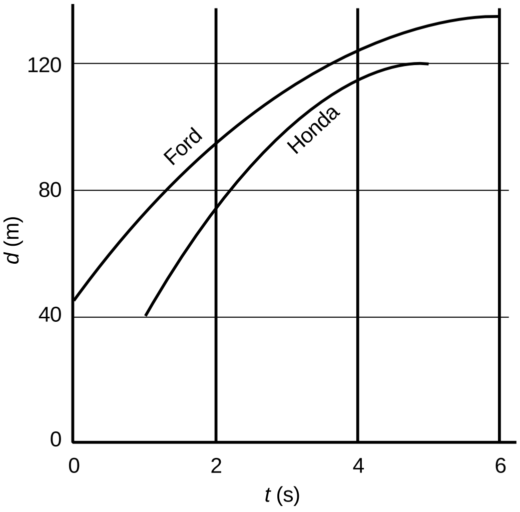
The second equation isn't valid before t_R, because it's a constant acceleration equation with a nonzero equation, but the acceleration is different (zero) before t_R. The the two equations for x_H are equal at $t = t_R$, and their derivatives (the velocity) are also equal at that time, which verifies that they match up correctly.

(c)

$$d = \left[\frac{1}{2}a_F - \frac{1}{2}a_H\right]t^2 + \left[v_{0F} - v_{0H} + a_H t_R\right]t + \left[x_{0F} - \frac{1}{2}a_H t_R^2\right]$$

Setting $d = 0$ gives a quadratic equation. We're interested in whether this has any real solutions, so all we need to do is convert the equation into the standard form $at^2 + bt + c = 0$ and check whether the discriminant b^2-4ac is positive. The coefficients are $a = 2.5\ \mathrm{m/s^2}$, $b = -20\ \mathrm{m/s}$, and $c = 50$ m. The discriminant comes out to be $-100\ \mathrm{m^2/s^2}$, which is negative, so there are no real solutions. The two vehicles will not collide.

(d)



The graphs don't intersect.

(e) The minimum of the function $d(t)$ will occur when its derivative is zero, i.e., when the *relative* velocity is zero. Differentiating the equation from part c, we have for the relative velocity, in terms of the *abc* coefficients,

$$d' = 2at + b.$$

Setting this equal to zero gives $t = -b/2a = 4.0$ s, which looks consistent with the graph. Plugging this time back in to the equation for d, we find $d = 10$ m, which also seems to match up with the graph.

2-i10 When solving a numerical problem like this, one should always do it purely algebraically first, then plug in numbers at the very end. Let $h = 10.0$ m and $v_0 = 10.0$ m/s. Let positive x be down.

(a) We have $h = (1/2)gt^2$, so $t = \sqrt{2h/g} = 1.4$ s (two sig figs because a generic value of g for anywhere on the earth's surface is only good to about two sig figs).

(b) $h = v_0 t + (1/2)gt^2$. Solving this quadratic gives

$$t = \frac{-v_0 \pm \sqrt{v_0^2 + 2gh}}{g}.$$

The positive root gives the positive time at which the rock hits the ground after we throw it. The negative root gives the time at which the rock would have hit the ground if we extrapolated back in time before the throw. The positive root gives $t = 0.74$ s. It makes sense that this is positive and less than the answer to part a.

(c) This is minus the time given by choosing the negative root in the quadratic formula, which comes out to be 2.8 s.

(d) Using the constant acceleration equation $v_f^2 = v_0^2 + 2a\Delta x$, we have $v_0^2/2g$ for the positive height above the roof. The height above the ground is $h + v_0^2/2g = 15$ m.

2-i11 (a) Let positive x be down. Then $h = (1/2)gT^2$.

(b) $h = -v_0 \cdot 2T + (1/2)g(2T)^2$, which gives $v_0 = (3/4)gT$. Note that units force this to be of the form gT multiplied by a unitless constant, so the only thing that really needed to be calculated was the 3/4. Parts c and d are like this as well.

(c) $g = v_0/t$, so $t = v_0/g = (3/4)T$.

(d) Let L be the additional height above the roof. Then $v_0^2 = 2gL$, so $L = v_0^2/2g = (9/32)gT^2$, and $h + L = (25/32)gT^2$.

2-i12 (a) The trip consists of four parts of equal duration. The duration, $T/4$, of each part is related to half the distance, d, to Proxima Centauri, $d/2 = \frac{1}{2}a(T/4)^2$, so $T = 4\sqrt{d/a}$.

(b) Plugging in to the formula from (a), we get 2.6×10^8 s = 7 years.

(c) The maximum speed is $(a)(T/4) = 6 \times 10^8$ m/s. This is more than twice the speed of light, which is not actually possible.

2-k1 $\Delta x = \frac{1}{2}at^2$, so for a fixed value of Δx, we have $t \propto 1/\sqrt{a}$. Translating this into the language of ratios gives $t_M/t_E = \sqrt{a_E/a_M} = \sqrt{3} = 1.7$.

2-k2 We have $v_f^2 = 2a\Delta x$, so the distance is proportional to the square of the velocity. To get up to half the speed, the ball needs 1/4 the distance, i.e., $L/4$.

2-k3 (a) We're not given any information about time, and we don't want to find a time, so the only constant-acceleration equation with the right variables in it is $v_f^2 = v_0^2 + 2a\Delta x$. The final velocity is zero, so we have $\Delta x = -v_0^2/2a$. We're assuming that v_0 is the same in both cases, so the factor of $-v_0^2/2$ is a constant, and we can get rid of the constant and rewrite the equation as a proportionality

$$H \propto a^{-1}.$$

Turning this proportionality into a statement about ratios, we have

$$\frac{H_P}{H_E} = \left(\frac{a_P}{a_E}\right)^{-1} = 15.$$

(b) Now the irrelevant variable is distance, so we want the constant acceleration equation $a = \Delta v/\Delta t$. Since the final velocity is zero, Δv equals minus the initial velocity, and is therefore constant. Throwing out this constant factor, we have

$$\Delta t \propto a^{-1},$$

so in this computation, as in part a, we're dealing with an inverse proportionality. The result is

$$\frac{\Delta t_P}{\Delta t_E} = \left(\frac{a_P}{a_E}\right)^{-1},$$

giving $\Delta t_P = 45$ s.

2-k4 We have $v^2 = 2a\Delta x$, and the factor of $2a$ is the same in both cases, so we can rewrite this as a proportionality $v^2 \propto \Delta x$. Solving for v gives $v \propto \sqrt{\Delta x}$, and converting this into the language of ratios results in $v_1/v_2 = \sqrt{\Delta x_1/\Delta x_2} = \sqrt{2}$.

2-k5 (a) We want to use a constant-acceleration equation to relate distance and time. We don't know the velocity, and we don't want to know it. The equation that has the right variables in it is $\ell = (1/2)at^2$ (using ℓ instead of x and taking $v_o = 0$). The rest of the method is the same as the standard one demonstrated in the appendix "Three essential mathematical skills" in the back of the book. Convert the equation to a proportionality by throwing out the constant factor of $a/2$, giving $\ell \propto t^2$. Solving for t gives $t \propto \ell^{1/2}$. Change this into a statement about ratios:

$$\frac{t_d}{t_h} = \left(\frac{\ell_d}{\ell_h}\right)^{1/2} = \sqrt{4} = 2 \qquad ,$$

where "d" refers to the dinosaur and "h" to a human.

(b) The distance each animal moves in a quarter stride is proportional to ℓ, so $v \propto \ell/t \propto \ell/\ell^{1/2} = \ell^{1/2}$. In terms of ratios,

$$\frac{v_d}{v_h} = \left(\frac{\ell_d}{\ell_h}\right)^{1/2} = \sqrt{4} = 2 \qquad .$$

In other words, the dinosaur's stride is longer by a factor of four, but it loses half of that advantage because its steps are slower by a factor of two.

2-k6 We want to use a constant-acceleration equation to relate acceleration and time for a fixed distance. We don't know the velocity, and we don't want to know it. The equation that has the right variables in it is $x = (1/2)at^2$ (setting $v_o = 0$). Solving for the acceleration gives $a = 2x/t^2$. The rest of the method is the same as the standard one demonstrated in the appendix "Three essential mathematical skills" in the back of the book. Convert the equation to a proportionality by throwing out the constant factor of $2x$, giving $a \propto t^{-2}$. Change this into a statement about ratios:

$$\frac{a_H}{a_r} = \left(\frac{t_H}{t_r}\right)^{-2} = (0.47)^{-2} = 4.5 \qquad ,$$

where "H" refers to Hatton and "r" to the reporter.

2-k7 We want to use a constant-acceleration equation to relate acceleration, speed, and distance. We don't know the time, and we don't want to know it. The equation that has the right variables in it is $v_f^2 = v_o^2 + 2a\Delta x$. The final velocity is zero. Solving for the acceleration gives $a = -v_o^2/2\Delta x$. The rest of the method is the same as the standard one demonstrated in the appendix "Three essential mathematical skills" in the back of the book. Convert the equation to a proportionality by throwing out the constant factor of $-1/2$, giving $a \propto v_o^2 \Delta x^{-1}$. Change this into a statement about ratios:

$$\frac{a_F}{a_S} = \left(\frac{v_F}{v_S}\right)^2 \left(\frac{\Delta x_F}{\Delta x_S}\right)^{-1} = (4.8)^2(1.9)^{-1} = 12$$

where "F" refers to the F-14 and "S" to the Sopwith.

2-k8 We want to use a constant-acceleration equation to relate acceleration and distance for a fixed final speed. We don't know the time, and we don't want to know it. The equation that has the right variables in it is $v_f^2 = 2a\Delta x$ (setting $v_o = 0$). Solving for the acceleration gives $a = v_f^2/2\Delta x$. The rest of the method is the same as the standard one demonstrated in the appendix "Three essential mathematical skills" in the back of the book. Convert the equation to a proportionality by throwing out the constant factor of $v_f^2/2$, giving $a \propto \Delta x^{-1}$. Change this into a statement about ratios:

$$\frac{a_H}{a_m} = \left(\frac{\Delta x_H}{\Delta x_m}\right)^{-1} = (1.62\,\text{m}/1.75\,\text{m})^{-1} = 1.08$$

where "H" refers to Harrell and "m" to a pitcher of the nominal minimum height.

2-k9 We want to use a constant-acceleration equation to relate acceleration and speed for a fixed distance. We don't know the time, and we don't want to know it. The equation that has the right variables in it is $v_f^2 = v_o^2 + 2a\Delta x$. The final velocity is zero. Solving for the initial speed gives $v_o = (-2a\Delta x)^{1/2}$. The rest of the method is the same as the standard one demonstrated in the appendix "Three essential mathematical skills" in the back of the book. Convert the equation to a proportionality by throwing out the constant factor of $\sqrt{-2\Delta x}$, giving $v_o \propto a^{1/2}$. Change this into a statement about ratios:

$$\frac{v_C}{v_D} = \left(\frac{a_C}{a_D}\right)^{1/2} = (1.09)^{1/2} = 1.04 \qquad ,$$

where "C" refers to the Chevy cop car and "D" to the Dodge.

2-n1 (a) The variables involved are distance, acceleration, and time, so we want the constant-acceleration equation that relates them, which is $\Delta x = \frac{1}{2}g\Delta t^2$. The balls are two separate objects, so we need to have two separate equations of this form.

$$\begin{aligned}
\Delta x_1 &= \frac{1}{2}g\Delta t_1^2 \\
\Delta x_2 &= \frac{1}{2}g\Delta t_2^2 \\
\Delta t_1 &= \sqrt{\frac{2\Delta x_1}{g}} \\
\Delta t_2 &= \sqrt{\frac{2\Delta x_2}{g}} \\
T &= \Delta t_2 - \Delta t_1 \\
&= \sqrt{\frac{2\Delta x_2}{g}} - \sqrt{\frac{2\Delta x_1}{g}} \\
g &= \left(\frac{\sqrt{2\Delta x_2} - \sqrt{2\Delta x_1}}{T}\right)^2
\end{aligned}$$

, (b) The units inside the parentheses are $\text{m}^{1/2}/\text{s}$, so squaring this quantity, we have m/s^2.

(c) When $\Delta x_1 = 0$, T is the same as Δt_2, and we have simply $g = (\sqrt{2\Delta x_2}/\Delta t_2)^2$, which is what we would have gotten if we had just solved $\Delta x_2 = \frac{1}{2}g\Delta t_2^2$ for g. This case reassures us that the result is correct, but it is useless in practical terms: we have to make Δx_1 great enough so that the ball will make a sound we can record.

(d) Mathematically, we get zero over zero, which is undefined. Physically, we cannot measure all the quantities in the equation with perfect precision, so we will not get exactly zero on the top or on the bottom; the result will be not an error on our calculator but rather a wildly unreliable answer! Nature is giving us a hint that the most accurate results will be obtained when the two heights are as different as possible, and will get less and less reliable as the heights are made closer to one another.

2-n2 We're given the distance it takes the flea to slow back down to a stop, $\Delta x = 30$ cm, so it's natural to use the equation $v^2 = 2a\Delta x$, which tells us the distance required in order to slow down from v to rest, or speed up from rest to v. The flea first accelerates rapidly, then, once its feet have left the ground, it begins decelerating. The acceleration probably isn't constant, but this is just an estimate anyway, so we'll assume it is. We then have $2g\Delta x = 2ah$, where a is the acceleration as it's taking off, and h is the distance its center of mass travels upward while its legs are straightening. The result is $a = g\Delta x/h$. Estimating $h = 0.1$ cm, we find $a \sim 300g$.

2-n3 (a)

$$v_f^2 = v_o^2 + 2a\Delta x$$

The initial velocity v_o, is the velocity of the spinning earth's surface, which at the equator is $(2\pi r)/(1 \text{ day}) \approx 500$ m/s. This is an order of magnitude less than the orbital velocity given in this problem. Since v_o is much less than v_f, v_o^2 is even more negligible compared to v_f^2, and we're justified in ignoring it for the purposes of this approximate calculation, whose accuracy is really limited by the assumed acceleration.

$$\begin{aligned} v_f^2 &= 2a\Delta x \\ \Delta x &= \frac{v_f^2}{2a} \\ &= 1\times 10^6 \text{ m} \\ &= 1000 \text{ km} \end{aligned}$$

(b) It's not very practical to build a railgun that would stretch halfway across a continent. The railgun idea is definitely not going to be practical for crewed launches, but it might work for lofting payloads that could handle more violent accelerations, such as raw materials. For instance, a payload that could handle $300g$ of acceleration would only need a 10-kilometer railgun.

2-n4 (a) Other than w, the only thing with units that can occur in our answer is g. If we want to combine a distance and an acceleration to produce a time, the only way to do so is like $\sqrt{w/g}$, possibly multiplied by a unitless constant.

(b) It is convenient to introduce the notations L for the length of one side of the vee and h for the height, so that $L^2 = w^2 + h^2$. The acceleration is $a = g\sin\theta = gh/L$. To travel a distance L with this acceleration takes time

$$t = \sqrt{2L/a} = \sqrt{\left(\frac{2w}{g}\right)\left(\frac{h}{w}+\frac{w}{h}\right)}.$$

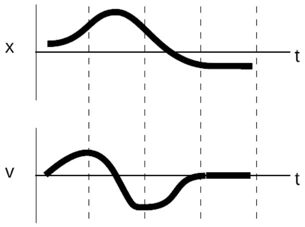
Let $x = h/w$. For a fixed value of w, this time is an increasing function of $x + 1/x$, so we want the value of x that minimizes this expression. Taking the derivative and setting it equal to zero gives $x = 1$, or $h = w$. In other words, the time is minimized if the angle is 45°.

(c) Plugging $x = 1$ back in, we have $t^* = 2t = 4\sqrt{w/g}$, so the unitless factor was 4.

2-n5 Let ℓ be the length of one of the chords and t the time the bead takes. If θ is the angle of the chord relative to the horizontal, then the bead's acceleration is $a = g\sin\theta$, and we have $\ell = (1/2)g\sin\theta t^2$, so that t depends on the ratio $\ell/\sin\theta$. But a little trigonometry shows that $\ell = 2r\sin\theta$, so the race is a tie. This fact was proved by Galileo.

2-o1 Since the lines are at intervals of one m/s and one second, each box represents one meter. From $t = 0$ to $t = 2$ s, the area under the curve represents a positive Δx of 6 m. (The triangle has half the area of the 2×6 rectangle it fits inside.) After $t = 2$ s, the area above the curve represents negative Δx. To get -6 m worth of area, we need to go out to $t = 6$ s, at which point the triangle under the axis has a width of 4 s and a height of 3 m/s, for an area of 6 m (half of 3×4).

2-o2



2-o3 (a) Each rectangle has a width of 1 s and a height of 1 m/s, so it represents a distance of 1 m. There are 14 rectangles between the curve and the t axis, so the change in position is 14 m. (b) The square below the axis counts as a negative change in position, so the result is 7 m.

2-o4 (a) When its velocity is negative it's coming back toward the hive, so at $t = 8.7$ s it is as far from the hive as it will ever get. (b) Between $t = 0$ and $t = 8.7$ s, the area under the curve is about 22 squares. Each square is 1 s wide and 1 m/s tall, so each one represents 1 m. The greatest distance from the hive is 22 m. (c) The area from $t = 8.7$ s to the end is under the axis, so it counts as negative. It's about 2 squares, so at the end the bee is about 20 m from the hive.

2-o5 (a) False. To find the acceleration from the velocity, we need to take the derivative, which is the slope of the tangent line. The area under the curve would be the integral, not the derivative.

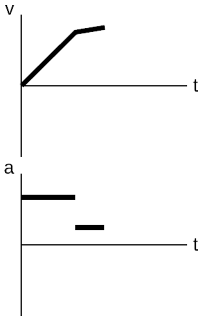
(b) True.

(c) False. Velocity is the indefinite integral of acceleration, not its derivative.

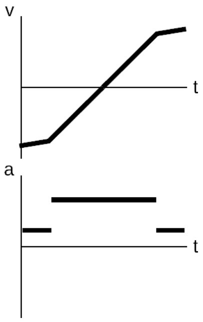
(d) True.

(e) False. The change in position is the definite integral of velocity, not acceleration.

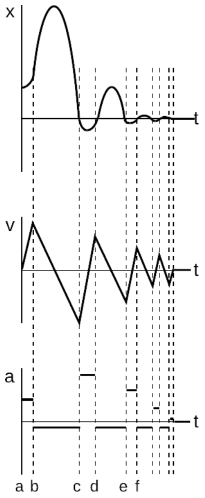
2-o6 (a) We choose a coordinate system with positive pointing to the right. Some people might expect that the ball would slow down once it was on the more gentle ramp. This may be true if there is significant friction, but Galileo's experiments with inclined planes showed that when friction is negligible, a ball rolling on a ramp has constant acceleration, not constant speed. The speed stops increasing as quickly once the ball is on the more gentle slope, but it still keeps on increasing. The a-t graph can be drawn by inspecting the slope of the v-t graph.



(b) The ball will roll back down, so the second half of the motion is the same as in part a. In the first (rising) half of the motion, the velocity is negative, since the motion is in the opposite direction compared to the positive x axis. The acceleration is again found by inspecting the slope of the v-t graph.



2-o7 This is a case where it's probably easiest to draw the acceleration graph first. While the ball is in the air (bc, de, etc.), the only force acting on it is gravity, so it must have the same, constant acceleration during each hop. Choosing a coordinate system where the positive x axis points up, this becomes a negative acceleration (force in the opposite direction compared to the axis). During the short times between hops when the ball is in contact with the ground (cd, ef, etc.), it experiences a large acceleration, which turns around its velocity very rapidly. These short positive accelerations probably aren't constant, but it's hard to know how they'd really look. We just idealize them as constant accelerations. Similarly, the hand's force on the ball during the time ab is probably not constant, but we can draw it that way, since we don't know how to draw it more realistically. Since our acceleration graph consists of constant-acceleration segments, the velocity graph must consist of line segments, and the position graph must consist of parabolas. On the x graph, I chose zero to be the height of the center of the ball above the floor when the ball is just lying on the floor. When the ball is touching the floor and compressed, as in interval cd, its center is below this level, so its x is negative.



2-p1 (a) Let f and g be functions. Then the chain rule states that if we construct the function $f(g(x))$, its derivative is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}.$$

On the right-hand side, the units of $\mathrm{d}g$ on the top cancel with the units of $\mathrm{d}g$ on the bottom, so the units do match up with those of $\mathrm{d}f/\,\mathrm{d}x$ on the left.

(b) The cosine function requires a unitless input and produces a unitless output. Therefore A must have units of meters, and b must have units of s^{-1} (inverse seconds, or "per second"). A is the distance the object moves on either side of the origin, and b is a measure of how fast it vibrates back and forth (how many radians it passes through per second).

(b) The derivative is $v = \mathrm{d}x/\,\mathrm{d}t = -Ab\sin(bt)$, where the factor of b in front comes from the chain rule. The product Ab does have units of m/s. If we hadn't put in the factor of b as required by the chain rule, the units would have been wrong. Physically, it also makes sense that a larger b, indicating a more rapid vibration, produces a greater v.

2-p2 (a) Solving for $\Delta x = \frac{1}{2}at^2$ for a, we find $a = 2\Delta x/t^2 = 5.51\ \mathrm{m/s^2}$. (b) $v = \sqrt{2a\Delta x} = 66.6\ \mathrm{m/s}$. (c) The actual car's final velocity is less than that of the idealized constant-acceleration car. If the real car and the idealized car covered the quarter mile in the same time but the real car was moving more slowly at the end than the idealized one, the real car must have been going faster than the idealized car at the beginning of the race. The real car apparently has a greater acceleration at the beginning, and less acceleration at the end. This make sense, because every car has some maximum speed, which is the speed beyond which it cannot accelerate.

2-p3 $v = \mathrm{d}x/\,\mathrm{d}t = 10 - 3t^2 = -17\ \mathrm{m/s}$.

2-p4 For both objects we have

$$a = kt,$$

where $k = 3.00\ \mathrm{m/s^3}$. Integration with respect to time gives

$$v = \frac{1}{2}kt^2 + v_0,$$

where the initial velocity v_0 is different for A than for B. A second integration results in

$$x = \frac{1}{6}kt^3 + v_0t + x_0,$$

with different values of x_0 for the two objects.

At $t = 2.00$ s, plugging in numbers results in:

$$\begin{aligned}&\text{(a) } v_A = 6.00 \text{ m/s}\\&\text{(b) } x_A = 4.00 \text{ m}\\&\text{(c) } v_B = -3.00 \text{ m/s}\\&\text{(d) } x_B = -9.00 \text{ m}\end{aligned}$$

(e) In this frame, A's velocity is zero, and B has the constant velocity $v_B - v_A = v_0$. The reason this works is that in the original frame of reference discussed in parts a-d, A and B have equal accelerations.

(f) This is easiest to determine in A's frame, rather than the frame assumed in parts a-d. In A's frame, B starts out to A's right, and travels leftward at constant velocity. A sits there for a while, and then B hits it.

2-p5 The velocity and acceleration are found by taking the first and second derivatives. Omitting units, these functions are:

$$\begin{aligned}v &= 3t^2 - 8t\\a &= 6t - 8.\end{aligned}$$

At the given times, we have:

t	v	a
-1	11	-14
0	0	-8
1	-5	-2
2	-4	4
3	3	10

If the particle is slowing down, its velocity and acceleration should have opposite signs. This occurs at $t = -1$ and $t = 2$.

2-p6 (a) Since b is added to t^2, b must have units of s^2. We want x to come out in units of meters, so a must have units of $\mathrm{m} \cdot \mathrm{s}^2$ (meters *times* seconds squared, not meters divided by seconds squared as in acceleration).

(b) From the given information, we can find her speed $v(t)$ by differentiation, and we can then find the maximum by the usual technique of setting the derivative equal to zero. That means taking two derivatives in a row: a second derivative. The second derivative of the position is the acceleration, so what it boils down to is finding the time when her acceleration is zero. (Note that the a in the original equation doesn't stand for acceleration.)

$$\begin{aligned}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[a\left(b+t^2\right)^{-1}\right]\\&= a\frac{\mathrm{d}}{\mathrm{d}t}\left[-\left(b+t^2\right)^{-2}(2t)\right]\\&= a\left[2\left(b+t^2\right)^{-3}(4t^2) - \left(b+t^2\right)^{-2}(2)\right]\\&= 0\end{aligned}$$

$$\begin{aligned}8\left(b+t^2\right)^{-3}(t^2) &= 2\left(b+t^2\right)^{-2}\\8t^2 &= 2\left(b+t^2\right)\\6t^2 &= 2b\\t &= \pm\sqrt{b/3}\end{aligned}$$

Plugging $t = \pm\sqrt{b/3}$ into the first derivative, we get $|v| = (3^{3/2}/8)ab^{-3/2}$.

(c) The $3^{3/2}/8$ is unitless, so the units come from the a and b factors: $\mathrm{m} \cdot \mathrm{s}^2 \cdot (\mathrm{s}^2)^{-3/2} = \mathrm{m/s}$.

2-p7 We expect the derivative to have units of m^3/s, since it represents a rate of change of a volume. The Leibniz notation confirms this, since $\mathrm{d}v/\mathrm{d}t$ represents an infinitesimal volume divided by an infinitesimal time. Computing the derivative, we get $4\pi c^3 t^2$, which has units of $(\mathrm{m/s})^3\mathrm{s}^2 = \mathrm{m}^3/\mathrm{s}$ as expected.

2-p8 (a) The input to the exponential function has to be unitless, so b must have units of s^{-1} (inverse seconds). The output of the exponential is also unitless, so if the right-hand side is to have units of meters, c must have units of m/s.

(b) Differentiating, we find that the velocity is

$$v = c(1 - bt)e^{-bt} \qquad .$$

We want to maximize this as a function of time, which we can do by taking its derivative and setting it equal to zero:

$$0 = bc(bt - 2)e^{-bt}$$

The maximum occurs at $t = 2/b$, and the velocity at this time ia $-ce^{-2}$, so the maximum speed is the absolute value of this, ce^{-2}. (Higher speeds occur at earlier times, but we're only finding the maximum speed as it comes back to the closed position.)
(c) We already know that c has units of m/s, so this does have the right units to be a velocity.

2-p9 (a) b: meters, c: meters/s (because you multiply it by t to get something with units of meters), k: units of s (because it doesn't make sense to take $e^{-t/k}$ unless the units of $-t/k$ cancel out)

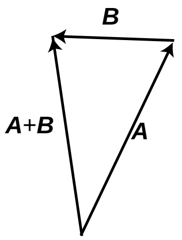
(b) $v = \mathrm{d}y/\mathrm{d}t = \mathrm{d}(-ct)/\mathrm{d}t + \mathrm{d}(-cke^{-t/k})/\mathrm{d}t = -c + ce^{-t/k}$

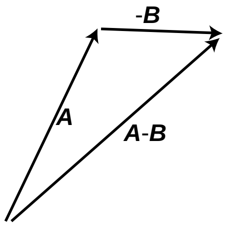
(c) For large values of t, $e^{-t/k}$ is nearly zero, so v approaches c. Thus, c can be interpreted as the maximum velocity she will ever reach (known as her terminal velocity).

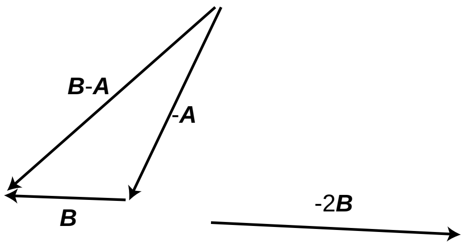
(d) Differentiating again, we get $a = \mathrm{d}v/\mathrm{d}t = \mathrm{d}(-c + ce^{-t/k})/\mathrm{d}t = -(c/k)e^{-t/k}$.

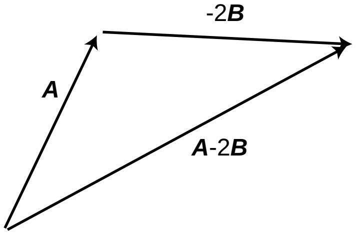
(e) For large t, the exponential becomes very small, so her acceleration is nearly zero.

3-a1









3-a2 (a) Let's use a coordinate system with positive x being east and positive y being north. In these coordinates, the vector from Bangkok to Phnom Penh has

$$\Delta x = 470 \text{ km} \qquad \Delta y = -250 \text{ km}$$

and the one from Phnom Penh to Hanoi has

$$\Delta x = 60 \text{ km} \qquad \Delta y = 1030 \text{ km} \qquad .$$

(b) The components of the vector from Bangkok to Hanoi equal the sum of the components of the two vectors above,

$$\Delta x = 530 \text{ km} \qquad \Delta y = 780 \text{ km} \qquad .$$

3-a3 Since the two given angles are both measured counterclockwise from east, we can just use a coordinate system with positive x being east and positive y north, and the positive and negative signs of the components will come out right if we just use cosines for x components and sines for y components. We find the components of the total Δr vector by adding the components of the two individual vectors:

$$\begin{aligned}\Delta x &= \Delta x_1 + \Delta x_2\\ &= (35\text{ km})\cos\ 25° + (22\text{ km})\cos 230°\\ &= 17.6\text{ km}\end{aligned}$$

(temporarily keeping 1 extra sig fig).

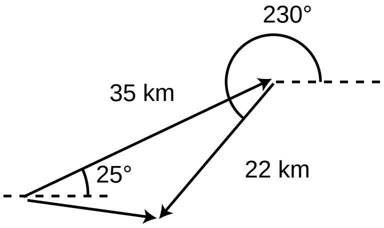
$$\begin{aligned}\Delta y &= \Delta y_1 + \Delta y_2\\ &= (35\text{ km})\sin 25° + (22\text{ km})\sin 230°\\ &= -2.1\text{ km}\end{aligned}$$

(temporarily keeping 1 extra sig fig).

The magnitude of the total Δr vector is 18 km, and the direction is

$$\tan^{-1}(-2.1\text{ km}/17.6\text{ km}) = -7° \text{ or } 173° \qquad .$$

Checking against the diagram, the correct arctangent is $-7°$.



3-a4 This problem has to be done analytically, not graphically, because it requires such high precision. The two given angles are both measured counterclockwise from the right, which is convenient; for angles defined this way, we can always find x components using a cosine, and y components using a sine, and the plus and minus signs will be come out automatically. The first vector's components are

$$\begin{aligned}\Delta x_1 &= (35.24\text{ mm})\cos 217.3°\\ &= -28.03\text{ mm}\\ \Delta y_1 &= (35.24\text{ mm})\sin 217.3°\\ &= -21.36\text{ mm} \qquad ,\end{aligned}$$

and the second one's are

$$\begin{aligned}\Delta x_2 &= (48.01\text{ mm})\cos 11.7°\\ &= 47.01\text{ mm}\\ \Delta y_2 &= (48.01\text{ mm})\sin 11.7°\\ &= 9.74\text{ mm} \qquad .\end{aligned}$$

Checking these against the figure, their sizes and signs look reasonable. Adding these would give the vector from the top hole to the right-hand hole, but she wants to go the opposite way, so the components we want are

$$\begin{aligned}\Delta x_3 &= -(\Delta x_1 + \Delta x_2)\\ &= -18.98\text{ mm}\end{aligned}$$

and

$$\begin{aligned}\Delta y_3 &= -(\Delta y_1 + \Delta y_2)\\ &= 11.62\text{ mm} \qquad .\end{aligned}$$

Again, the signs and sizes look about right. The distance she has to move is given by the Pythagorean theorem,

$$\begin{aligned}|\Delta\mathbf{r}_3| &= \sqrt{\Delta x_3^2 + \Delta y_3^2}\\ &= 22.25\text{ mm} \qquad .\end{aligned}$$

The direction is

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{\Delta y_3}{\Delta x_3}\right)\\ &= -31.4° \text{ or } 148.5°\end{aligned}$$

(measured counterclockwise from the right). We can tell from looking at the figure that the second arctangent is the correct one. (I've given the final answers with four sig figs of accuracy, although you could argue that it should be three, since the least accurate piece of given data is the 11.7° angle. In reality, however, that angle isn't any less accurate than the 217.3° one, i.e., changing either one by a tenth of a degree will have about the same effect on the final result.)

3-a5 To prove that this operation is not rotationally invariant, it suffices to find one example in which converting the inputs $\mathbf{A}$ and B into a different, rotated coordinate system causes the output $\mathbf{C}$ to be a completely different vector, not just the same vector represented in a different system of coordinates. Almost any example will show this, unless we really manage to shoot ourselves in the foot with an unusual choice like $\mathbf{A} = 0$, $B = 0$. As an example, let's take $B = 1$, and $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}$ in the original coordinate system, resulting in $\mathbf{C} = 2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$. Now rotate the coordinate system by 180 degrees about the z axis. In this new coordinate system, $\mathbf{A} = -\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$, but the scalar B stays the same, giving $\mathbf{C} = 0$. This new $\mathbf{C}$ is clearly not just the same vector expressed in a differently oriented coordinate system, since its magnitude has changed from a nonzero value to zero.

3-d1

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta_{ab}\\ \theta_{ab} &= \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)\\ &= 12.9^\circ\end{aligned}$$

3-d2 Let $\mathbf{v} = c\hat{\mathbf{x}} + d\hat{\mathbf{y}}$, where c and d are to be found. We don't expect to be able to determine both c and d uniquely, because for any $\mathbf{v}$ that works, so will any nonzero scalar multiple of it. That is, we only expect to be able to find the ratio c/d, which expresses how big c is in proportion to d.

We want $\mathbf{v}$ to be perpendicular to $\mathbf{u}$, so $\mathbf{u}\cdot\mathbf{v} = 0$, or $ac + bd = 0$. Solving for the unknown ratio gives $c/d = -b/a$ (assuming $a \neq 0$). Concretely, a choice that works is $c = b$ and $d = -a$, so that

$$\mathbf{v} = b\hat{\mathbf{x}} - a\hat{\mathbf{y}}.$$

(Writing it this way also works in the special case where $a = 0$.) To check that this is a solution, we can compute $\mathbf{u}\cdot\mathbf{v} = ab - ba = 0$, as desired.

3-g1 The easiest way to find such a vector is to take the cross product.

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})_x &= A_yB_z - B_yA_z = -3\\ (\mathbf{A}\times\mathbf{B})_y &= A_zB_x - B_zA_x = 6\\ (\mathbf{A}\times\mathbf{B})_z &= A_xB_y - B_xA_y = -3\end{aligned}$$

So our vector is $-3\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 3\hat{\mathbf{z}}$. We could also use any other vector lying along the same line, i.e., any scalar multiple of this one. I checked my result by computing its dot product with each of the two given vectors, verifying that it was zero in each case.

3-g2 (a) The cross product inside the parentheses results in a vector, and the second cross product results in a vector. (b) The cross product gives a vector, and then dotting it with the other vector gives a scalar. (c) The dot product gives a scalar, but you can't take a cross product of a scalar with a vector. This one is nonsense.

3-g3 (a) To find $2\mathbf{B}$ we double both the components of $\mathbf{B}$. Then to calculate $\mathbf{A} - 2\mathbf{B}$, we subtract those from the components of $\mathbf{A}$. This gives $\mathbf{A} - 2\mathbf{B} = (-7.0\hat{\mathbf{x}} - 28.0\hat{\mathbf{y}})$ meters. The pythagorean theorem results in a magnitude of 28.9 m.

(b) $\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = -33.0\ \text{m}^2$.

(c) Since $\mathbf{A}$ and $\mathbf{B}$ both lie in the x-y plane, the cross product will only have a z component: $(\mathbf{A}\times\mathbf{B})_z = A_xB_y - A_yB_x = 56.0\ \text{m}^2$.

(d) This is a little less work to compute if we first simplify the expression to $\mathbf{A}^2 - \mathbf{B}^2$, where the square indicates taking the dot product of the vector with itself, i.e., finding the square of its magnitude. Subtracting the squares of the magnitudes, we get $-144\ \text{m}^2$.

(e) $\theta = \cos^{-1}(\mathbf{A}\cdot\mathbf{B}/|\mathbf{A}||\mathbf{B}|) = 121^\circ$

3-g4 The definition of the vector cross product in terms of the components is $(\mathbf{A}\times\mathbf{B})_x = A_yB_z - B_yA_z$, and similarly for y and z. Reversing the roles of $\mathbf{A}$ and $\mathbf{B}$ interchanges the positive and negative terms on the right, so the sign flips.

3-g5 (a) False. An angle of zero would mean that the vectors were parallel, but unless $a = b$ it will not be true that these two vectors are parallel.

(b) True. If we arrange them as in the graphical definition of vector subtraction, they form a triangle.

(c) True. It doesn't matter if we flip $\mathbf{B}$, it can still be the same side of the same triangle.

(d) True. This is part of the definition of the cross product. (We could consider it to be an ex-

ception where the cross product is zero, in which case the angles we have in mind are undefined.)
(e) True. The dot product is defined as a component, and a component has a sign, which is defined to be negative in this situation.
(f) True. This is how unit vectors are defined.

3-g6 Since it's not associative in general, we expect that almost any set of three vectors, chosen at random, will provide a suitable counterexample. I decided to use combinations of the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, since they're easy to calculate with. The first example I tried was $\hat{\mathbf{x}}\times(\hat{\mathbf{y}}\times\hat{\mathbf{z}})$, which demonstrated Murphy's law by being one of those rare examples in which associativity *does* hold, because both $\hat{\mathbf{x}}\times(\hat{\mathbf{y}}\times\hat{\mathbf{z}})$ and $(\hat{\mathbf{x}}\times\hat{\mathbf{y}})\times\hat{\mathbf{z}}$ are zero. My next example worked: $\hat{\mathbf{y}}\times(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = \hat{\mathbf{y}}\times\hat{\mathbf{x}} = -\hat{\mathbf{z}}$, but $(\hat{\mathbf{y}}\times\hat{\mathbf{y}})\times\hat{\mathbf{z}} = 0\times\hat{\mathbf{z}}$ equals zero, which is different.

3-g7 Our geometrical definition of the cross product in three dimensions was that its magnitude was the area of the parallelogram formed by the two vectors, and its direction was perpendicular to both of them (and right-handed).

There is nothing problematic about extending the definition of the magnitude to four dimensions, because the parallelogram is formed within the two-dimensional plane containing the two vectors, so we never even “see” the existence of the fourth dimension.

But the definition of the direction fails to carry over properly, because the direction isn't uniquely determined. As a concrete example, let's label our axes x, y, z, and w. The cross product $\hat{\mathbf{x}}\times\hat{\mathbf{y}}$ would have to be a vector with magnitude 1 and a direction perpendicular to both $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. But there is more than one vector that would satisfy these requirements, e.g., either $\hat{\mathbf{z}}$ or $\hat{\mathbf{w}}$.

We also gave an expression for the cross product in terms of components. In three dimensions, this could also be used as a definition, and the choice of one or the other is a matter of style. However, it's conceivable that one of the definitions would generalize nicely to four dimensions while the other wouldn't. But it's not too hard to show that this doesn't work. Suppose someone tells you that the generalization produces a certain result for $\hat{\mathbf{x}}\times\hat{\mathbf{y}}$. This result would have to lie in the z-w plane. But even if the directions of the x and y axes have already been fixed, we still have freedom in defining our coordinate system so that z and w point in any directions we like, as long as they're perpendicular to x and y and to each other. That is, we can rotate the z and w axes freely. Therefore if someone wrote down a formula for the z and w components of $\hat{\mathbf{x}}\times\hat{\mathbf{y}}$, the formula would not be rotationally invariant.

Remark: It turns out that one can generalize the cross product to four dimensions, but the generalization can't be a vector. It is an object called a tensor, which is essentially a matrix.

3-g8 Since it's rotationally invariant, it must be built out of cross products, dot products, scalar products, and vector sums. Because its definition has to make sense regardless of the units of the vectors, the given information is enough to fix its behavior under scaling of the vectors. (I'll revisit this point below to make it more clear.) The given scaling information shows that if we write out an expression for $f(\mathbf{A},\mathbf{B})$, each term must contain three factors of $\mathbf{A}$ and a single factor of $\mathbf{B}$. Using only cross products, we can write several expressions that fit these criteria:

$$\begin{gathered}\mathbf{A}\times(\mathbf{A}\times(\mathbf{A}\times\mathbf{B}))\\ \mathbf{A}\times(\mathbf{A}\times(\mathbf{B}\times\mathbf{A}))\\ \mathbf{A}\times(\mathbf{B}\times(\mathbf{A}\times\mathbf{A}))\\ \mathbf{B}\times(\mathbf{A}\times(\mathbf{A}\times\mathbf{A}))\end{gathered}$$

Of these, the first two differ only in sign, while the third and fourth vanish identically. The first one fits all the desired criteria.

No other expression involving only cross products can fit the given criteria, since the only ones that have the right number of factors of $\mathbf{A}$ and $\mathbf{B}$ are the ones given, or trivial variations on them that can be put into one of these forms by using the anticommutativity of the cross product.

We also need to prove that no other expression exists that satisfies the given criteria and that is writen using some combination of dot products, cross products, scalar products, and vector sums. In fact, it is possible to write other expressions that satisfy the given criteria, but they are equivalent to the expression already found. Two examples of this type are $(\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\times\mathbf{A})$ and $(1/2)\mathbf{A}\times(\mathbf{A}\times(\mathbf{A}\times\mathbf{B}))+(1/2)(\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\times\mathbf{A})$.

In any term appearing in such an expression, there must be three multiplications. If none of these was a cross product, the result would be a scalar, and similarly if more than one was a dot product. Therefore the only possibility other than three cross products is a dot, a cross, and a scalar, as in the example already found. Rearranging the factors in every possible way fails to give anything that satisfies the given criteria.

It may be easier to understand now why it was necessary to assume that the function makes sense regardless of the units of the input vectors. We've shown that the function $A^2(\mathbf{B}\times\mathbf{A})$ satisfies the given criteria, where A^2 means $\mathbf{A}\cdot\mathbf{A}$. Suppose that we changed this to $(A^2+P(A^2))(\mathbf{B}\times\mathbf{A})$, where $P(x)$ is a polynomial having zeroes at $x=1$ and 4. This would also fit the given data. However, the condition $P(1)=0$ can only be meaningful if we define the units in which $x=1$.

3-j1 We want to find out about the velocity vector v_{BG} of the bullet relative to the ground, so we need to add Annie's velocity relative to the ground v_{AG} to the bullet's velocity vector v_{BA} relative to her. Letting the positive x axis be east and y north, we have

$$\begin{aligned} v_{BA,x} &= (140\ \text{mi/hr})\cos 45^\circ \\ &= 100\ \text{mi/hr} \\ v_{BA,y} &= (140\ \text{mi/hr})\sin 45^\circ \\ &= 100\ \text{mi/hr} \end{aligned}$$

and

$$\begin{aligned} v_{AG,x} &= 0 \\ v_{AG,y} &= 30\ \text{mi/hr}. \end{aligned}$$

The bullet's velocity relative to the ground therefore has components

$$\begin{aligned} v_{BG,x} &= 100\ \text{mi/hr} \qquad \text{and} \\ v_{BG,y} &= 130\ \text{mi/hr}. \end{aligned}$$

Its speed on impact with the animal is the magnitude of this vector

$$\begin{aligned} |v_{BG}| &= \sqrt{(100\ \text{mi/hr})^2+(130\ \text{mi/hr})^2} \\ &= 160\ \text{mi/hr} \end{aligned}$$

(rounded off to 2 significant figures).

3-j2 The boat's velocity relative to the land equals the vector sum of its velocity with respect to the water and the water's velocity with respect to the land,

$$\mathbf{v}_{BL}=\mathbf{v}_{BW}+\mathbf{v}_{WL}.$$

If the boat is to travel straight across the river, i.e., along the y axis, then we need to have $\mathbf{v}_{BL,x}=0$. This x component equals the sum of the x components of the other two vectors,

$$\mathbf{v}_{BL,x}=\mathbf{v}_{BW,x}+\mathbf{v}_{WL,x},$$

or

$$0=-|\mathbf{v}_{BW}|\sin\theta+|\mathbf{v}_{WL}|.$$

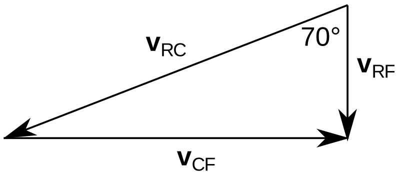
Solving for θ, we find

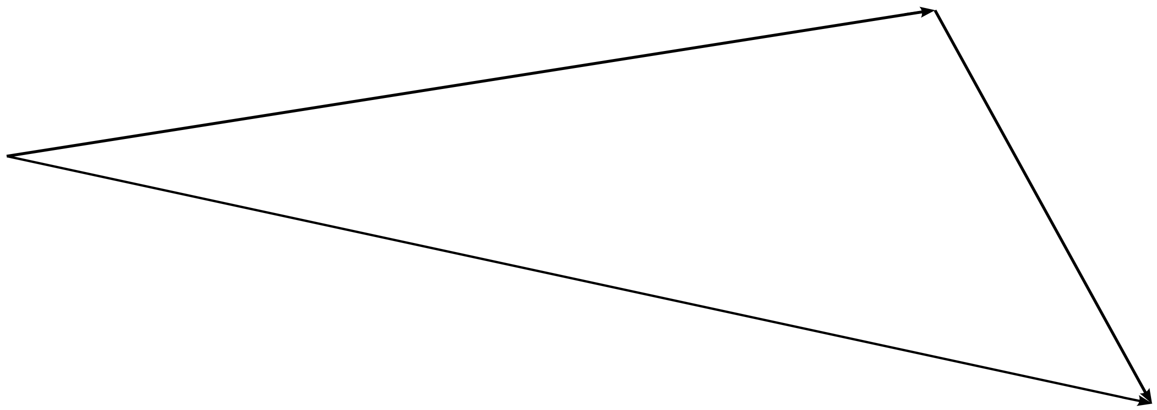
$$\sin\theta=|\mathbf{v}_{WL}|/|\mathbf{v}_{BW}|,$$

so

$$\theta=\sin^{-1}\frac{|\mathbf{v}_{WL}|}{|\mathbf{v}_{BW}|}.$$

3-j3 The addition of the three velocity vectors looks like this:





Problem 3-j7.

Here R is the rain, C is the car, and F is the freeway.

(a) For the magnitudes we have $v_{RF} = v_{CF}/tan 70° = 38$ km/hr.

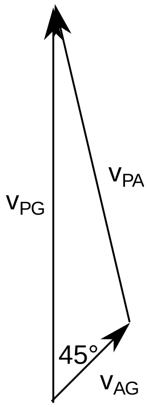
(b) $v_{CF} = v_{RC}/\sin 70° = 112$ km/hr

3-j4 (a) Let R be the rottweiler, C the collie, and D the dirt. Then $\mathbf{v}_{CD} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})v/\sqrt{2}$, $\mathbf{v}_{RD} = -v\hat{\mathbf{y}}$, and $\mathbf{v}_{RC} = \mathbf{v}_{RD} - \mathbf{v}_{CD}$, which results in

$$\mathbf{v}_{RC} = \left[-\frac{1}{\sqrt{2}}\hat{\mathbf{x}} - \left(1 + \frac{1}{\sqrt{2}}\right)\hat{\mathbf{y}}\right] v.$$

(b) $t = D/|v_{RC}| = D/\left[v\sqrt{2+\sqrt{2}}\right]$.

3-j5 (a) Let P be the plane, G the ground, and A the air. Then the situation is as shown in this figure:



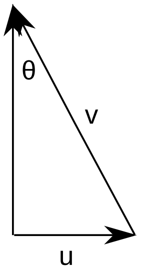
We need the x components of $\mathbf{u}$ and $\mathbf{v}$ to cancel out, so in terms of the magnitudes, $u \sin\theta = v \sin 45°$, giving

$$\theta = \sin^{-1}\left(\frac{1}{3\sqrt{2}}\right) = 14°$$

(b) $v_{\text{PG},x} = 0$, so all we need to calculate is

$$\begin{aligned} v_{\text{PG},y} &= v\cos 45° + u\cos 14° \\ &= 180 \text{ m/s} \end{aligned}$$

3-j6 The addition of the velocity vectors looks like this:



(a) We have $\theta = \sin^{-1}(u/v)$.

(b) The velocity of the plane relative to the ground is the left side of the triangle, $w = \sqrt{v^2 - u^2}$, so the time to get to the city is $t = D/w = D/\sqrt{v^2 - u^2}$.

(c) The units are $\text{m}/\sqrt{(\text{m}^2/\text{s}^2)}$, which does turn out to be seconds.

(d) If $u = v$ we get an infinite result. This makes sense, because in this situation the plane can only stand still by flying directly into the wind.

3-j7 See the figure. I chose a scale of 0.5 cm ↔ 10^{-7} m/s. The vector sum is 14.3 cm in length, which converts to 2.9×10^{-7} m/s. The direction, measured with a protractor, is about 12° south of east.

3-j8 To keep the writing simple, let u be the speed at which they swim, v their running speed, w the speed of the water, and ℓ the width of the river.

(a) Andrés's speed relative to the town is $\sqrt{u^2 - w^2}$, so $T_A = \ell/\sqrt{u^2 - w^2} = 25.0$ s.

(b) We have $t_1 = \ell/u$, and during this time Brenda moves downstream a distance $d = t_1 w$, so $t_1 = 20.0$ s.

(c) Running back to the town takes her $t_2 = t_1 w/v = \ell w/uv = 3.0$ s.

(d) $T_B = (\ell/u)(1 + w/v) = 23.0$ s.

3-j9 Let ℓ be the width of the river, and let César swim at an angle θ relative to the line cutting perpendicularly across the river. His total time is

$$\begin{aligned} T = t_1 + t_2 &= \frac{\ell}{u\cos\theta} + \frac{(w - u\sin\theta)\frac{\ell}{u\cos\theta}}{v} \\ &= \frac{\ell}{v}\left[\frac{v+w}{u}\sec\theta - \tan\theta\right]. \end{aligned}$$

It's simpler to find the angle that minimizes this expression if we define the unitless variable $y = Tv/\ell$ and the unitless constant $\beta = (v + w)/u$, so that

$$y = \beta\sec\theta - \tan\theta.$$

As claimed in the question, the optimal values of y and θ are only going to depend on the sum $v+w$ that appears in β, not on v or w individually.

The problem only asks that we prove this fact, but it may also be of interest to sketch the computation of the optimal time.

Setting $\mathrm{d}y/\,\mathrm{d}\theta = 0$ gives

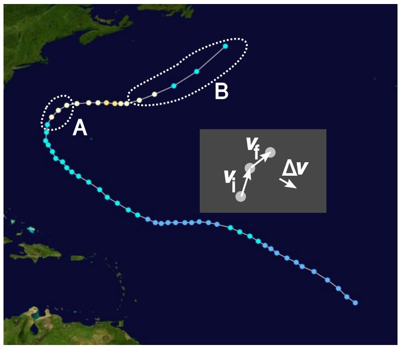
$$0 = \sec\theta\,[\beta\tan\theta - \sec\theta] \implies \theta = \sin^{-1}\beta^{-1}$$

Drawing a right triangle, we can see this gives $\sec\theta = \beta(\beta^2 - 1)^{-1/2}$ and $\tan\theta = (\beta^2 - 1)^{-1/2}$, and therefore $y = (\beta^2 - 1)^{1/2}$, or

$$T = \frac{\ell}{v}\sqrt{\left(\frac{v+w}{u}\right)^2 - 1}$$

3-m1 Yes, it would be slowing down.

3-m2 The velocity vector points in the direction of motion. There are two time periods during which the direction of motion is northeast:



During period A, around Aug. 15, the path is curving to the right. The $\Delta\mathbf{r}$ vectors connecting the points in dot-to-dot fashion can be thought of as synonyms for the $\mathbf{v}$ vectors, since $\mathbf{v} \approx \Delta\mathbf{r}/\Delta t$, and Δt is just a constant (6 hours). As shown in the inset figure, vector subtraction of these $\mathbf{v}$ vectors gives a $\Delta\mathbf{v}$ that points south-east, so the acceleration is to the southeast.

During period B, a similar vector subtraction gives an acceleration to the northwest, which is not what is required by the question.

3-m3 The original velocity vector had $v_x =$ 21.1 m/s, $v_y = 0$. The later one has $v_x =$ (21.1 m/s) × cos 7° = 20.94 m/s and $v_y =$ (21.1 m/s) × sin 7° = 2.57 m/s. The change in the velocity vector is

$$\begin{aligned} \Delta v_x &= -0.16 \text{ m/s} \qquad \text{and} \\ \Delta v_y &= 2.57 \text{ m/s} \qquad , \end{aligned}$$

so the acceleration vector has components

$$\begin{aligned} a_x &= \Delta v_x/\Delta t = -0.16 \text{ m/s}^2 \qquad \text{and} \\ a_y &= \Delta v_y/\Delta t = 2.57 \text{ m/s}^2 \qquad , \end{aligned}$$

giving a magnitude of

$$\sqrt{(0.16\ \text{m/s}^2)^2 + (2.57\ \text{m/s}^2)^2} = 2.6\ \text{m/s}^2 \qquad .$$

Since the x component is negative and the y component is positive, it has to be pointing northwest, and the angle measured counterclockwise from east is $\tan^{-1}(2.57/-0.16) = 92°$. (Your calculator probably gives you the other arctangent, $-88°$, which is not the appropriate one.)

3-m4 The acceleration is mainly vertical. Each car's v_y starts out zero, increases as the bump's slope gets steeper, reaches a maximum, and then decreases back to zero as the car reaches the top of the bump. Riding back down the bump is similar.

The equation $a = \Delta v/\Delta t$ isn't strictly true here, since a isn't constant. However, we can still use it to get a comparison between the two cars. We then have $a_P/a_M = (\Delta v_P/\Delta v_M)(\Delta t_M/\Delta t_P)$. The first factor is clearly equal to 37/25, but so is the second one, since the Maserati's lower speed causes it to take more time to go through the same motion. Therefore $a_P/a_M = (37/25)^2 = 2.2$.

3-p1 (a) The horizontal and vertical motions are independent, so they hit simultaneously. (b) While she's in the air, the only force on her is gravity. There is no horizontal force on acting on her, so her horizontal acceleration is zero, meaning she keeps a constant speed. Her speed on impact is the same as when she went over the edge, 30 km/hr. (c) The same as when he went over the edge, 40 km/hr. (d) They have the same vertical velocities, but different horizontal velocities, so Bill's speed is greater.

3-p2 In Miss Lettuce's frame of reference, the float is at rest and the sidewalk is moving. The bouquet is initially at rest, so when she drops it, it falls straight down and lands at her feet.

In her fans' frame, the float is moving and the sidewalk is at rest. The bouquet is initially moving to the right, and even after she drops it, it retains that horizontal motion, because there is no horizontal force that would change it. They see a parabolic arc going to the right.

3-s1 (a) The units of the first term are just the units of b, so b must have units of meters. The units of c have to be m/s, so that the seconds will cancel and give meters. Similarly d must have units of m/s^2, i.e., acceleration.

(b)

$$\begin{aligned}\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ &= c\hat{\mathbf{y}} + 2dt\hat{\mathbf{z}} \\ \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= 2d\hat{\mathbf{z}}\end{aligned}$$

(c) The only force acting on the bullet is gravity, so its acceleration $2d$ equals g, and $\hat{\mathbf{z}}$ points down. The bullet's velocity at $t = 0$ is $c\hat{\mathbf{y}}$, so c is its initial velocity, and $\hat{\mathbf{y}}$ points to the west. If the coordinate system is right-handed, then $\hat{\mathbf{x}}$ must point south, and b is simply the bullet's location in the north-south direction, measured from some arbitrary origin.

3-s2 (a) The horizontal and vertical motions are independent, so we can write down separate equations for them:

$$x = v_x t \qquad \text{[motion with constant velocity]}$$

$$y = y_\text{o} - \frac{1}{2}gt^2 \qquad \text{[motion with constant acceleration]}$$

We're not given information about time, and we don't want to find it either, so we need to eliminate it. Solving the first equation for t and substituting into the second equation gives

$$y = y_\text{o} - \frac{1}{2}g\left(\frac{x}{v_x}\right)^2 \qquad .$$

We're interested in $x = L$, and the variable d corresponds to $y_\text{o} - y$, so we have

$$d = \frac{1}{2}g\left(\frac{L}{v_x}\right)^2 \qquad .$$

(b) Convert everything to mks to start with. 73.3 mi/hr=32.8 m/s, 60.0 ft=18.3 m. Plugging in, we get $d = 1.53$ m=5.00 ft. I calculated this using the two-sig-fig value $g = 9.8$ m/s^2, so that ends up being the limiting factor for accuracy, and the result should be rounded off to two sig figs, $d = 5.0$ ft.

3-s3 (a) For the vertical motion, we have $h = (1/2)gt^2$, which gives the air time $t = \sqrt{2h/g}$. You need to travel a horizontal distance $h\tan\theta$ in that time, so $v = h\tan\theta/t = \sqrt{gh/2}\tan\theta$.

(b) The units are $\sqrt{\mathrm{m^2/s^2}} = \mathrm{m/s}$, which makes sense.

(c) A higher g means you have to run faster, which makes sense physically. A higher h also means you have to run faster, which makes sense, e.g., you can't jump off of the top of a mountain and land at the base, but you could easily jump from a small scale model of the mountain. A bigger θ requires a higher speed, which makes sense. In the case of $\theta = 0$, we have $v = 0$, which makes sense because you can just step off the cliff. For $\theta = 90°$, the velocity becomes infinite, and that also makes sense: the "cliff" is now just flat ground, and you can't jump off of Kansas.

(d) Plugging in numbers gives 1.8 m/s.

3-s4 (a) $R = (2v^2/g)\sin\theta\cos\theta$ (c) 45 °

3-s5 (a) Let $v_0 = 15$ m/s and $\theta = 45°$. We have $v_{0,y} = v_0\sin\theta$. The final velocity when it hits the ground is minus this, so we have $\Delta v = -2v_0\sin\theta$. The definition of acceleration $a = \Delta v/\Delta t$ then gives $\Delta t = \Delta v/(-g) = 2(v_0/g)\sin\theta = 2.2$ s.

(b) Now let's take "final" to be the top. Using $v_f^2 = v_i^2 + 2ay$, we find $y = v_0^2\sin^2\theta/2g = 5.7$ m.

(c) During the time Δt found in part a, the horizontal distance traveled is $v_x\Delta t = v_0 t\cos\theta = 2(v_0^2/g)\sin\theta\cos\theta = 23$ m.

3-s6 (a) For its motion from the hand to the top of its arc, it has a constant acceleration equal to g, so $v^2 = 2gy$, and $y = v^2/2g$. (b) Only the vertical component, $v_y = v\sin\theta$, of the initial velocity has any effect on its vertical motion, so the equation becomes $y = v^2\sin^2\theta/2g$.

3-s7 To find the time in the air, we can use the constant-acceleration equation $a = \Delta v/\Delta t$. The initial vertical velocity is $v_0\sin\theta$, and the change in velocity is therefore $-2v_0\sin\theta$, giving $\Delta t = 2v_0\sin\theta/g$. The distance traveled horizontally is $L = v_0\cos\theta t$. Plugging in $\theta = 45°$ gives $v_0 = \sqrt{Lg} = 1.4\times10^3$ m/s, which is high, but surely achievable for one as mighty as Kim.

3-s8 (a) The vertical and horizontal motions are independent of each other. Since the balls reach the same height, their vertical motion is the same, so the initial y velocity of the green ball is also v_0.

(b) The times are the same, for the same reasons as in part a.

(c) The x velocity is $v_x = \sqrt{(2v_0)^2 - v_0^2} = \sqrt{3}v_0$. The time in the air is $2v_0/g$. Multiplying these gives the range, $2\sqrt{3}v_0^2/g$, and since $v_0^2 = 2gH$, this range can be expressed in terms of H as $4\sqrt{3}H$.

3-s9 Let h be the height of the building. The vertical component of the impact velocity is given by $v_y^2 = 2gh$. Relating the magnitude of the impact velocity to its components, we have $(2v_0)^2 = v_0^2 + 2gh$, which gives $v_0 = \sqrt{2gh/3} = 8.08$ m/s.

3-s10 The horizontal motion is at the constant velocity v_0, while the vertical motion is with constant acceleration g. The vertical component of the impact velocity is given by $v_y^2 = 2gh$. Given the x and y components of the vector, we can use the inverse tangent function to find its direction. The result is

$$\theta = \tan^{-1}\left(\frac{\sqrt{2gh}}{v_0}\right)$$

3-s11 (a) Using the constant-acceleration equation $v^2 = 2a\Delta x$, we have

$$\begin{aligned} v_{0y}^2 &= 2g(y-h) \\ y &= \frac{v_{0y}^2}{2g} + h \\ &= \frac{v_0^2}{8g} + h \end{aligned}$$

(b) The strategy is that first we find the time spent in the air, and then we multiply by the horizontal velocity to find the distance traveled. For the time t at which the rock hits the ground, we have

$$-\frac{1}{2}gt^2 + v_{0y}t + h = 0.$$

Solving this for t gives a quadratic. Of the two roots, we want the one that happens after the throw. The result is

$$t = \frac{v_0}{2g}\left[1 + \sqrt{1 + \frac{8gh}{v_0^2}}\right].$$

Multiplying by $v_x = (\sqrt{3}/2)v_0$, we find

$$x = \frac{\sqrt{3}v_0^2}{4g}\left[1 + \sqrt{1 + \frac{8gh}{v_0^2}}\right].$$

3-s12 (a) Let positive x be to the right, and positive y up. The horizontal motion is at constant velocity $u\cos\alpha$, so

$$x = ut\cos\alpha.$$

The vertical motion has initial velocity $u\sin\theta$ and acceleration $-g$, so

$$y = ut\sin\alpha - \frac{1}{2}gt^2.$$

Setting $x = \ell$ in the first equation and solving for t gives $t = \ell/u\cos\alpha$. Now we set $y = h$ in the second equation and substitute for t, which gives

$$h = \ell\tan\alpha - \frac{1}{2}g(\ell/u\cos\alpha)^2.$$

Solving this for u results in

$$u = \sqrt{\frac{gl}{2\cos^2\alpha(\tan\alpha - h/\ell)}}.$$

(b) The ratio h/ℓ is unitless, so it makes sense to subtract it from $\tan\alpha$. The units of the quantity inside the square root come solely from the factor gl, which has units of m^2/s^2. Taking the square root gives m/s, which makes sense.

(c) Physically, if gravity is stronger we'll need to roll the ball faster to get it up to the hole. Mathematically, g is on top, so increasing it increases u.

(d) Physically, if the height is increased we'll need to increase the speed. Mathematically, h/ℓ is subtracted on the bottom, so increasing h reduces the bottom, which increases the result.

(e) If $\alpha = 90°$, then u blows up to infinity. This makes sense, because if the ball is launched straight up, then it doesn't matter how fast we roll it, it will never cover the horizontal distance h.

(f) When $\tan\alpha = h/\ell$, the ramp is pointed straight at the hole. If we were playing the game in zero gravity, this would be just fine, but when there is gravity, the path of the ball can't be straight, so there is no way it can get to the hole when it's aimed this way. Mathematically, our result for u blows up under these conditions, which makes sense.

(g) Plugging in gives $u = 4.1$ m/s.

3-s13 The time when it crosses the y axis is

$$t_y = -\frac{x_0}{v_x} = 5.0 \text{ s}.$$

It crosses the x axis at

$$t_x = -\frac{2v_{0y}}{a_x} = 1.6 \text{ s}.$$

Since it crosses the x axis first, we need to calculate where that happens.

$$x = x_0 + v_x t_x = 6.8 \text{ m}.$$

3-s14 (a) For motion along the ramp, $v_f^2 = v_0^2 - 2aL$, where L is the length of the ramp, so the height of the ramp is $h = L\sin\theta = (v_0^2 - v_f^2)/4a = 1.4$ m.

(b) For the time t when it hits the ground, $0 = -(1/2)gt^2 + v_{fy}t + h$, where $v_{fy} = v_f/2$. The quadratic has two solutions, and the one we want is the one for $t > 0$, which is

$$t = \frac{v_f + \sqrt{v_f^2 + 8gh}}{2g} = 0.70 \text{ s}.$$

3-s15 (a) Let the speed at launch be v_1. Then application of the constant-acceleration equation $v_f^2 = v_0^2 + 2ax$ leads immediately to $v_1 = \sqrt{v_0^2 - 2gL\sin\theta}$.

(b) Applying the same constant-acceleration equation to the motion in the air gives $y = L\sin\theta + v_1^2\sin^2\theta/2g$. Substituting for v_0 gives

$$y = L\sin\theta + \frac{v_0^2 - 2gL\sin\theta}{2g}\sin^2\theta.$$

3-s16 The problem is stated for a cylinder, but the minimum velocity will be one that is possible only in the case where the bullet flies in the plane of the figure, so we can reduce the problem to two dimensions.

The relevant physical conditions are the radius r of the cylinder and the strength g of the gravitational field. Based on units, the solution must be of the form $\sqrt{gr}$ multiplied by a unitless constant. Therefore we can simplify the math by taking $g = 1$ and $r = 1$, and waiting until the end of the problem to put the factor of $\sqrt{gr}$ back in.

There are two qualitatively different ways in which the bullet could fail to clear the cylinder. It could spend a finite time in the air and then hit, or it could simply fail to lift off. If we only consider the second condition, then there are several methods that allow a very short (even one-line) solution, but they involve tricks (such as approximating the bullet's motion as circular motion) or fancy math (such as matching the curvatures of the circle and parabola by setting the second-order terms of their Taylor series equal). We will instead take an approach that uses straightforward algebra and no tricks, and that addresses both possible modes of failure.

Let v be the initial speed of the bullet, and choose Cartesian coordinates with the usual orientation and the origin at the center of the circle. Then its motion is given by $x = vt$, $y = 1 - (1/2)t^2$, and eliminating t gives $y = 1 - (1/2)(x/v)^2$, the equation of the bullet's parabolic trajectory. For big enough values of v, this parabola will only intersect the circle at $x = 0$. For too-small values, it will intersect it in at least one other place. By symmetry, any such intersections will occur in pairs on the left and the right, but we consider only $x > 0$. If the bullet achieves lift-off but then hits the circle, then there should be two intersections for $x > 0$, one where the bullet enters the circle and one where it exits again at the bottom. If it never achieves lift-off, then there should be only one, where it exits.

To locate and count these intersections, we equate the equations of the circle and the parabola, $1-(1/2)(x/v)^2 = \pm\sqrt{1-x^2}$. The only positive solution is $x = 2v\sqrt{1-v^2}$, which exists if the square root is real. Since there is never more than one positive solution, we have shown rigorously that the only way in which we can fail is if the bullet never lifts off.

The solution fails to exist if $v > 1$, so the minimum speed is 1. Reinserting the factors that contain units, we have $v = \sqrt{gr}$ for the minimum speed.

3-s17 We have $v_{o,y} = u\sin\theta$, so ignoring signs, $\Delta v = 2_v o, y$, and the time spent aloft is $t = (2u/g)\sin\theta$. During this time, the horizontal distance traveled is $L = v_x t = (2u^2/g)\sin\theta\cos\theta$. Since we want to solve for θ, it is convenient to use a double-angle identity to rewrite this as $L = (u^2/g)\sin 2\theta$, which results in

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{Lg}{u^2}\right) .$$

Since the physically admissible values of θ must be between 0 and 90 degrees, the arcsine has to be between 0 and 180 degrees. In a typical case, there are two such arcsines, one in the first quadrant and the other in the second quadrant. The values of θ are such that $\theta_1 = 90° - \theta_2$.

When $L > u^2/g$, there are no solutions: the target is out of range. When $L = u^2/g$, there is exactly one solution, and $\theta = 45°$.

Regarding units, the quantity inside the arcsine is unitless. Call this x. From the given variables L, g, and u, the only unitless quantities we can form are x and functions of x. Since θ is unitless (radians are not really units), we can tell from units that θ can depend only on x, but we can't tell what form this dependence will take.

4-a1 (a) The force of gravity on an object can't just be g, both because g doesn't have units of force and because the force of gravity is different for different objects.

(b) The force of gravity on an object can't just be m either. This again has the wrong units, and it also can't be right because it should depend on how strong gravity is in the region of space where the object is.

(c) If the object happened to be free-falling, then the only force acting on it would be gravity, so by Newton's second law, $a = F/m$, where F is the force that we're trying to find. Solving for F, we have $F = ma$. But the acceleration of a free-falling object has magnitude g, so the magnitude of the force is mg. The force of gravity on an object doesn't depend on what else is happening to the object, so the force of gravity must also be equal to mg if the object doesn't happen to be free-falling.

4-a2 (a) This is a measure of the box's resistance to a change in its state of motion, so it measures the box's mass. The experiment would come out the same in lunar gravity.
(b) This is a measure of how much gravitational force it feels, so it's a measure of weight. In lunar gravity, the box would make a softer sound when it hit.
(c) As in part a, this is a measure of its resistance to a change in its state of motion: its mass. Gravity isn't involved at all.

4-a3 (a) A liter of water has a mass of 1.0 kg. The mass is the same in all three locations. Mass indicates how much an object resists a change in its motion. It has nothing to do with gravity. (b) The term "weight" refers to the force of gravity on an object. The bottle's weight on earth is $F_W = mg = 9.8$ N. Its weight on the moon is about one sixth that value, and its weight in interstellar space is zero.

4-a4 The partner's hands are not touching the climber, so they don't make any force on him. The hands have an indirect effect through the rope, but our concept of force only includes direct effects (see p. 43).

The corrected table looks like this:
force of the earth's gravity, ↓
force from the rope, ↑

The student is also wrong to claim that the upward and downward forces are unbalanced. The climber is moving down at constant speed, so his acceleration is zero, and the total force acting on him is zero. The upward and downward forces are of equal strength, and they cancel.

4-a5 $a = \Delta v/\Delta t$, and also $a = F/m$, so

$$\begin{aligned}\Delta t &= \frac{\Delta v}{a} \\ &= \frac{m\Delta v}{F} \\ &= \frac{(1000\ \text{kg})(50\ \text{m/s} - 20\ \text{m/s})}{3000\ \text{N}} \\ &= 10\ \text{s}\end{aligned}$$

4-a6 No, we can only conclude that if there are forces acting on it, their sum is zero.

4-a7 We have $x = (1/2)at^2$, so $x \propto t^2$ for the cumulative distance. This gives $x_2/x_1 = 4$, so $x_2 = 8.0$ m, and the additional distance traveled in the second time interval is 6.0 m.

4-d1 $F = ma = kt$, so $a = (k/m)t$. Integrating once gives its velocity, $v = (k/2m)t^2 + v_o$. The constant of integration, v_o, equals zero, because the initial speed is zero, so $v = (k/2m)t^2$. Integrating again gives $x = (k/6m)t^3 + x_o$.

4-g1 (a) The maximum acceleration is 5 N, so $F_{\text{max}} = ma_{\text{max}} = 5$ N.
(b) The change in velocity Δv equals the area under the curve, which is 15 m/s. For the average force, we have $F_{\text{av}} = ma_{\text{avg}} = m\Delta v/\Delta t = 3.8$ N.

4-g2 (a) Newton's second law says that $a = (F_s - W)/m$, where F_s is the force from the scale pushing up on her feet. Let the positive direction be upward. We're given that the elevator is slowing down while rising, so $a < 0$ in this coordinate system, and we must have $F_s < W$, so $F_s = 0.97W$.
(b) Plugging in numbers, we get $a = -.03W/m = -.03g = -.3\ \text{m/s}^2$. The magnitude of the acceleration is $.3\ \text{m/s}^2$. (The sign is only negative for the particular coordinate system we chose.)

4-g3 (a) The velocity is constant, so Newton's first law applies: the total force is zero. This means that the upward force of the elevator on the person's feet is the same strength as the downward force of gravity, or 890 N.
(b) Newton's second law gives $a = (F_s - mg)/m$, where F_s is the force from the scale on the person's feet. We have $F_s = m(g + a)$, and g and a are opposite in sign, so the result is 440 N.

4-g4 (a) We have $v^2 = 2aL$ for the average acceleration, so the average force is $F = ma = mv^2/2L$.
(b) The units on the right-hand side are $\text{kg}\cdot(\text{m/s})^2/\text{m} = \text{kg}\cdot\text{m/s}^2$, which is the same as newtons.
(c) Mathematically, increasing m increases F. That makes sense, because more force would be required in order to accelerate a more massive bullet. The dependence on v also makes sense, because accelerating the bullet up to a higher speed should take more force. Raising L in our answer would lower F, and that makes sense, because if we have a greater distance over which to accelerate the bullet, we can do it more gradually.

4-g5 Let F_f be the magnitude of the frictional force. Newton's second law gives $a = (F - F_f)/M$, and setting this equal to $F/3M$ gives $F_f = (2/3)F$.

4-g6 (a) We have $m = F/a$, so $m \propto 1/a$. Since $a_1 > a_2$, we must have $m_1 < m_2$.
(b) For the acceleration when the force is applied to the combined mass, we have

$$\begin{aligned} a &= \frac{F}{m_1 + m_2} \\ &= \frac{F}{F/a_1 + F/a_2} \\ &= (1/a_1 + 1/a_2)^{-1} \end{aligned}$$

4-j1 The mass is increased by a factor of 1.5, so the acceleration is reduced by a factor of 1.5, to 2 m/s^2.

4-j2 For motion with constant acceleration, starting from rest, we have $\Delta x = (1/2)at^2$. For a fixed Δx, this becomes $t \propto a^{-1/2}$. Newton's second law gives $a \propto F$, so $t \propto F^{-1/2}$. The result is

$$\frac{t_1}{t_2} = \left(\frac{F_1}{F_2}\right)^{-1/2} = (1/1.9)^{-1/2} = 1.4$$

4-j3 For motion with constant acceleration, starting from rest, we have $v^2 = 2a\Delta x$, or $v \propto a^{1/2}$. Newton's second law gives $a \propto m^{-1}$, so $v \propto m^{-1/2}$. The result is

$$\frac{v_2}{v_1} = \left(\frac{340\ \text{kg} + 60\ \text{kg}}{340\ \text{kg}}\right)^{-1/2} = 0.92$$

4-j4 (a) $F_W = mg$, so reducing g by 0.01 m/s^2 reduces his weight by

$$72\ \text{kg} \times 0.01\ \text{m/s}^2 = 0.72\ \text{N}$$

His mass stays the same.

(b) For motion with constant acceleration, $v_f^2 = v_i^2 + 2ax$. At the top of his jump, his vertical velocity is zero, so $v_f = 0$, and $0 = v_i^2 + 2ax$. His acceleration equals $-g$. Solving for x, the height to which he can jump is, $x = v_i^2/2g$. Since the height to which he can jump is inversely proportional to g, a 0.1% decrease in g will result in a 0.1% increase in height, or 2 mm.

4-j5 For the average acceleration a we have $v^2 = 2a\Delta x$, so $a \propto 1/\Delta x$, and therefore $F \propto 1/\Delta x$ as well.

$$\frac{F}{W} = \frac{\Delta x_1}{\Delta x_2} = 350$$

(This is actually a calculation of the *total* force acting on the person during the impact, so we could say that the force from the ground was $351W$. But to the 2 sig fig precision of the problem, this is irrelevant.)

4-j6 We have $x = (1/2)at^2$, so $x \propto a \propto 1/m$. The heavier book travels half the distance, or 1.0 m.

4-m1 (a) The swimmer's acceleration is caused by the water's force on the swimmer, and the swimmer makes a backward force on the water, which accelerates the water backward. (b) The club's normal force on the ball accelerates the ball, and the ball makes a backward normal force on the club, which decelerates the club. (c) The bowstring's normal force accelerates the arrow, and the arrow also makes a backward normal force on the string. This force on the string causes the string to accelerate less rapidly than it would if the bow's force was the only one acting on it. (d) The tracks' backward frictional force slows the locomotive down. The locomotive's forward frictional force causes the whole planet earth to accelerate by a tiny amount, which is too small to measure because the earth's mass is so great.

4-m2 Newton's third law says their forces on each other are equal. Since $a = F/m$, the less massive old lady has a greater acceleration.

4-m3 When you drop an object, the earth does accelerate up to meet it, but the acceleration is very small because the earth's mass is so big.

4-m4 Yes, they are equal and opposite, because otherwise the total force on you would not be zero, and you would accelerate vertically. No, Newton's third law does not apply, because there are three objects involved, not just two: the earth, your feet, and the floor. Newton's third law says that two objects' forces on each other are equal and opposite, so it never applies to two forces both acting on the same object.

4-m5 Their claim doesn't make sense. The branch isn't accelerating, so evidently the total force on it is zero. The two sides of the shears must be making equal forces on the two sides of the branch, in opposite directions. This has to be true regardless of the type of shears.

4-m6 The scale reads too high when the backpack is on the way down and decelerating, and also when it's on the way up and accelerating. It reads too low when the backpack is on the way up and decelerating, and also when it's on the way down and accelerating. In other words, when I'm making an upward force on the backpack that's greater than the force required to support its weight, Newton's third law says the backpack must be making a correspondingly large force downward on me, which registers on the scale. Conversely, when I make an unusually small upward force on the pack, it makes an unusually small downward force on me.

4-p1 (a) We are supposed to find an equation for mass based on information about forces and accelerations. Newton's second law, $a = F_{total}/m$, is what relates these three quantities. There are two forces acting on the elevator: a downward force from gravity and an upward force from the cable. Let's use positive numbers for upward forces and negative numbers for downward ones. The force of gravity will then be $-mg$, and the maximum force the cable can exert will be $+T$, giving $F_{total} = -mg + T$. Substituting this into Newton's second law gives

$$a = \frac{-mg+T}{m} \qquad .$$

Solving for m gives

$$m = \frac{T}{a+g} \qquad .$$

(b) If $a = 0$, then the equation becomes $m = T/g$, i.e. $T = mg$. Interpretation: if the elevator is not accelerating, then the maximum mass is the mass that results in a gravitational force as great as the maximum tension in the cable. This is intuitively obvious if the elevator is at rest, but perhaps a little surprising for the case where the elevator is, say, moving upward at constant speed. Most people would not expect that the tension in the cable would be the same whether the car was at rest or moving upward at constant speed.

If the car is accelerating downward at 9.8 m/s^2, that would be $a = -g$, since we are using positive for up and negative for down. Plugging this in to the equation gives $m = \infty$ (division by zero). Interpretation: this is the case where the elevator is simply dropping like a rock. The cable does not need to exert any force in this situation, and so there is no limit on the mass of the car.

4-p2 Let's use a coordinate system in which positive x is down. Then the downward force of gravity on Sally is positive, and the blaster's upward force on Sally is negative. (a)

$$\begin{aligned} a &= \frac{F_{total}}{m} \\ &= \frac{mg - F_B}{m} \\ &= g - F_B/m \\ h &= \frac{1}{2}at^2 \\ t &= \sqrt{\frac{2h}{a}} \\ &= \sqrt{\frac{2h}{g - F_B/m}} \end{aligned}$$

(b) The units of g are m/s^2. F_B has units of newtons, and dividing newtons by kilograms gives m/s^2 (as in Newton's second law). In the denominator, we're subtracting m/s^2 from m/s^2. That's good, because you can't subtract things that have different units. Dividing h, with units of m, by m/s^2 gives units of s^2, and taking the square root of that gives units of seconds, which matches up correctly with the units of the variable t on the left side.
(c) For large values of F_B, the quantity inside the square root becomes negative, and the result for the time becomes an imaginary number. The point where this bogus behavior sets in is the point at which the denominator goes from positive to negative, i.e., when the denominator is zero: $g - F_B/m = 0$, or $F_B = mg$. In other words, if F_B is greater than the force of gravity, she'll fly up, and will never even hit the ground.

4-p3 (a) Newton's second law says $F_{air} - F_{weight} = ma$, or $F_{air} - mg = ma$. Since the forces are constant, this will be motion with constant acceleration, so $v = at = (F_{air} - mg)t/m$.
(b) 7.8 m/s

4-p4 (a)

$$\begin{aligned} v_f^2 &= v_o^2 + 2a\Delta x \\ 0 &= v^2 + 2ad \\ a &= -\frac{v^2}{2d} \\ F &= ma \\ &= -\frac{mv^2}{2d} \end{aligned}$$

(b) The quantity $-\frac{mv^2}{2d}$ has units of

$$\frac{\text{kg}\cdot\text{m}^2/\text{s}^2}{\text{m}} = \text{kg}\cdot\text{m/s}^2 \qquad ,$$

which are the same as the units of force (mass multiplied by acceleration).
(c) If m gets bigger, F gets bigger. That makes sense, because more force would be required in order to stop a more massive object in the same distance. If v increases, so does F, and that also makes sense. Dividing by a bigger number gives a smaller result, so if d gets bigger, F gets smaller. That also makes sense, because a smaller force would take more distance to stop a moving object.
(d) The only thing to be careful about here is that if we want an answer in SI (meter-kilogram-second) units, we have to make sure to convert all the raw data to SI before plugging in:

$$\begin{aligned} F &= -\frac{mv^2}{2d} \\ &= -\frac{\left(6.7\times10^{-27}\ \text{kg}\right)\left(2.0\times10^{7}\ \text{m/s}\right)^2}{2\left(7.1\times10^{-4}\ \text{m}\right))} \\ &= -1.9\times10^{-9}\ \text{N} \end{aligned}$$

4-p5 Taking the car and truck to be a single object, Newton's second law results in $a = F_R/4M$. Let F_B the force between the bumpers of the car and the truck. Applying the second law to the truck along gives $F_B = 3M \cdot a = (3/4)F$.
4-p6 (a) Let, e.g., F_{23} mean 2's force on 3. By Newton's first law, the total force on block 3 is zero, so $F_{23} - M_3g = 0$, so $F_{23} = M_3g$. Newton's

third law then gives $F_{32} = -F_{23} = -M_3 g$.
(b) Applying Newton's first law to block 2, we have $F_{12} + F_{32} - M_2 g = 0$, so $F_{12} = (M_2 + M_3)g$. This makes sense because 1 is supporting the entire weight of 2 and 3.

4-s1 (a) Friction would be parallel to the ceiling, but all the motion is vertical. (b) If the rice were to stay stuck onto the ceiling of the cup, it would have to accelerate along with the cup. When you're banging on something like this the acceleration can be huge, perhaps thousands of meters per second squared. This is much more than anything gravity could supply. (c) The cup is the only object touching a grain of rice that could be exerting a force on it, and Newton's second law would then require that the this force equal ma. As the cup decelerates, we therefore need a large upward force from the cup on the rice if the rice is to decelerate along with it. This would be some kind of sticky force, and there's a limit on how strong such a force can be. If it isn't strong enough, then the grain's acceleration can't match that of the cup, and it will come loose.

4-s2 The acceleration a in Newton's second law is the acceleration of the center of mass. The atoms at the point of contact between the ball and the bat move together and have equal accelerations, but that doesn't mean the centers of mass have equal accelerations. The ball and the bat both start squashing on impact.

5-a1

5-a2

5-a3

5-a4

5-a5

5-a6 If kinetic friction were stronger than static friction, then we could have, for example, a situation in which 4 N was the force necessary to break static friction and make an object slide across a horizontal surface, but 6 N was necessary in order to produce a force greater than kinetic friction in order to produce an acceleration. This wouldn't make sense, because what would happen if you applied 5 N? It would break static friction, but then there would be no acceleration because the force would be weaker than kinetic friction. The way you would actually measure a static frictional force experimentally would be to measure how much force you had to apply in order to get the object to accelerate, so there's no way you could have determined the 4 N force for static friction in the first place.

5-d1 In each case, the y component, T_y, of the tension has to be enough to support half the boy's weight. The y component of the tension is going to be given by either the sine or the cosine, and we can see that it must be the cosine because that gives the right answer for the first positoin, $T_y = T\cos\theta = T$. Since T_y is always the same, the tension must be proportional to $1/\cos\theta$. The tensions are in the ratios of 1 to $1/\cos 45°$ to $1/\cos 60°$, or 1 to 1.41 to 2.

5-d2 (a) Let the positive x axis be to the right and positive y up. The x components of the forces cancel by symmetry, so they don't give us any useful information. The y components of the three forces are $S\cos\theta/2$, $S\cos\theta/2$, and $-L$. Setting the sum of these equal to zero, we find $S = L/(2\cos\theta/2)$. (b) For $\theta = 0$, we find $S = L/2$, which makes sense physically because two forces S are cooperating perfectly to cancel L. (c) For $\theta = 180°$, we have $S = \infty$, which is not physically possible, but does make sense. If we were to grab the two anchor lines in our hands and pull them to the right and left away from each other, it would take infinite force to make $\theta = 180°$ exactly. (c) This occurs when $2\cos\theta/2 = 1$, i.e., $\theta = 2\cos^{-1}(1/2) = 120°$.

5-d3 (a) Let positive x be to the right and positive y up. Zero total force acts on the carabiner in the y direction,

$$-L + 2T\cos(\theta/2) = 0 \qquad .$$

Zero total force also acts on the left-hand anchor, so

$$\begin{aligned} S_y - T\cos(\theta/2) &= 0 \qquad \text{and} \\ S_x + T + T\sin(\theta/2) &= 0 \qquad . \end{aligned}$$

Algebra leads to the result

$$S = L\frac{\sqrt{(1+\sin(\theta/2))/2}}{\cos(\theta/2)} \qquad .$$

Problem 5-a1:

force acting on magnet			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
magnetism	up	car	magnetism	down	magnet
gravity	down	planet earth	gravity	up	magnet
normal	down	car	normal	up	magnet

Problem 5-a2:
(a) A monkey climbing a palm tree.

force acting on monkey			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
normal	right	tree	normal	left	monkey (right hand)
normal	left	tree	normal	right	monkey (left hand)
gravity	down	planet Earth	gravity	up	monkey
static friction	up	tree	static friction	down	monkey (right hand)
static friction	up	tree	static friction	down	monkey (left hand)

Problem 5-a2:
(b) A piece of tape stuck on the ceiling.

force acting on tape			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	tape
sticky	up	ceiling	sticky	down	tape

Problem 5-a3:

force acting on rower			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
normal	↑	seat	normal	↓	rower
gravity	↓	Earth	gravity	↑	rower
normal	→	oar	normal	←	rower
static friction	←	seat	static friction	→	rower
normal	↖	footrest	normal	↘	rower

Problem 5-a4:

force acting on farmer			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
static friction	up	cow	static friction	down	farmer
static friction	up	wall	static friction	down	farmer
gravity	down	planet earth	gravity	up	farmer
normal	left	cow	normal	right	farmer
normal	right	wall	normal	left	farmer

Problem 5-a5:

(The "lift" force below is really not so different from a normal force.)

force acting on plane			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
normal?	east	air	normal?	west	plane (propeller)
friction	west	air	friction	east	plane
lift	up	air	lift	down	plane (wing)
gravity	down	earth	gravity	up	plane

(b) When $\theta = 0$, we have $S = L/\sqrt{2}$, which makes sense because in this situation $T = L/2$, and each anchor experiences two forces equal in magnitude to T, acting at right angles. This is worse than the correct setup, in which $\theta = 0$ gives $S = L/2$.

5-d4 (a) There are three forces acting on the part of the rope right under the person's feet: the two tensions and the normal force of the person's feet on the rope, which equals the person's weight. The vector sum of the three forces must be zero, i.e. their x components must add up to zero and so must their y components. The x components will just naturally cancel out by symmetry, so that's not helpful information. The y components of the two tension forces are both equal to each other, so double the y component of the tension in one side must equal the person's weight. The y component of the tension in one side is $T\sin\theta$, where θ is the angle of the rope above horizontal. Therefore, we must have $Mg = 2T\sin\theta$. The length of each half of the tightrope is given by the Pythagorean theorem, $\sqrt{h^2+(L/2)^2}$, and using trig, $\sin\theta = h/\sqrt{h^2+(L/2)^2} = 1/\sqrt{1+(L/2h)^2}$. Solving for T, $T = Mg/(2\sin\theta) = \frac{1}{2}Mg\sqrt{1+(L/2h)^2}$.
(b) Examining the equation $T = Mg/(2\sin\theta) = \frac{1}{2}Mg\sqrt{1+(L/2h)^2}$, if h was zero, $L/2h$ would be infinite, so the tension would have to be infinite. Physically, the rope would break at some nonzero value of h.

5-d5 (a) If there was no friction, the angle of repose would be zero, so the coefficient of static friction, μ_s, will definitely matter. We also make up symbols θ, m and g for the angle of the slope, the mass of the object, and the acceleration of gravity. The forces form a triangle just like the one in example **??** on p. **??**, but instead of a force applied by an external object, we have static friction, which is less than $\mu_s|\mathbf{F}_N|$. As in that example, $|\mathbf{F}_s| = mg\sin\theta$, and $|\mathbf{F}_s| < \mu_s|\mathbf{F}_N|$, so

$$mg\sin\theta < \mu_s|\mathbf{F}_N|.$$

From the same triangle, we have $|\mathbf{F}_N| = mg\cos\theta$, so

$$mg\sin\theta < \mu_s mg\cos\theta.$$

Rearranging,

$$\theta < \tan^{-1}\mu_s.$$

(b) Both m and g canceled out, so the angle of repose would be the same on an asteroid.

5-d6 The main reason that this problem is difficult is that you have to realize there are two separate cases. Pressing the block and the wall together can result in either an upward or a downward frictional force on the block. If you're applying almost enough force to slide the block up the wall, then the frictional force is downward, but if you're applying just barely enough to keep it from sliding down, then the frictional force is upward. Because the vertical component of the frictional force has an opposite signs in the two cases, it is necessary to do a separate version of the algebra in each case.

Case A: upward friction on the block

Let the positive x axis be to the right and the positive y axis up. Then both the x and y components of the force on the block must equal zero. Let $\mathbf{F}_N$ be the normal force of the wall on the block. The maximum force of static friction is given by $|\mathbf{F}_{s,max}| = \mu_s|\mathbf{F}_N|$, and it is the maximum in which we're interested, since we're looking for the largest possible value of $|\mathbf{F}_H|$.

zero total force in the x direction:

$$|\mathbf{F}_H|\cos\theta - |\mathbf{F}_N| = 0$$

zero total force in the y direction:

$$|\mathbf{F}_H|\sin\theta - mg + \mu_s|\mathbf{F}_N| = 0$$

We solve the first equation for $|\mathbf{F}_N| = |\mathbf{F}_H|\cos\theta$, then substitute into the second equation and solve to find

$$|\mathbf{F}_H| = \frac{mg}{\sin\theta + \mu_s\cos\theta}$$

Case B: downward friction on the block

The solution of this part is exactly analogous, except that the total force in the y direction is $|\mathbf{F}_H|\sin\theta - mg - \mu_s|\mathbf{F}_N| = 0$, with a minus sign in front of the frictional force. The result is

$$|\mathbf{F}_H| = \frac{mg}{\sin\theta - \mu_s\cos\theta}$$

In the case of $\theta = 90°$, both the maximum and the minimum are mg, which makes sense. The maximum force blows up to infinity for $\theta = \tan^{-1}\mu_s$. With this angle, no amount of force will make the block slip, because the greater normal force leads to a greater frictional force, which is great enough to counteract the greater vertical force from the hand. For angles smaller than this, the equation for the maximum force gives a negative result, which is not physically meaningful because it was supposed to represent the magnitude of the force.

5-d7 (a) The tension at the center describes a force that each half of the wire exerts on the other half. Therefore we should analyze the forces on one half of the wire, say the left half.

(Here "cohesive" is just a name I made up for the force that holds the wire together and keeps it from breaking.)

Let the magnitude normal force be F_N, and let the tension in the middle of the wire be T_m. By Newton's first law, the total force acting on the left half of the wire must be zero. Let's do analytic addition, which means adding components. Let the positive x axis be to the right, and the positive y axis up. Then adding the horizontal components gives

$$T_m - F_N\cos\theta = 0 \qquad ,$$

and adding the vertical ones gives

$$F_N\sin\theta - \frac{1}{2}mg = 0 \qquad .$$

The result is $T_m = mg/2\tan\theta$.

(b) The tension at the ends equals F_N. Solving the original two equations for this unknown, we have $F_N = mg/2\sin\theta$. Since $\sin\theta$ is always less than $\tan\theta$ for angles between 0 and 90°, we find $F_N > T_m$. The tension at the ends is greater than in the middle. This is easy to verify intuitively by imagining the case of $\theta \approx 90°$.

5-d8 Let a small section of the rope, extending from $-\mathrm{d}\theta/2$ to $+\mathrm{d}\theta/2$, have tension $T - \mathrm{d}T/2$ and $T + \mathrm{d}T/2$ applied at its ends. Within this section, there are normal forces that point in different directions, but their directions are so similar that we can consider them as a single force of

Problem 5-d7:

force acting on left half of wire			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet earth	gravity	up	left half
cohesive	right	right half	cohesive	left	left half
normal	up and left	hook	normal	down and right	left half

magnitude dN acting in the direction normal to the post at $\theta = 0$. Since the total force in the normal direction must be zero, we have $\mathrm{d}N = (T - \mathrm{d}T/2)\sin(\mathrm{d}\theta/2) + (T + \mathrm{d}T/2)\sin(\mathrm{d}\theta/2)$. Applying the small-angle approximation $\sin x \approx x$, we have $\mathrm{d}N = T\,\mathrm{d}\theta$. The total force in the tangential direction must also be zero, so $\mu_s\,\mathrm{d}N = \mathrm{d}T$. Combining the two results, we find $\mu_s T\,\mathrm{d}\theta = \mathrm{d}T$. Separating variables, we have $\mathrm{d}T/T = \mu_s\,\mathrm{d}\theta$. Integration of both sides gives $\ln T = \mu_s\theta + c$, which upon exponentiation of both sides gives $T_2/T_1 = e^{\mu_s\theta}$

5-d9 (a) Let the mass of the block be M. The downward force Mg of gravity on it is canceled by the upward normal force from the ledge, so the strength of the normal force is $F_N = Mg$. The maximum force of static friction is $\mu_s F_N = \mu_s Mg = \mu_s\rho Vg$. Since we assume there is no friction between the rope and the lip of the cliff, this is the maximum tension that can be sustained in the rope. Equating this to the climber's body weight gives $\mu_s = m/\rho V$.

(b) The units are mass over mass, which cancels out. That makes sense, because the definition of μ_s implies that it is unitless.

(c) If m goes up, our equation says that μ_s also goes up. That makes sense, because a heavier climber would need more friction. If ρ or V gets bigger, the result gets smaller. That also makes sense, because increasing either of these would make the block heavier, which would allow it to hold without such a high coefficient of friction. (d) The result is 0.04, which is smaller by an order of magnitude than the reference value for granite on granite. This is a large margin of safety.

5-d10 (a) There is no theoretical limit on how much normal force F_N the climber can make on the walls with each foot, so the frictional force can be made arbitrarily large. This means that with any $\mu > 0$, we can always get the vertical forces to cancel. The theoretical minimum value of μ will be determined by the need for the horizontal forces to cancel, so that the climber doesn't pop out of the corner like a watermelon seed squeezed between two fingertips. The horizontal component of the frictional force is always less than the magnitude of the frictional force, which is turn is less than μF_N. To find the minimum value of μ, we set the static frictional force equal to μF_N.

Let the x axis be along the plane that bisects the two walls, let y be the horizontal direction perpendicular to x, and let z be vertical. Then cancellation of the forces in the z direction is not the limiting factor, for the reasons described above, and cancellation in y is guaranteed by symmetry, so the only issue is the cancellation of the x forces. We have $2F_s\cos(\theta/2) - 2F_N\sin(\theta/2) = 0$. Combining this with $F_s = \mu F_N$ results in $\mu = \tan(\theta/2)$.

(b) For $\theta = 0$, μ is very close to zero. That is, we can always theoretically stay stuck between two parallel walls, simply by pressing hard enough, even if the walls are made of ice or polished marble with a coating of WD-40. As θ gets close to 180°, μ blows up to infinity. We need at least some dihedral angle to do this technique, because otherwise we're facing a flat wall, and there is nothing to cancel the wall's normal force on our feet.

(c) The result is 99.0°, i.e., just a little wider than a right angle.

5-g1 (a)

top spring's rightward force on connector
...connector's leftward force on top spring
bottom spring's rightward force on connector
...connector's leftward force on bottom spring
hand's leftward force on connector
...connector's rightward force on hand

Looking at the three forces on the connector, we see that the hand's force must be double the force of either spring. The value of $x - x_\text{o}$ is the same for both springs and for the arrangement as a whole, so the spring constant must be $2k$. This corresponds to a stiffer spring (more force to produce the same extension).

(b) Forces in which the left spring participates:
hand's leftward force on left spring
...left spring's rightward force on hand
right spring's rightward force on left spring
...left spring's leftward force on right spring

Forces in which the right spring participates:
left spring's leftward force on right spring
...right spring's rightward force on left spring
wall's rightward force on right spring
...right spring's leftward force on wall

Since the left spring isn't accelerating, the total force on it must be zero, so the two forces acting on it must be equal in magnitude. The same applies to the two forces acting on the right spring. The forces between the two springs are connected by Newton's third law, so all eight of these forces must be equal in magnitude. Since the value of $x - x_\text{o}$ for the whole setup is double what it is for either spring individually, the spring constant of the whole setup must be $k/2$, which corresponds to a less stiff spring.

5-g2 The parallel case is trivial: $k = k_1 + k_2$ by the same reasoning as before. The series case requires more delicacy, because the strings don't get elongated by the same amount. For simplicity of notation, let $x_\text{o} = 0$ for both springs. Since their forces on each other are equal by Newton's third law, the elongations are related by $F = k_1x_1 = k_2x_2$. The equivalent spring constant is $F/(x_1 + x_2) = k_1x_1/(x_1 + x_2) = k_1x_1/(x_1 + x_1k_1/k_2) = k_1k_2/(k_1 + k_2)$. Another way of writing this is $1/(1/k_1 + 1/k_2)$.

5-g3 (a) Spring constants in parallel add, so the spring constant has to be proportional to the cross-sectional area. Two springs in series give half the spring constant, three springs in series give 1/3, and so on, so the spring constant has to be inversely proportional to the length. Summarizing, we have $k \propto A/L$. (b) With the Young's modulus, we have $k = (A/L)E$.The spring constant has units of N/m, so the units of E would have to be N/m^2.

5-g4 (a) From problem 16 we have $k_{whole} = (A/L)E$. The simplest way to work this is to imagine a fiber consisting of only two atoms. Then we have $A = b^2$ and $L = b$, and $k = k_{whole}$, resulting in $k = bE$. (b) The result is on the order of 1 N/m. (c) The stretching of the bond is $x = F/k = mg/k$. A typical mass of an atom would be on the order of 10^{-26} kg, as you can find, for example, by looking up the masses of subatomic particles in the back of the book. (Another method would be to multiply the atomic volume b^3 by a typical density of solid matter.) The resulting extension of the length of the bond is on the order of 10^{-25} m, which is negligible, even compared to b. In other words, individual molecules won't be distorted significantly by gravitational stresses.

5-j1 Since its velocity vector is constant, it has zero acceleration, and the sum of the force vectors acting on it must be zero. There are three forces acting on the plane: thrust, lift, and gravity. We are given the first two, and if we can find the third we can infer its mass. The sum of the y components of the forces is zero, so

$$\begin{aligned}0 &= F_{thrust,y} + F_{lift,y} + F_{W,y}\\ &= |\mathbf{F}_{thrust}|\sin\theta + |\mathbf{F}_{lift}|\cos\theta - mg.\end{aligned}$$

The mass is

$$\begin{aligned}m &= (|\mathbf{F}_{thrust}|\sin\theta + |\mathbf{F}_{lift}|\cos\theta)/g\\ &= 7.0 \times 10^4 \text{ kg}\end{aligned}$$

5-j2 (a) We have the following three equations to start off with:

$$\begin{aligned}v^2 &= 2ax\\ F &= \mu_k mg\\ a &= F/m\end{aligned}$$

Eliminating F and a gives

$$\mu_k = \frac{v^2}{2gx} = 0.037.$$

5-j3 (a) Let positive be down. At $t = 0$, $v = 0$, so $a = g$. At the half-way point, we're told that the styrofoam has already reached terminal velocity, so its velocity is constant, and $a = 0$.
(b) 1.50 m/5.00 s = 0.300 m/s.
(c) $b = g/v_{\text{term}} = 32.7\ \text{s}^{-1}$.
(d) $g - bv = 4.9\ \text{m/s}^2$.
(e) We have $\mathrm{d}v/\,\mathrm{d}t = g - bv$. Separating variables gives

$$\frac{\mathrm{d}v}{g - bv} = dt.$$

We now integrate both sides.

$$\ln\left(\frac{g - bv}{\text{const}}\right) = -bt,$$

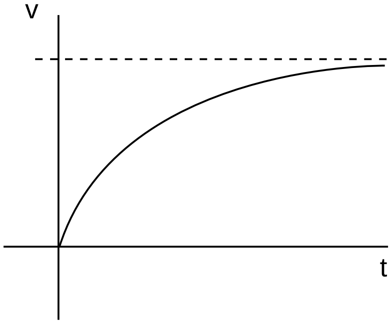
where the constant in the denominator is actually an additive constant, i.e., a constant of integration, if we break it out of the log. Solving for v gives

$$v = \frac{1}{b}\left[g - (\text{const.})e^{-bt}\right].$$

Since we have $v = 0$ at $t = 0$, the constant most equal g, so

$$\begin{aligned} v &= \frac{g}{b}\left[1 - e^{-bt}\right] \\ &= v_{\text{term}}\left[1 - e^{-gt/v_{\text{term}}}\right] \end{aligned}$$

The graph looks like this:



5-m1 (a) By Newton's third law, the forces are F and $-F$. Pick a coordinate system in which skater 1 moves in the negative x direction due to a force $-F$. Since the forces are constant, the accelerations are also constant, and the distances moved by their centers of mass are $\Delta x_1 = (1/2)a_1T^2$ and $\Delta x_2 = (1/2)a_2T^2$. The accelerations are $a_1 = -F/m_1$ and $a_2 = F/m_2$. We then have

$$\begin{aligned} \ell_f - \ell_0 &= \Delta x_2 - \Delta x_1 \\ &= \frac{1}{2}F\left(\frac{1}{m_1} + \frac{1}{m_2}\right)T^2, \end{aligned}$$

resulting in

$$T = \sqrt{\frac{2(\ell_f - \ell_0)}{F\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}}$$

(b)

$$\sqrt{\frac{\text{m}}{\text{N/kg}}} = \sqrt{\frac{\text{m}}{\text{kg}\cdot\text{ms}^{-2}\text{kg}^{-1}}} = \text{s}$$

(c) If the force is bigger, we expect physically that they will reach arm's length more quickly. Mathematically, a bigger F on the bottom results in a smaller T.
(d) If one of the masses is very small, then $1/m_1+1/m_2$ gets very big, and T gets very small. This makes sense physically. If you flick a flea off of yourself, contact is broken very quickly.

5-m2 (a) Since the wagon has no acceleration, the total forces in both the x and y directions must be zero. There are three forces acting on the wagon: $\mathbf{F}_T$, $\mathbf{F}_W$, and the normal force from the ground, $\mathbf{F}_N$. If we pick a coordinate system with x being horizontal and y vertical, then the angles of these forces measured counterclockwise from the x axis are $90° - \phi$, $270°$, and $90° + \theta$, respectively. We have

$$\begin{aligned} F_{x,total} &= |\mathbf{F}_T|\cos(90° - \phi) + |\mathbf{F}_W|\cos(270°) + |\mathbf{F}_N|\cos(90° + \theta) \\ F_{y,total} &= |\mathbf{F}_T|\sin(90° - \phi) + |\mathbf{F}_W|\sin(270°) + |\mathbf{F}_N|\sin(90° + \theta), \end{aligned}$$

which simplifies to

$$\begin{aligned} 0 &= |\mathbf{F}_T|\sin\phi - |\mathbf{F}_N|\sin\theta \\ 0 &= |\mathbf{F}_T|\cos\phi - |\mathbf{F}_W| + |\mathbf{F}_N|\cos\theta. \end{aligned}$$

The normal force is a quantity that we are not given and do not wish to find, so we should choose it to eliminate. Solving the first equation for $|\mathbf{F}_N| = (\sin\phi/\sin\theta)|\mathbf{F}_T|$, we eliminate $|\mathbf{F}_N|$ from the second equation,

$$0 = |\mathbf{F}_T|\cos\phi - |\mathbf{F}_W| + |\mathbf{F}_T|\sin\phi\cos\theta/\sin\theta$$

and solve for $|\mathbf{F}_T|$, finding

$$|\mathbf{F}_T| = \frac{|\mathbf{F}_W|}{\cos\phi + \sin\phi\cos\theta/\sin\theta}.$$

Multiplying both the top and the bottom of the fraction by $\sin\theta$, and using the trig identity for $\sin(\theta+\phi)$ gives the desired result,

$$|\mathbf{F}_T| = \frac{\sin\theta}{\sin(\theta+\phi)}|\mathbf{F}_W|.$$

(b) The case of $\phi = 0$, i.e., pulling straight up on the wagon, results in $|\mathbf{F}_T| = |\mathbf{F}_W|$: we simply support the wagon and it glides up the slope like a chair-lift on a ski slope. In the case of $\phi = 180° - \theta$, $|\mathbf{F}_T|$ becomes infinite. Physically this is because we are pulling directly into the ground, so no amount of force will suffice.

5-m3 (a) The two static frictional forces must together cancel out the elevator's weight $F_W = Mg$, so

$$F_s = \frac{Mg}{2}$$

The maximum amount of static friction is given by $F_s < \mu_s F_N$, so

$$F_N > \frac{Mg}{2\mu_s}$$

(b) If friction is weaker, we'd need to press more firmly. The equation demonstrates this behavior, since dividing by a smaller value of μ_s will give a greater result.

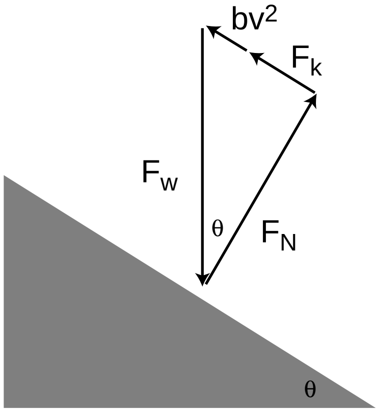
5-m4 (a) There are four forces acting on the skier, a normal force F_N from the snow, a gravity force F_W, a kinetic friction force $F_k = \mu_k F_N$, and an air friction force bv^2. If the skier is to approach a constant velocity, the total force on her must become nearly zero. Laying the vectors tip-to-tail results in the right triangle shown in the figure. At first the bv^2 force will not be big enough to close the gap in the triagle, and the skier will be accelerating, but eventually as her velocity grows and the bv^2 force gets stronger the total force will approach zero and she will stop accelerating. Trigonometry gives

$$\begin{aligned} bv^2 + F_k &= F_W\sin\theta \qquad \text{and} \\ F_N &= F_W\cos\theta \qquad , \end{aligned}$$

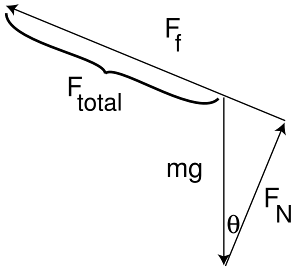
and using these two equations plus $F_W = mg$ and $F_k = \mu_k F_N$ we find

$$v = \sqrt{\frac{mg}{b}\left(\sin\theta - \mu_k\cos\theta\right)} \qquad .$$

(b) For θ less than $\tan^{-1}\mu_k$, we'd be taking the square root of a negative number. These are angles so shallow that the skier would slow to a halt.



5-m5 (a) See table on last page.



Problem 5-m5:

force acting on car			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	car
static friction	uphill	road	static friction	downhill	car
normal	up and forward	road	normal	down and backward	car

(b) From the figure, we see that the total force is equal to the force of static friction, minus the length of the short leg of the right triangle, $F_s - mg\sin\theta$. The maximum possible deceleration will be obtained if static friction has its maximum value $F_s = \mu_s F_N = \mu_s mg\cos\theta$. The result for the total force is $\mu_s mg\cos\theta - mg\sin\theta = mg(\mu_s\cos\theta - \sin\theta)$. By Newton's second law, the maximum deceleration is the total force divided by the mass, or $g(\mu_s\cos\theta - \sin\theta)$.
(c) Increasing the car's mass increases its inertia, which makes it harder to stop, but it also increases the gravitational force on it, which causes a greater normal force, and therefore a greater maximum value for the static friction force.
(d) At $\theta = 0$, we find that the maximum deceleration of a car on a flat road is $\mu_s g$. For $\theta < 0$, the $\sin\theta$ is negative, which means the result for the deceleration is *greater* than on a flat road. That makes sense: it's easier to make a panic stop when you're going uphill.
(e) As θ increases, the $\cos\theta$ term gets smaller and the $\sin\theta$ term gets bigger. At some angle, the result for the deceleration will become zero. This is the steepest slope the car can be on without slipping. For angles greater than this, static friction must give way to kinetic friction, and our result from part b becomes incorrect, since we derived it assuming static friction.

5-m6 (a) See table.

(b) The vertical forces cancel, so $F_N = mg$. Picking a coordinate system in which positive is forward, the horizontal force of friction produces an acceleration $a = F_k/m = -\mu_k F_N/m = -\mu_k g$. Applying the constant-acceleration equation $v_f^2 = v_i^2 + 2ax$, we find

$$x = \frac{v_i^2}{2\mu_k g} \qquad .$$

(c) The coefficient of friction is unitless, because it's defined as the ratio of two forces. The units of v_i^2 are $\mathrm{m}^2/\mathrm{s}^2$, and dividing by m/s^2 gives meters.

(d) Mathematically, a higher v_i gives a higher result, and this makes sense physically, because you can go farther if you start out going faster. A greater value of μ_k gives a smaller final result, and this also makes sense; increasing friction (e.g., by roughening the ice) would make the skater stop sooner. Increasing g would also decrease the final result; this is because a stronger gravitational field would press the skater more firmly against the ice, increasing F_N, and thereby indirectly increasing friction.

(e) The result is 2.4 km.

(f) This is clearly unrealistic. Probably the biggest reason for the unrealistic result is that we've neglected air resistance. Air resistance increases rapidly with speed, so that could be a reasonable approximation at lower speeds, but at these speeds (about 50% faster than an olympic sprinter), it's probably a poor approximation. When you see pictures of olympic-level speed skaters, they're always wearing special suits that are designed to cut down on air resistance; this shows that air resistance is a very important effect at the speeds they get up to.

Problem 5-m6:

force acting on skater			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	Earth	gravity	up	skater
normal	up	ice	normal	down	skater
kin. fr.	back	ice	kin. fr.	forward	skater

5-m7

$$\begin{aligned} v &= \sqrt{2aL} \\ &= \sqrt{2\frac{F_k}{m}L} \\ &= \sqrt{2\frac{\mu_k F_N}{m}L} \\ &= \sqrt{2\frac{\mu_k mg}{m}L} \\ &= \sqrt{2\mu_k gL} \end{aligned}$$

Note that the cop doesn't even need to know anything about the car other than the coefficient of kinetic friction between its tires and the asphalt μ_k, and μ_k probably doesn't vary all that much from one car to another.

5-m8 Let the positive x axis be to the right, and positive y up.

(a) The block isn't popping up into the air or sinking into the surface, so the y forces cancel:

$$\begin{aligned} &F_N - Mg - F\sin\theta = 0 \\ &F_N = Mg + F\sin\theta \end{aligned}$$

(b) The case where it's just about to slip is the one where it's in equilibrium horizontally and the force of static friction is at its maximum value $\mu_s F_N$, so that $F\cos\theta - \mu_s F_N = 0$. Substituting the answer from part a for F_N and solving for F, we find

$$F = \frac{Mg}{\cos\theta/\mu_s - \sin\theta}.$$

(c) This is the same as part b, but with μ_k.

(d) The answer to part b misbehaves starting at the critical angle for which the denominator is zero, which is $\tan^{-1}(1/\mu_k)$. Beyond this angle, the result for F is negative, which is unphysical.

5-m9 (a) Let positive x be the direction parallel to the ramp and uphill, and $+y$ perpendicular to the ramp and upward. The total force in the y direction is zero, so the normal force between the pack and the ramp is $F_N = mg\cos\theta$. The force in the x direction is $F - mg\sin\theta - mg\mu_k\cos\theta$. Using $L = (1/2)at^2$, we find

$$t = \sqrt{\frac{2Lm}{F - mg\sin\theta - mg\mu_k\cos\theta}}.$$

(b) We now have

$$\begin{aligned} F_N &= mg\cos\theta + F\sin\theta \\ F_x &= F\cos\theta - mg\sin\theta - \mu_k(mg\cos\theta + F\sin\theta) \\ t &= \sqrt{\frac{2Lm}{F\cos\theta - mg\sin\theta - \mu_k(mg\cos\theta + F\sin\theta)}}. \end{aligned}$$

5-m10 (a) See tables.

(b) Let N_1 be the normal force between the ground and block 1, and N_2 the normal force between the two blocks. Then because block 2's acceleration is zero, the total force on it is zero, and $T - \mu_k N_2 = 0$, so $T = \mu_k M_2 g$.

(c) Newton's second law gives

$$\begin{aligned} M_1 a &= F - \mu_k N_1 - \mu_k N_2 \\ &= F - \mu_k(M_1 + M_2)g - \mu_k M_2 g \\ &= F - \mu_k(M_1 + 2M_2)g \\ a &= \frac{F - \mu_k(M_1 + 2M_2)g}{M_1} \end{aligned}$$

5-m11 (a) Three forces act on the mass: a gravitational force Mg, a normal force from the

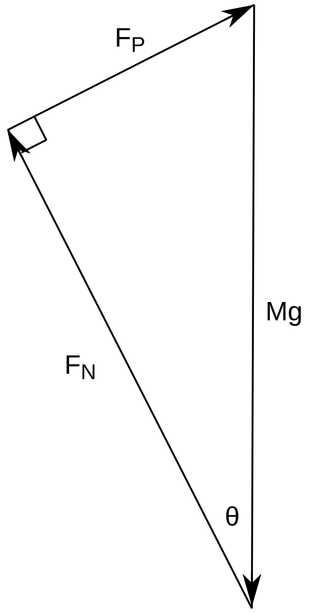
Problem 5-m10:

force acting on block 1			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	↓	Earth	gravity	↑	block 1
normal	↑	ground	normal	↓	block 1
kin. fr.	←	ground	kin. fr.	→	block 1
normal	↓	block 2	normal	↑	block 1
kin. fr.	←	block 2	kin. fr.	→	block 1
not given	→	not given	not given	←	block 1

Problem 5-m10:

force acting on block 2			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	↓	Earth	gravity	↑	block 2
normal	↑	block 1	normal	↓	block 2
kin. fr.	→	block 1	kin. fr.	←	block 2
normal?	←	rope	normal?	→	block 2

ramp, and a force from the rope that equals the reading F_P on the force probe. As shown in the figure, these three forces, when assembled tip-to-tail for vector addition, make a right triangle with Mg on the hypotenuse and F_P on the leg opposite to the angle θ. The result is that $F_P = Mg\sin\theta$.



(b) The expression found in part a gives 0 when $\theta = 0$ and Mg when $\theta = 90°$. Both of these make sense.

5-m12 Let positive x be the direction parallel to the ramp and uphill, and $+y$ perpendicular to the ramp and upward. Let F_N be the normal force between the ramp and the box, and F the force you apply. Applying Newton's second law in the x and y directions gives

$$\begin{aligned} F_x &= -mg\sin\theta + F\cos(\varphi-\theta) - \mu F_N \\ 0 &= F_N - mg\cos\theta + F\sin(\varphi-\theta). \end{aligned}$$

Eliminating F_N gives

$$\begin{aligned} F_x = -mg\sin\theta + F\cos(\varphi-\theta) - \mu mg\cos\theta \\ + \mu F\sin(\varphi-\theta). \end{aligned}$$

We want to choose φ so as to maximize F_x. This is made simpler if we change the independent variable to $\alpha = \varphi - \theta$ and recognize that the only part of F_x that is relevant is proportional to

$$A = \cos\alpha + \mu\sin\alpha.$$

Taking the derivative and setting it equal to zero gives $\alpha = \tan^{-1}\mu$, or

$$\varphi = \theta + \tan^{-1}\mu.$$

One way of interpreting this is as follows. If there is no friction, then we simply set $\varphi = \theta$, so that the force in the direction of motion is maximized. If there is friction, then we can win by raising φ slightly higher than θ. For small values of α, this gives almost no reduction in the x component of our force, while producing a significant reduction in the normal force.

5-m13 Let m be the mass of the board, x the displacement of the board relative to the center, and N_1 and N_2 the two normal forces.

Suppose that (1) the board is not rotating, and (2) kinetic friction obtains on both sides. Then the total torque is zero, $mgx + r(N_1 - N_2) = 0$, and the total horizontal force on the board is $\mu_k(N_1 - N_2) = -mg\mu_k x/r$. This is a restoring force proportional to x, so we have simple harmonic motion, provided that the assumptions hold: no rotation, and purely kinetic friction. The frequency is $\omega = \sqrt{g\mu_k/r}$. This is a funny kind of simple harmonic motion, because the kinetic frictional forces are continually dissipating energy as heat, and yet the motion is undamped.

Under what conditions can assumptions 1 and 2 be violated? Let the amplitude of the simple harmonic motion be A.

For $A > r$, the board will tip as its center moves beyond the top of one of the wheels, violating assumption 1. The board might either fall on the ground or make a bobble and come back.

Assumption 2, purely kinetic friction, can also be violated. The maximum velocity in simple harmonic motion is $A\omega$. If this velocity ever becomes as high as the tangential velocity of the wheels, Ωr, then kinetic friction would be replaced by static friction on one side, and we would no longer have a self-consistent solution in simple harmonic motion. One possibility is that under these conditions, the board can accelerate up to velocity Ωr, then stick at that velocity for a while before its weight shifts onto the other wheel sufficiently, at which point it will start to decelerate again; the motion is periodic but not simple harmonic.

We can also have possibilities that combine the features of the tipping solutions with those of the static-frictional solutions.

5-p1 (a) (b) (c) Let's choose a coordinate system in which positive is forward. The forces acting on the tugboat give

$$a = \frac{F_f - T}{m}$$

and the ones acting on the ship result in

$$a = \frac{T}{m} \qquad .$$

Note that we use the same symbol a in both cases: they're tied together, so their accelerations must be the same. To solve for T, we need to eliminate a, which we can do by setting the two equations equal:

$$\frac{F-T}{m} = \frac{T}{M}$$

Bringing all the T stuff to one side,

$$\frac{F}{m} = \left(\frac{1}{m} + \frac{1}{M}\right)T \qquad ,$$

Problem 5-p1:
Part a

force acting on tugboat			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
fluid fr.	forward	water	fluid fr.	backward	tugboat
tension	back	ship	tension	forward	tugboat

Problem 5-p1:
Part b

force acting on ship			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
tension	forward	tugboat	tension	back	ship

so

$$\begin{aligned} T &= \frac{F}{m(\frac{1}{m}+\frac{1}{M})} \\ &= \frac{F}{1+m/M} \\ &= \frac{M}{M+m}F \qquad . \end{aligned}$$

(d) For $M=0$, the tension is zero, which makes sense. For $M=\infty$, we have $M/(M+m)=1$, since the bottom becomes essentially the same as M, so $T=F$. The interpretation is that the tugboat is so light that essentially none of the force from the water on its propellers is required in order to accelerate it.

5-p2 (a) See table on last page.

(b) Let up be positive, and let a be the (positive) acceleration of mass m. Newton's second law applied to mass m gives

$$a = \frac{T-mg}{m} \qquad ,$$

and for mass M,

$$-a = \frac{T-Mg}{M} \qquad .$$

We have two equations in two unknowns, T and a. To solve them, we need to eliminate one variable. Let's eliminate a.

$$\begin{aligned} \frac{T-mg}{m} &= -\frac{T-Mg}{M} \\ \frac{T}{m}-g &= -\frac{T}{M}+g \\ \left(\frac{1}{m}+\frac{1}{M}\right)T &= 2g \\ (M+m)T &= 2gMm \\ T &= \frac{2gMm}{M+m} \end{aligned}$$

Now we can find a.

$$\begin{aligned} a &= \frac{T-mg}{m} \\ &= \frac{T}{m}-g \\ &= \frac{2gM}{M+m}-g \\ &= \left(\frac{2M}{M+m}-1\right)g \\ &= \left(\frac{2M}{M+m}-\frac{M+m}{M+m}\right)g \\ &= \left(\frac{M-m}{M+m}\right)g \end{aligned}$$

(c) No. If the masses are equal, the acceleration will be zero, not the velocity. The equal

Problem ??:

force acting on mass m			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
normal	up	string	normal	down	mass m
gravity	down	planet Earth	gravity	up	mass m

masses could be left at rest, and would stay at rest, regardless of their relative heights. They could also be left in motion, and would stay in motion, without speeding up or slowing down.
(d) (found above)
(e) Mathematically, the expression for T vanishes when either mass is zero. Physically this makes sense, because if either mass is not present, the rope will go slack.
5-p3 (a) See the tables for an analysis of the forces involved.

Applying Newton's second law to the climber of mass M, and letting the positive direction be downward, we have

$$Mg - T = Ma \qquad .$$

The climber of mass m doesn't sink into the snow or fly up in the air, so

$$F_N - mg = 0 \qquad .$$

Newton's second law for m, with positive to the right, gives

$$T - \mu_k F_N = ma \qquad .$$

These are three equations in three unknowns, a, T, and F_N. Eliminating $F_N = mg$ reduces it two a system of two equations in two unknowns:

$$\begin{aligned} Mg - T &= Ma \\ T - \mu_k mg &= ma \end{aligned}$$

Next we solve the first equation for $T = M(g-a)$ and substitute into the other equation, giving one equation with one unknown:

$$M(g-a) - \mu_k mg = ma$$

Solving this for a, we find

$$a = \frac{M - \mu_k m}{M+m} g \qquad .$$

(b) The definition of μ_k is $F_k = \mu_k F_N$, so μ_k is unitless. That means that it makes sense to do the subtraction $M - \mu_k m$, since both terms have units of kg. The quotient $(M - \mu_k m)/(M+m)$ is unitless, and multiplying it by g gives units of acceleration, which checks out.

(c) Mathematically, increasing g increases a. That makes sense because more gravity should make them accelerate faster. Mathematically, increasing μ_k decreases a, and this checks out because increasing friction should keep them from accelerating as fast. Increasing m decreases the numerator and increases the denominator, and both of these effects will reduce a. This makes sense because adding extra weight to m will help both by increasing friction and by increasing the inertia of the whole system, which is measured by $M + m$. M is the hardest variable to analyze mathematically. Increasing M increases both the numerator and the denominator, so there are two countervailing effects, and it's not obvious which is the greater. But since the numerator is smaller than the denominator, increasing M by a certain amount will cause a greater percent change in the numerator, so the net effect will be to increase a. This makes sense physically, because if M was miniature French poodle, we wouldn't expect the acceleration to be as fast.
5-p4 (a) See tables.

(b) Applying Newton's second law to block B, and letting the positive direction be downward, we have

$$M_B g - T = M_B a \qquad .$$

Problem 5-p3:

force acting on climber of mass m			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	climber
kinetic friction	left	snow	kinetic friction	right	climber
normal	up	snow	normal	down	climber

Problem 5-p3:

force acting on climber of mass M			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	climber
normal	up	rope	normal	down	climber

Problem 5-p4:

force acting on block A			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	block A
kinetic friction	left	snow	kinetic friction	right	block A
normal	up	snow	normal	down	block A

Problem 5-p4:

force acting on block B			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	block B
normal	up	rope	normal	down	block B

We're told that $a = 0$ in the situation described in part b, but I'll put off making use of that fact until we're done with the setup, so that we can recycle it for part c. Block A doesn't sink into the table or fly up in the air, so

$$F_N - M_A g = 0 \qquad .$$

Newton's second law for M_A, with positive to the right, gives

$$T - \mu_k F_N = M_A a \qquad .$$

Now let's set $a = 0$. These are three equations in three unknowns, μ_k, T, and F_N. Eliminating $F_N = M_A g$ reduces it two a system of two equations in two unknowns:

$$\begin{aligned} M_B g - T &= 0 \\ T - \mu_k M_A g &= 0 \end{aligned}$$

Next we solve the first equation for $T = M_B g$ and substitute into the other equation, giving one equation with one unknown:

$$M_B - \mu_k M_A = 0$$

Solving this for μ_k, we find $\mu_k = M_B/M_A$.

(c) We now go back to the original setup, with $a \neq 0$, and also substitute $2M_A$ for M_A. The three unknowns are now a, T, and F_N. Eliminating $F_N = 2M_A g$ reduces it two a system of two equations in two unknowns:

$$\begin{aligned} M_B g - T &= M_B a \\ T - 2\mu_k M_A g &= 2M_A a \end{aligned}$$

Next we solve the first equation for $T = M_B(g - a)$ and substitute into the other equation, giving one equation with one unknown:

$$M_B(g - a) - 2\mu_k M_A g = 2M_A a$$

Solving this for a, we find

$$a = \frac{M_B - 2\mu_k M_A}{2M_A + M_B} g,$$

and substituting $\mu_k = M_B/M_A$ from part b, we get

$$a = \frac{-g}{2M_A/M_B + 1}.$$

The negative sign indicates that the motion slows down.

5-p5 (a) See table on last page.

(b) See table on last page.

(c) The vertical forces on Ginny cancel out, so the normal force between Ginny's pants and the sled has magnitude $N_2 = Mg$. Similarly, the normal force between the sled and the snow is $N_1 = (M + m)g$, since the snow supports the total mass of both Ginny and the sled. In the situation where she can just barely accomplish what she wants to do, the initial acceleration is vanishingly small; to find the limiting case, we set it to zero. That means there must be zero total horizontal force on her, and zero total horizontal force on the sled. There are two frictional forces on the sled, F_2 from Giny and F_1 from the snow. If they're canceling, their magnitudes must be equal, $|F_2| = |F_1|$. If the sled just barely starts slipping over the snow before Ginny's pants start slipping over the sled, then both these forces are at their maximum value for static friction,

$$\begin{aligned} \mu_2 N_2 &= \mu_1 N_1 \\ \mu_2 M g &= \mu_1 (M + m) g \\ \mu_2 &= \mu_1 (1 + m/M) \\ M &= \frac{m}{\mu_2/\mu_1 - 1} \end{aligned}$$

(d) For $\mu_2 = \mu_1$, M becomes infinite. For smaller values of μ_2, we have a negative value for M, which is also impossible. In other words, for $\mu_2 \leq \mu_1$, it wouldn't matter if Ginny carried a backpack full of lead bricks; she'd still slip off the sled.

5-p6 (a) The normal force at the top surface is Mg, and the one at the bottom surface is $(M + m)g$. Since there is slipping at both surfaces, the coefficient of kinetic friction is the one we care about, and the two frictional forces are $\mu_k M g$ at the top surface (acting on both blocks) and $\mu_k (M + m) g$ at the bottom. The middle block decelerates, while the top one accelerates. The velocity of the middle block is $v - \mu_k(2M/m +$

Problem 5-p5:
Part a:

force acting on Ginny			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	Ginny
static friction	left	ropes	static friction	right	Ginny
normal	up	sled	normal	down	Ginny
static friction	right	sled	static friction	left	Ginny

Problem 5-p5:
Part b:

force acting on sled			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
gravity	down	planet Earth	gravity	up	sled
normal	down	Ginny	normal	up	sled
static friction	left	Ginny	static friction	right	sled
normal	up	snow	normal	down	sled
friction	right	snow	friction	left	sled

1)gt, and the velocity of the top block is $\mu_k gt$. Setting these equal to each other, we find

$$t = \frac{v}{2\mu_k g(M/m+1)} \qquad .$$

(b) The units of the right-hand side are $(\mathrm{m/s})/(\mathrm{m/s^2}) = \mathrm{s}$.
(c) If m gets bigger, M/m gets smaller, and t gets bigger. This makes sense physically. The acceleration of the top block doesn't depend on the mass of the middle block at all. The acceleration of the middle block depends on its own mass for two reasons: (1) if it has more mass it has more inertia; (2) if it has more mass then there will be more friction at the bottom surface. Although these two effects are in the opposite direction, they're not of equal strength. Doubling the mass of the middle block, for example, doubles its inertia, but doesn't double the frictional force at the bottom surface, and has even less of an effect on the total force felt by the block.
(d) If M gets bigger, its own acceleration of $\mu_k g$ is unchanged, because the greater frictional force cancels out the effect of the greater inertia. On the other hand, the frictional force on the middle block is increased, so the time should get shorter. Mathematically, this is the behavior we see.
(e) Making g bigger makes the time shorter. That makes sense, because if gravity presses the surfaces together more firmly, there will be more friction.

5-p7 (a) In the case where the block just barely sticks, the static frictional force is at the maximum possible value,

$$F_s = \mu_s F_N \qquad .$$

(This is static friction because there is no slipping.) Let positive x be to the right and positive y be up. Applying Newton's second law in the x direction gives

$$-F_N \sin\theta + F_s \cos\theta = ma \qquad ,$$

and in the y

$$F_N \cos\theta + F_s \sin\theta - mg = 0 \qquad .$$

These are three equations in the three unknowns F_s, F_N, and a. Eliminating F_s and F_N and solving for a, we find

$$a = \left(\frac{\mu_s \cos\theta - \sin\theta}{\mu_s \sin\theta + \cos\theta} \right) g \qquad .$$

(b) Coefficients of friction are unitless, so the whole expression on the right consists of unitless stuff, except for the factor of g. That makes the right-hand-side come out to have units of acceleration, which makes sense.

(c) Mathematically, increasing θ decreases $\cos\theta$ and increases $\sin\theta$. That means that it decreases the numerator of our expression for a — until the numerator reaches zero, at the critical angle $\theta_c = \tan^{-1}\mu_s$. This is the angle at which it becomes impossible to accelerate the wedge at all without making the block slip, because the block is barely able to stick even when the wedge is sitting still.

The denominator is a little trickier. For small values of θ, $\cos\theta$ is nearly constant, while $\sin\theta$ increases approximately in proportion to θ, so the denominator is increasing. But is it increasing for all θ of interest? If it stops increasing at some θ, then at that θ $d(\mu \sin\theta + \cos\theta) = 0$, but this occurs at θ_c. Therefore the denominator is increasing for all physically relevant values of θ.

Since the numerator is decreasing and the denominator is increasing, the result is decreasing. This makes sense physically, because if the wedge is steeper, it should be harder to keep the block from slipping.

5-p8 (a) Each block has three forces acting on it: gravity, a normal force from the ramp, and a force from the rope. The components perpendicular to the slope cancel out, while the ones parallel to it cause an acceleration a, which will be the same for both blocks provided that the rope doesn't go slack. We therefore have $ma = mg\sin\theta + T$ and $ma = mg\sin\phi - T$. Eliminating a and solving for T, we find $T = \frac{1}{2}mg(\sin\phi - \sin\theta)$.

(b) The units of mg are newtons, and the sines are unitless, so the result does have units of newtons, as expected.

(c) For $\phi = \theta$, the tension is zero, i.e., the rope goes slack. This makes sense. For $\phi < \theta$, the tension becomes negative; this is unphysical, but that makes sense, because the assumption we made (that the rope would stay taut) has failed. For $\theta = 0$ and $\phi = 90°$, the result is $T = mg/2$, which we can readily verify is a correct solution. It accelerates the top block across its horizontal surface with $a = g/2$, and it cancels out half of the gravitational force on the bottom block, also causing $a = g/2$.

5-p9 (a) Most of the forces on the box are either parallel to or perpendicular to the slope, so it's convenient to pick coordinate axes aligned with the slope. Let positive x be uphill, and positive y perpendicular to that (up and to the left). Four forces act on the box: a normal force F_N in the positive y direction, a kinetic frictional force F_k in the $-x$ direction, a force from the rope with magnitude T in the $+x$ direction, and a downward force of gravity with magnitude Mg. The gravitational force is the only one with both an x and a y component; it has x component $-Mg\sin\theta$ and y component $Mg\cos\theta$.

Since the box doesn't hop off of or sink into the slope, the total force on the box in the y direction must be zero:

$$0 = F_N - Mg\cos\theta$$

To predict the box's acceleration, we apply Newton's second law in the x direction:

$$a = (T - Mg\sin\theta - \mu_k F_N)/M$$

These are two equations in the three unknowns F_N, a, and T. Since we don't have enough equations to solve for the unknowns, we apply Newton's second law to the person:

$$a = (mg - T)/m \qquad ,$$

where we make use of the facts that the tension in a rope is unaffected by going around a frictionless pulley and that the person's acceleration must match that of the box, since they're tied together. Solving these equations for a results in

$$a = (m - M\sin\theta - \mu_k M\cos\theta)g/(M+m) \qquad .$$

(b) All the terms in the factor $m - M\sin\theta - \mu_k M\cos\theta$ have units of kilograms, and these cancel with the units of the $M + m$, giving nothing but acceleration units from the factor of g.

(c) When $M = 0$, the rope should be slack and the person should simply free-fall with acceleration g. This is indeed what happens when we plug in $M = 0$.

When $\theta = -90°$, the slope is rotated clockwise so that it becomes a vertical wall above the person's head. There is no normal force between the box and the slope, so there is no friction. The person and the box should both simply free-fall with acceleration g. Plugging in $\theta = -90°$ does produce this result.

5-p10 (a) Newton's first law applied to the person gives $T = mg$. Applied to the block, in the direction parallel to the ramp, it gives $T = Mg\sin\theta$. Eliminating T gives the result $m/M = \sin\theta$.

(b) Newton's first law applied to the block, in the direction perpendicular to the ramp, tells us that the normal force between the block and the ramp is $F_N = Mg\cos\theta$. In the parallel direction, it gives $T - Mg\sin\theta \pm \mu_s F_N$. The result is

$$\frac{m}{M} = \sin\theta \mp \mu_s\cos\theta.$$

5-p11 (a) Let positive x be to the right and positive y up. The block is acted on by four forces: the two external forces of magnitude F, along with a normal force and a frictional force from the surface. If the block is to remain at rest, then the forces in the x direction must cancel, $F - F\cos\theta - \mu_s F_N = 0$, and similarly in the y direction, $F_N - F\sin\theta = 0$. Eliminating F_N and solving for the coefficient of friction results in

$$\mu_s = \frac{1-\cos\theta}{\sin\theta}.$$

(b) At $\theta = 0$ we expect that the block should be able to remain at rest without friction, since the horizontal forces cancel by symmetry. Our expression for μ_s is an indeterminate form when $\theta = 0$, but application of l'Hôpital's rule gives

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = \lim\frac{\sin\theta}{\cos\theta} = 0,$$

as required.

5-s1 Because the pulleys are frictionless, the rope that the person is grabbing has equal tension F throughout. Therefore the pulley immediately above the load has an upward force $2F$ acting on it from the two strands of rope above it. This tells us that the system has a mechanical advantage of 2, and the person can lift a weight $2F$, which corresponds to a mass $2F/g$.

5-s2 Let the tension in the rope connected to the right-hand weight be T. By arguments similar to the ones in the example in section 5.6, we know that the tension in the rope connected to the left-hand weight is $2T$. The accelerations of the two weights are also related. For every centimeter that the weight on the left moves up, the pulley in the middle will move one centimeter to the right. This causes one centimeter of rope to come around the middle pulley, lengthening the part the goes over the right-hand pulley and down to the weight. The right-hand weight therefore drops two centimeters: one centimeter because an extra centimeter of rope is being fed around the middle pulley, and another centimeter because the middle pulley is moving a centimeter to the right. Therefore the acceleration of the right-hand weight is twice as much as the acceleration of the left-hand one, and is in the opposite direction, $a_2 = -2a_1$. Applying Newton's second law to each weight (with positive being up), we have the following set of three equations in three unknowns:

$$\begin{aligned} a_2 &= -2a_1 \\ a_1 &= \frac{2T - mg}{m} \\ a_2 &= \frac{T - mg}{m} \end{aligned}$$

Substituting the first equation into the third one reduces this to two equations in two unknowns:

$$a_1 = 2T/m - g$$
$$-2a_1 = T/m - g$$

Now we solve the second equation for $T/m = -2a_1 + g$ and substitute into the first one, which makes it one equation in one unknown:

$$a_1 = 2(-2a_1 + g) - g$$

Solving this, we find $a_1 = g/5$. The positive sign indicates that this mass (the one on the left) goes up.

5-s3 Let a be the acceleration of the mass on the right and a' the acceleration of the mass on the left. For every centimeter that the mass on the left moves down, the left-hand segment of the rope attached to it lengthens by 1 cm, while the three other segments all shorten by 1/3 cm. Therefore

$$a' = -3a \qquad .$$

Let the tension in this piece of rope be T. Since the middle pulley is massless, the total force acting on it must be zero, so the tension in the piece of rope connecting it to the crossbar must be $2T$. Since the crossbar is massless, the total force on it must be zero, so the rope connecting to the right-hand mass must have tension $3T$. Newton's second law gives

$$T - mg = ma'$$
$$3T - mg = ma \qquad .$$

This is a set of three equations in three unknowns. The result is $a = g/5$.

5-s4 Applying Newton's second law to each of the three masses, we have

$$2T - m_3 g = m_3 a_3$$
$$T - m_1 g = m_1 a_1$$
$$T - m_2 g = m_2 a_2 \qquad .$$

Mass 1's acceleration relative to the earth equals the sum of its acceleration a_{1P} relative to the lower pulley and that pulley's acceleration relative to the earth. The pulley's acceleration relative to the earth equals $-a_3$. Doing this for mass 2 as well, we have

$$a_1 = a_{1P} - a3$$
$$a_2 = a_{2P} - a3$$
$$a_{1P} = -a_{2P} \qquad .$$

These are six equations in six unknowns. Eliminating a_{1P} and a_{2P} reduces the second block of equations to simply

$$a_3 = -(a_1 + a_2)/2 \qquad ,$$

i.e., mass 3's acceleration relates to the average of the other two masses'. Eliminating $T = m_1(a_1 + g)$, and then eliminating a_2, we find

$$\frac{a_1}{g} = \frac{(3/2)m_2 m_3/m_1 - (2m_2 + m_3/2)}{(1/2)m_2 m_3/m_1 + (2m_2 + m_3/2)} \qquad .$$

This can be simplified considerably by setting

$$\beta = m_1 \left(\frac{4}{m_3} + \frac{1}{m_2} \right) \qquad ,$$

which yields

$$\frac{a_1}{g} = \frac{3 - \beta}{1 + \beta} \qquad .$$

We can verify that this gives correct results in some simple cases:

$m_1 = m_2 = 1$, $m_3 = 2$, $\beta = 3$, $a_1 = 0$
$m_1 = m_2 = 1$, $m_3 = 0$, $\beta = \infty$, $a_1 = -g$
$m_1 = m_2 = 1$, $m_3 = \infty$, $\beta = 1$, $a_1 = g$
$m_1 = m_3 = 1$, $m_2 = 0$, $\beta = \infty$, $a_1 = -g$

6-a1 $(\text{m/s})^2/\text{m} = \text{m}^2/\text{s}^2/\text{m} = \text{m/s}^2$

6-a2 If neither the seat nor the seatbelt is applying a force to the pilot, then the only force on him is the earth's gravity, i.e. his weight. When the only force on an object is gravity, its acceleration is g. The speed needs to be such that the acceleration equals g. The acceleration is v^2/r, so $v^2/r = g$, or $v = \sqrt{gr} = 99$ m/s.

6-a3 The force is $F = ma = m\omega^2 r = m(2\pi T)^2 r = 1.3 \times 10^{47}$ newtons. This is an absurd amount of force, and it clearly isn't being exerted by any physical object. There isn't actually any force, and Sirius isn't actually accelerating. It just appears to accelerate in this noninertial frame of reference.

6-a4 The "radius" concept is a bit fuzzy here, because the train has a finite width to it. Although the inside of the outer rail is where the force has to be applied when the train is starting to tip, its radius from the center of the curve isn't quite the same as the radius of the circle traveled by the train's center of mass. Then again, maybe the train's center of mass would be above the outer rail if it was just on the verge of flipping over. Anyhow, ignoring these niceties, we assume that there is a maximum acceleration, which is the same in both cases. Converting $a = v^2/r$ into a proportionality by throwing out the factor of a, we have $v^2 \propto r$, or $v \propto \sqrt{r}$. Rewriting this in terms of ratios gives $v_1/v_2 = \sqrt{r_1/r_2}$. Since we're doing a ratio, it doesn't matter whether we do diameter or radius. The result is $(0.95\ \text{m/s})\sqrt{662\ \text{mm}/1067\ \text{mm}} = 0.75$ m/s. We know we set up the ratio the right way, since it makes sense that the speed comes out slower on the tighter curve.

6-a5 (a) Let's start by calculating the angular velocity:

$$\begin{aligned}\omega &= \frac{2\pi}{(24\ \text{hr})(3600\ \text{s})}\\ &= 7.272\times10^{-5}\ \text{s}^{-1}\end{aligned}$$

(keeping an extra sig fig to avoid accumulating rounding errors). For the speed, we have

$$\begin{aligned}v &= \omega r\\ &= 384\ \text{m/s}.\end{aligned}$$

(b) The acceleration is

$$\begin{aligned}a &= \omega^2 r\\ &= 0.0279\ \text{m/s}^2,\end{aligned}$$

or about 0.3% of g.

6-a6 (a) We have $\omega = 2\pi/T$, where T is the period, so $a = \omega^2 r = 4\pi^2 r/T^2 = 15\ \text{m/s}^2$.

(b) For a fixed a, we have $T \propto \sqrt{r}$, so quadrupling r changes the minimum period to 4.0 s.

6-a7 (a) $r = v^2/a = 50$ m.

(b) $T = (2\pi r)/v = 31$ s.

(c) From the given information, it's initially at the bottom of the circle, on the negative y axis, and moving counterclockwise. After a quarter of a revolution, it's at the right edge of the circle, so its velocity is in the positive y direction, at $\theta = 90°$.

6-a8 (a) Newton's second law applied to the driver's body gives

$$a = \frac{F_\text{total}}{m}.$$

Their acceleration is

$$a = \frac{v^2}{R}.$$

The total force acting on the driver's body is

$$\begin{aligned}F_\text{total} &= mg - N\\ &= \frac{2}{3}mg.\end{aligned}$$

Eliminating a and F_total gives

$$v = \sqrt{\frac{2}{3}gR}.$$

(b)

$$\frac{\text{m}}{\text{s}} = \sqrt{\left(\frac{\text{m}}{\text{s}^2}\right)\cdot\text{m}}$$

(c) If g is bigger, v gets bigger. That makes sense, because if gravity is stronger, the car has to be going faster to get the same feeling of partial lift-off. If R is bigger, v gets bigger. That also makes sense, because a very large R indicates a nearly flat road, and you can't get any lift-off on a flat road.

6-a9 We have $a = v^2/r$, so $v = \sqrt{ar} = \sqrt{5.0gr} = 120$ m/s.

6-d1 (a) $\theta = vt/r$, so $\mathbf{r} = r(\cos(vt/r))\hat{\mathbf{x}} + r(\sin(vt/r))\hat{\mathbf{y}}$.

(b) We need to differentiate twice with respect to t, using the chain rule and the fact that

$d(\sin x)/\,dx = \cos x$ and $d(\cos x)/\,dx = -\sin x$. The first differentiation gives a velocity vector

$$\begin{aligned}\mathbf{v} &= \left(\frac{v}{r}\right) r \left(-\sin\frac{vt}{r}\right)\hat{\mathbf{x}} + \left(\frac{v}{r}\right) r\left(\cos\frac{vt}{r}\right)\hat{\mathbf{y}}\\ &= v\left(-\sin\frac{vt}{r}\right)\hat{\mathbf{x}} + v\left(\cos\frac{vt}{r}\right)\hat{\mathbf{y}}.\end{aligned}$$

The second differentiation gives

$$\begin{aligned}\mathbf{a} &= \left(\frac{v}{r}\right) v\left(-\cos\frac{vt}{r}\right)\hat{\mathbf{x}} + \left(\frac{v}{r}\right) v\left(-\sin\frac{vt}{r}\right)\hat{\mathbf{y}}\\ &= \frac{v^2}{r}\left[-\cos\frac{vt}{r}\hat{\mathbf{x}} - \sin\frac{vt}{r}\hat{\mathbf{y}}\right] \qquad .\end{aligned}$$

(c) The magnitude of the vector in square brackets equals one, since $\cos^2 + \sin^2 = 1$. Multiplying by the scalar v^2/r therefore gives a vector with magnitude v^2/r.

6-g1 (a) The inward normal force must be sufficient to produce circular motion, so

$$|\mathbf{F}_N| = mv^2/r.$$

We are searching for the minimum speed, which is the speed at which the static friction force is just barely able to cancel out the downward gravitational force. The maximum force of static friction is

$$|\mathbf{F}_s| = \mu_s|\mathbf{F}_N|,$$

and this cancels the gravitational force, so

$$|\mathbf{F}_s| = mg.$$

Solving these three equations for v gives

$$v = \sqrt{\frac{gr}{\mu_s}}.$$

(b) Greater by a factor of $\sqrt{3}$.

6-g2 (a) Newton's second law in the vertical direction:

$$T\sin\theta - Mg = 0.$$

Newton's second law in the horizontal direction:

$$T\cos\theta = Mv^2/R.$$

Eliminating T and solving for v gives

$$v = \sqrt{\frac{gR}{\tan\theta}}$$

(b)

$$\frac{\mathrm{m}}{\mathrm{s}} = \sqrt{\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)\cdot\mathrm{m}}$$

(c) If g is bigger, v gets bigger. That makes sense, because if gravity is stronger, Tommy has to whirl the brick faster to get it to rise as high. If R is bigger, v gets bigger. That also makes sense, because intuitively this would be very hard to do with a long rope. If θ gets bigger, v gets smaller. That makes sense, because the faster he whirls the brick, the higher it will rise. As $\theta \to 0$, $v \to \infty$, so Tommy can never get the brick to spin in a perfectly horizontal plane.

6-g3 (a)

equal and opposite force

force acting on ball involved in Newton's 3rd law

type of object exerting object

force direction the force type direction exerting it

tension in and up string tension down and out ball

gravity down earth gravity up ball

(b)

Newton's second law, horizontal component:

$$v^2/r = F_T\sin\theta/m \qquad (1)$$

Newton's second law, vertical component:

$$0 = (F_T\cos\theta - mg)/m \qquad (2)$$

geometry:

$$r = L\sin\theta \qquad (3)$$

$$v = 2\pi r/P \qquad (4)$$

The variable v is not given and not desired, so we eliminate it using equation (4):

$$4\pi^2 mr/P^2 = F_T\sin\theta(5)$$

We next use equation (3) to eliminate r:

$$4\pi^2 mL/P^2\sin\theta = F_T\sin\theta \qquad (6)$$

$$4\pi^2 mL/P^2 = F_T \qquad , \qquad (7)$$

which has the unintended side-effect of making θ disappear. The equation is still useful, however, because it tells us F_T in terms of variables we know. We can use it to eliminate F_T in equation (2):

$$0 = 4\pi^2 L/P^2 \cos\theta - g \qquad , \qquad (8)$$

which can be solved for θ:

$$\theta = \cos^{-1}\left(\frac{gP^2}{4\pi^2 L}\right) (9)$$

(c) We can't take the arccosine of a number that's greater than one, so there would be no solution for sufficiently large values of P. Physically, if P approached infinity, the ball would be standing still, in which case θ would equal zero. In the case where the angle is zero, step (6) → (7) is invalid, since we can't divide both sides of the equation by zero.

6-g4 Applying Newton's second law and $a = v^2/r$ to each block independently, we have

$$v_1^2/L_1 = (T_1 - T_2)/m_1$$
$$v_2^2/(L_1 + L_2) = T_2/m_2$$

We can eliminate $v_1 = 2\pi L_1/P$ and $v_2 = 2\pi(L_1+L_2)/P$, and straightforward algebra then yields

$$T_1 = \left(\frac{4\pi^2}{P^2}\right)[m_1 L_1 + m_2(L_1 + L_2)]$$
$$T_2 = \left(\frac{4\pi^2}{P^2}\right) m_2(L_1 + L_2)$$

6-g5 (a) Let positive x be inward and positive y up. Then $a_x = F_x/m$ gives $v^2/r = F_N/m$, and $a_y = F_y/m$ results in $0 = \mu_s F_N - mg$. Solving for r, we find $r = \mu_s v^2/g$. (b) $(\text{m/s})^2/(\text{m/s}^2) = \text{m}$. (c) Mathematically, increasing v increases r; this makes sense because it should be harder to do if r is bigger, so you should have to run faster. Increasing g decreases r, and that makes sense because, e.g., if g were 0, you could do this with an unlimited r. Increasing μ_s increases r; this makes sense because, e.g., in the absence of friction, it should be impossible to do the stunt, and the maximum r should be 0, i.e., it shouldn't be possible for any positive r. (d) The result is 50 meters, which is counterintuitively huge.

6-g6 (a) Newton's second law in the vertical direction:

$$F_N - mg = 0.$$

Newton's second law in the horizontal direction:

$$\mu_s F_N = \frac{4\pi^2 mR}{T^2}.$$

Eliminating F_N gives

$$\mu_s = \frac{4\pi^2 R}{gT^2}.$$

6-g7 Let r be the distance of the heart from the axis, and let f be the rate of rotation in revolutions per second, $f = 296/360\ \text{s}^{-1}$. The time for one rotation is $1/f$. The speed at which the heart is circling the axis is (circumference)/(time) = $2\pi rf$. We then have $a/g = v^2/r$, or

$$\frac{a}{g} = \frac{4\pi^2 f^2 r}{g} = 1.8.$$

6-j1 (a) Only an inward force, i.e., a force perpendicular to the surface of contact, can have an effect on whether the batter separates from the beater. Friction is always parallel to the surface of contact.

(b) See table on last page.

(c) An inward force is required to keep the batter from following Newton's first law and flying off straight, on a tangent to the circle. The amount of force required is given by Newton's second law, $F = ma = mv^2/r$. If the speed increases, so does amount of inward sticky force needed to continue deflecting the batter off of its desired straight-line path. There's a maximum amount of force that the stickum can supply, beyond which it becomes unstuck.

6-j2 The apparent paradox arises if you assume, incorrectly, that v and T are the same even though r is different. You can't change r while leaving v and T the same. A should have said that *for objects moving at the same speed,*

Problem 6-j1:
Part b:

force acting on tape			*equal and opposite force involved in Newton's 3rd law*		
type of force	*direction*	*object exerting the force*	*type*	*direction*	*object exerting it*
sticky	left	beater	sticky	right	tape

the smaller circle gives a greater acceleration. B should have said that *for objects that take the same amount of time per cycle*, the smaller circle gives a smaller acceleration.

6-j3 (a) The force on the hamster is $k(L-b)$, where L is the length of the spring (which is the same as the radius of the circle). Newton's second law gives $v^2/L = k(L-b)/m$. We can eliminate $v = 2\pi L/T$, and straightforward algebra then gives

$$L = \frac{b}{1-\frac{4\pi^2 m}{kT^2}}$$

(b) For a certain value of T, the denominator becomes zero. If we were to speed up the motor gradually, L would grow without bound as we approached this value of T. Physically, the spring would have to break. The reason for this surprising behavior will become clearer after you learn about the phenomenon of resonance.

6-j4 (a) Let the tension in each of the top bars be T, and in each bottom one U. The vertical components of the forces on the mass M must cancel, so

$$2U\cos\theta = Mg \qquad ,$$

and similarly for the vertical forces on each mass m,

$$(T-U)\cos\theta = mg \qquad .$$

The horizontal force of the bars on each mass m must satisfy Newton's second law,

$$(T+U)\sin\theta = \frac{4\pi^2 mL\sin\theta}{P^2} \qquad .$$

We now have three equations in the three unknowns U, T, and θ. We use the first equation to eliminate U, which reduces this to two equations in two unknowns:

$$T + \frac{Mg}{2\cos\theta} = \frac{4\pi^2 mL}{P^2}$$
$$T\cos\theta - \frac{Mg}{2} = mg$$

Solving the second equation for $T = (mg + Mg/2)/\cos\theta$, substituting into the first equation, and solving for θ, we find the result claimed for θ.

(b) Low speeds correspond to large values of the period, P. At low enough speeds, the equation derived in part a ends up being the arccosine of a number greater than one, i.e., there is no solution of that form. What happens at these speeds is the same things that happens at zero speed: everything hangs straight down. In this solution, the radius of the circle traveled by each mass m is zero. (The mathematical loophole here was that we canceled a $\sin\theta$ on both sides of one of the equations, but you can't divide both sides of an equation by zero.) By making a nonzero, we make such a zero-radius solution impossible, because even when the engine isn't spinning, each mass m is already at some distance from the axis.

6-j5 (a) Newton's second law in the vertical direction:

$$F_N - mg = 0.$$

Newton's second law in the horizontal direction:

$$\mu F_N = \frac{v^2}{r}$$

Eliminating F_N gives

$$v = \sqrt{\mu g r} = 30\text{ m/s}$$

(one sig fig, because μ only had one sig fig).

(b)

$$\begin{aligned} a_r &= v^2/r \\ &= 4.00\ \text{m/s}^2 \\ a_t &= 3.00\ \text{m/s}^2 \\ |\mathbf{a}| &= \sqrt{a_r^2 + a_t^2} \\ &= 5.00\ \text{m/s}^2 \end{aligned}$$

(c) The forces acting on the pine tree are gravity and the force of the string it hangs on. The tree's acceleration is backward and to the right, so the total force acting on it must be a horizontal vector that points backward and to the right. That means that the tree swings forward and to the left.

(d) Graphical addition of the string's force and the gravitational force produces a right triangle with its vertical leg being mg and its horizontal leg F_total, which equals ma. Therefore we have $\theta = \tan^{-1}(a/g) = 27.0°$. (Although $g = 9.8\ \text{m/s}^2$ has only two sig figs, varying it by $\pm 0.02\ \text{m/s}^2$ only changes θ by $0.05°$, so the answer really does have three sig figs.)

6-j6 Because the bead slides frictionlessly on the string, there can't be any difference in tension between the top and bottom segments. The vector sum of the two forces acting on the ball lies in the direction that bisects the two strings, and this direction is the direction of a simple conical pendulum. As found in another problem, this angle lies below the horizontal by an angle $\beta = \tan^{-1}(g/\omega^2 r)$. Geometry then gives $\phi = \theta + 2\beta = \theta + 2\tan^{-1}(g/\omega^2 r)$.

6-j7 (a) As the bead speeds up, a greater force toward the axis is required to keep it from flying off straight. This means that the normal force has to get stronger, and therefore the kinetic friction force will get stronger.

(b) Define a coordinate system with axes t, r, and p. Axis r points radially outward away from the central axis; t downhill and tangent to the wire; and p perpendicular to t and r. The velocity has only a component $v_t = v$. The motion, projected into the horizontal plane, is circular motion with speed $v\cos\theta$, where θ is the angle of the downhill slope, $\tan\theta = \alpha$, where $\alpha = d/2\pi r$. The acceleration is purely in the r direction, with $a_r = -v^2\cos^2\theta/r$.

The frictional force has only a t component. The normal force has both an r and a p component. The weight has both t and p components.

Newton's second law in the r direction gives $F_{n,r} = -mv^2/r\cos^2\theta$. Cancellation of the forces in the p direction gives $F_{n,p} = -F_{w,p} = mg\cos\theta$. We therefore have

$$F_{f,t} = -\mu_k F_n = -m\mu_k\cos\theta\sqrt{g^2 + \left(\frac{v^2\cos\theta}{r}\right)^2} \quad .$$

This has to cancel with $F_{w,t} = mg\sin\theta$, and solving for v we find

$$v = \sqrt{gr\sqrt{(1+\alpha^2)(1-\frac{\alpha^2}{\mu_k^2})}} \quad .$$

(c) In the limit of r approaching infinity, the problem is the same as that of an object sliding down an inclined plane. Under those circumstances, the only possible way of getting a constant velocity would be to have the slope chosen to have some particular value that would match properly with the coefficient of friction. A quick calculation shows that the necessary condition is $\mu_k = \tan\theta$. This matches up properly with our expression for v in the case of the helical wire, because the only way to get a finite limiting value for v as r approached infinity would be to let the factor $1 - \alpha^2/\mu_k^2$ approach zero.

6-m1 The relevant constant-acceleration equation is $\omega_f^2 = \omega_0^2 + 2\alpha\theta$. Setting $\omega_f = 0$ and solving for θ, we have $\theta = -\omega_0^2/2\alpha$. This is the number of radians. Ignoring the sign, the number of revolutions is $|\theta|/2\pi = \omega_0^2/4\pi\alpha = 290$ (2 sig figs).

6-m2 One revolution is 2π radians, so $2\pi = (1/2)\alpha t^2$. The result is $t = 2\sqrt{\pi/\alpha} = 35$ s.

6-m3 The constant acceleration equation $\theta = \omega_0 t + (1/2)\alpha t^2$ tells us that the number of revolutions is

$$\frac{\omega_0 t + (1/2)\alpha t^2}{2\pi\ \text{radian/rev}} = 54811747219.1399\ \text{rev.}$$

The fraction of a revolution is real. The method used in the observations was capable of measuring small fractions of a rotation. The effect of the deceleration is much larger than this, however — about 100 revolutions over this period of time.

6-m4 The rotational analog of the constant-acceleration equation $v_f^2 = v_0^2 + 2ax$ is $\omega_f^2 = \omega_0^2 + 2\alpha\theta$. The angular displacement is $\theta = 2\pi N$. Setting $\omega_f = 0$ and ignoring the sign, we have $\alpha = \omega^2/4\pi N$.

6-m5 (a) $\omega = \mathrm{d}\theta/\,\mathrm{d}t = 3At^2 - B$.
(b) $\alpha = \mathrm{d}\omega/\,\mathrm{d}t = 6At$.
(c) Setting $\omega = 0$ and solving for t gives $t_+ = \sqrt{B/3A}$.
(d)

$$\begin{aligned}\omega_{\text{av}} &= \frac{\Delta\theta}{\Delta t} \\ &= At_+^2 - B \\ &= -2B/3\end{aligned}$$

6-m6 The angular velocity is the indefinite integral of the angular acceleration:

$$\begin{aligned}\omega &= \int \alpha\,\mathrm{d}t \\ &= 6t^2 - 6t^4,\end{aligned}$$

with a zero constant of integration because we're given that the wheel is initially at rest. Setting $\omega = 0$, we find that the wheel is at rest again at $t = 1$ s. The number of revolutions is

$$\begin{aligned}N &= \Delta\theta/2\pi \\ &= \frac{1}{2\pi}\int_0^{1\text{ s}} \omega\,\mathrm{d}t \\ &= \frac{2}{5\pi} \\ &= 0.13\end{aligned}$$

6-p1 (a) For the Earth's rotation, $\omega = \Delta\theta/\Delta t = (2\pi)/(24\text{ hr}) = 7.3 \times 10^{-5}\text{ s}^{-1}$, and similarly for its yearly revolution around the sun, $\omega = 2.0 \times 10^{-7}\text{ s}^{-1}$.
(b) The acceleration is $\omega^2 r$. The factor of ω^2 would tend to make the acceleration due to rotation bigger by a factor of $365^2 = 1.3 \times 10^5$, but the r factor would contribute a factor of (radius of orbit)/(radius of Earth) $= 2.3\times10^4$ in the opposite direction. The former effect is bigger, so the acceleration due to rotation is bigger. It's surprising, however, that the two numbers would even end up being near the same order of magnitude, given how different the ω and r values were.

6-p2 (a) The easiest way to figure this out is to consider the case $r = 0$. Here the bug is just spinning in place, and clearly will not slip. (There is not even a preferred direction in which it would slip.) So the bug is more likely to slip at large values of r.

(b) The vertical forces on the bug cancel, so the normal force of the turntable acting on the bug equals mg, where m is the mass of the bug. The maximum force of static friction is $\mu_s mg$. Equating this to $m\omega^2 r$ gives $r = \mu_s g/\omega^2$.

6-p3 The maximum force of static friction is $\mu_s mg$. The acceleration vector has radial component $\omega^2 r$ and tangential component αr, so its magnitude is $r\sqrt{\omega^4 + \alpha^2}$. Applying Newton's second law, we get

$$\omega = \left[\left(\frac{\mu_s g}{r}\right)^2 - \alpha^2\right]^{1/4}$$

6-p4 (a) $\omega = \mathrm{d}\theta/\,\mathrm{d}t = 24t - 12t^2$.
$\alpha = \mathrm{d}\omega/\,\mathrm{d}t = 24 - 24t$.
(b) Setting $\omega = 0$ gives $t_2 = 2$ s. The maximum angular velocity occurs when the angular acceleration is zero, which is at $t_1 = 1$ s.
(c) Because $\alpha = 0$ at this time, $a_t = 0$. The radial acceleration is $a_r = \omega^2 r = 7.2\text{ m/s}^2$.
(d) $N = \Delta\theta/2\pi = 2.5$.
(e) Plugging in to the answer from part a gives $\omega = 12\text{ s}^{-1}$. The period is $T = 2\pi/\omega = 0.52$ s, and the frequency in revolutions per minute is

$$\frac{\text{rad}}{\text{s}} \times \frac{\text{s}}{\text{min}} \times \frac{\text{rev}}{\text{rad}} = \omega \cdot \frac{60}{1} \cdot \frac{1}{2\pi} = 115\text{ min}^{-1}.$$

6-p5 (a) $\alpha = \omega_0/t$, so $t = \omega_0/\alpha$.
(b) At time $t/2$, the angular velocity is $\omega_0/2$, and

equating $a_r = \omega^2 r$ to $a_t = \alpha r$ gives $\alpha = (\omega/2)^2$. Plugging this value of α in to the result from part a, we find $t = 4/\omega_0 = 0.10$ s.

6-s1 (a) Changing the pitch by a given factor simply scales up the rate at which we move outward to larger circumferences, so the angular acceleration must be directly proportional to p.

(b) We expect the angular acceleration, which has units of s^{-2}, to depend on v, p, and r, and since the only one of these variables that contains seconds is v, the result must be proportional to v^2. Given that $\alpha \propto v^2 p$, units constrain the dependence on r as well, so that $\alpha \propto v^2 p r^{-3}$.

(c) The given information is that

$$\omega r = v = \text{constant}$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{p}{2\pi} = \text{constant},$$

To save writing we introduce the symbol $h = p/2\pi$. The spiral can be either left-handed or right-handed, and this can be accomodated by assigning a sign to p and h. By the chain rule, we have

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t},$$

so that

$$\frac{1}{h} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot r = v.$$

Separating variables gives $r\,\mathrm{d}r = vh\,\mathrm{d}t$, and integrating this, we find $\frac{1}{2}r^2 = vht$, where the constant of integration can be taken to be zero, since it won't matter in our final answer what time we define to be $t = 0$. We now find

$$\omega = (v/2ht)^{1/2},$$

and differentiating this gives

$$\begin{aligned}\alpha &= -\frac{1}{2}(v/2h)^{1/2} t^{-3/2} \\ &= -v^2 h r^{-3} \\ &= -\frac{1}{2\pi} v^2 p r^{-3},\end{aligned}$$

so the unknown unitless constant was $-1/2\pi$.

(d) I found this easiest to figure out by imagining the spiral to stand still, while a point moves along it like a bead on a wire. When we reverse the bead's direction of motion, nothing happens to the geometrical variables p and r, v reverses its sign, and α keeps the same sign (similar to the way that a rock thrown up in the air has the same acceleration on the way up and the way down). Because our equation contains v^2, it continues to make sense when we flip the direction of motion: neither side changes sign.

(e) To complete our check that the answer has the right symmetry properties, we need to account for the minus sign in the result. At first glance, this appears to have the *wrong* symmetry, since it seems to say that the motor's torque on the CD would always be in the clockwise direction — but anything that looks clockwise from one side is counterclockwise from the back. The solution to this puzzle is that p also has a sign. If we view the CD player from the back rather than the front, both p and α flip signs.

6-s2 Application of Newton's laws gives a normal force mv^2/r and a tangential acceleration $(\mu/r)v^2$ due to friction. If we let x be the distance traveled around the circumference, then the motion satisfies the differential equation

$$\left(\frac{r}{\mu}\right) x'' = \pm (x')^2,$$

where the sign depends on the direction of motion. Since only the derivatives of x occur, not x itself, we can substitute $v = x'$ and rewrite this as a first-order equation

$$A\frac{\mathrm{d}v}{\mathrm{d}t} = \pm v^2,$$

where $A = r/\mu$. Separating variables and integrating twice, we find

$$x = \mp A \ln \ell(t),$$

where ℓ is any linear function.

7-a1 (a) To lift the bag at constant velocity, the person must cancel out the force of gravity

mg. The work done is therefore about $mgd =$ 50 kJ. (b) The calculation comes out the same, and this is what we expect based on conservation of energy if the pulley is frictionless: the bag still gains the same amount of energy. (In reality there would be some friction in the pulley, so the force would be slightly more than the given value, the work would be slightly greater, and the extra energy would go into heating the pulley and the rope.)

7-a2

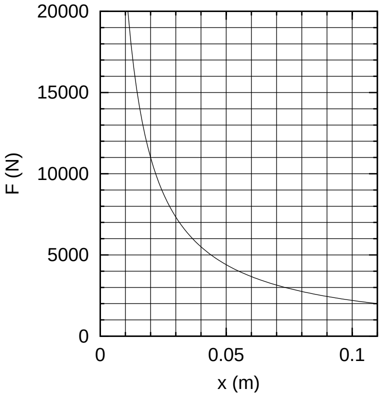
$$\begin{aligned} W_{wp} &= \int_0^x F_\parallel \, \mathrm{d}x' \\ &= \int_0^x (a+bx')\,\mathrm{d}x' \\ &= ax + \frac{1}{2}bx^2 \end{aligned}$$

By Newton's third law, the plane's force on the air is of the same strength, but in the opposite direction, and therefore $W_{wp} = -W_{pw}$.

7-a3 (a) See graph.

(b) 47 rectangles × 0.01 m × 1000 N = 470 J.

(c) The engine's power is the work done per unit time, which is $6 \times 2400 \times 470$ J/s $=$ 1.1×10^5 W = 150 horsepower.



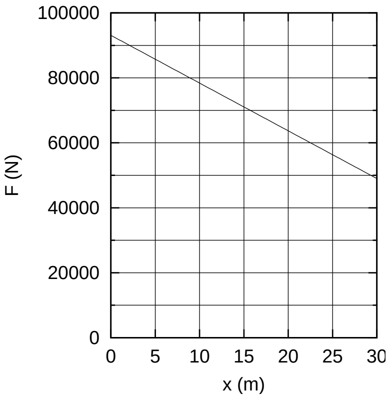
(d) Assuming the same value of a, if you doubled both x_1 and x_2, the x axis of the graph would be stretched out to twice the width, but since the force is inversely proportional to x, the force on the piston would be half as much, and the F axis of the graph would be shrunk to half the height. The area would not change. The argument fails for $F \propto x^{-1.4}$, because then the force would decrease by more than 50%.

7-a4 (a) The mass of 30 m of chain is 30 m × 150 kg/m = 4500 kg. The total mass is 5000 kg+ 4500 kg = 9500 kg, so the total weight is that times g, or 9.3×10^4 N.

(b) The mass has decreased by 150 kg, to 9350 kg, giving a weight of 9.2×10^4 N.

(c) There area under the curve is about 43 rectangles, each representing 5 m×10000 N = 5×10^4 J, so the total work done is 2.1×10^6 J.

(d) 2.1×10^6 J/(4186 J/kcal) = 500 kcal.



7-a5 (a) We want to find the area under the graph from $x = 0$ to $x = cL$. This area can be split up into a rectangle plus a triangle sitting on top of it. To make all the math easier, let's start by leaving out the factors of L and T_o, since these just set the scale on each axis; we'll start by calculating everything as if we had $L = 1$ and $T_\mathrm{o} = 1$, and then throw those two factors back in at the end. In these fake units, the point at the graph at the corner, where the muscle can't contract anymore, has coordinates $(c, 1-c)$. The rectange has width c, height of $1-c$, and area $c(1-c)$. The triangle has base c, height c, and area $(1/2)c^2$. The total area is $c - c^2/2$. Putting the factors of $L = 1$ and $T_\mathrm{o} = 1$ back in, we find that the work is $W = T_\mathrm{o}L(c - c^2/2)$.

(b) T has units of newtons, L meters, and c is unitless, so the result has units of newton-meters, which is equivalent to joules.

(c) The $c = 0$ indicates a muscle that is "frozen" in its extended position and can't flex at all; physically it shouldn't be able to do any work, and mathematically we do get $W = 0$ in this case. In the $c = 1$ case, the area would be that of a complete triangle with width L and height T_o, giving an area of $(1/2)T_oL$, and this is indeed what we get if we plug in $c = 1$ into the general equation.

(d) For a cylinder, we have $V = LA$, and with $T_o = kA$ the result for the work becomes $kV(c - c^2/2)$. The whole thing is strictly proportional to the muscle's volume, V, which is why bodybuilders talk about "bulking up."

(e) Converting the given data to mks, 200 cm^3 = 2×10^{-4} m^3, and $k = 10^6$ N/m^2. Plugging in, we get 96 J.

7-d1 No. The mass can't be negative, and neither can v^2.

7-d2 A force is an interaction between two objects, so while the bullet is in the air, there is no force. There is only a force while the bullet is in contact with the book. There is energy the whole time, and the total amount doesn't change. The bullet has some kinetic energy, and transfers some of it to the book as heat, sound, and the energy required to tear a hole through the book.

7-d3 $\text{KE}_B = \text{KE}_P \implies (1/2)m_Bv_B^2 = (1/2)m_Pv_P^2$. Solving for v_P, we get $v_P = v_B\sqrt{m_B/m_P} = 160$ m/s.

7-d4 The initial KE is $(1/2)mv^2$, and the final KE is $(1/2)m(2v/3)^2 = (1/2)(4/9)mv^2$. The final KE is 4/9 of the initial KE, and so $1 - 4/9 = 5/9$ of the initial kinetic energy was lost.

7-d5 We're testing conservation of energy,

$$E_{total,i} \quad =? \quad E_{total,f}.$$

The only form of energy we can determine from the photo is kinetic energy, so this amounts to testing whether

$$KE_{1,i} \quad =? \quad KE_{1,f} + KE_{2,f},$$

where the initial time is before the collision and the final time is after the collision is over. Using $KE = (1/2)mv^2$, and taking advantage of the fact that the masses are equal, this becomes

$$v_{1,i}^2 \quad =? \quad v_{1,f}^2 + v_{2,f}^2.$$

Now $v = d/t$, where t is the time between flashes, and the common factor of t^{-2} can be divided out on both sides, so we end up with

$$d_{1,i}^2 \quad =? \quad d_{1,f}^2 + d_{2,f}^2,$$

where d is the distance between one position and the next. That is, the square of the distance is our measure of energy.

But rather than measuring directly from one position to the next, it's more accurate to measure the distance over several of the time intervals in succession. In four flashes, the incoming ball moves 6.0 cm. In the same amount of time after the collision, it moves 2.8 cm, and the target ball moves 5.0 cm.

Since we're leaving out lots of factors that have units, I'll write the numbers below without units.

$$\begin{aligned} 6.0^2 &=? \quad 2.8^2 + 5.0^2 \\ 36.0 &=? \quad 7.8 + 25.0 \\ 36.0 &=? \quad 32.8 \end{aligned}$$

There appears to have been a slight loss of energy, but it's really at the limit of our ability to measure things from the figure. We also know that there will be some extra energy after the collision in the form of heat and sound. The only thing that would really disprove conservation of energy in this collision would be if the final KE was *greater* than the initial KE, and if the increase was significant given the limited accuracy of the measurements.

7-d6 Converting the speeds to mks, we have 18 m/s and 36 m/s. (a) $(1/2)mv^2 = 2.4 \times 10^5$ J; (b) Four times greater, 9.6×10^5 J. In this sense, driving twice as fast is four times more dangerous, not twice as dangerous as you might think.

7-d7 We don't have actual masses and velocities to plug in to the equation, but that's OK. We just have to reason in terms of ratios and proportionalities. Kinetic energy is proportional to mass and to the square of velocity, so B's kinetic energy equals

$$(13.4\text{ J})(3.77)/(2.34)^2 = 9.23\text{ J}$$

7-d8 (a) Plugging the data into $KE = (1/2)mv^2$ gives a kinetic energy of 23 kilojoules.
(b) The hundred-times-greater mass increases the KE by a factor of 100, but KE depends on the square of the velocity, so reducing the speed by 1/100 cuts the kinetic energy by 1/10,000. The combined effect is a factor of 100/10,000=1/100. The boulder has 100 times less KE than the foam.
(c) Our intuition doesn't do well with things that are squared. We don't expect the foam to have so much kinetic energy, but its big speed, squared, results in a really really big KE.

7-d9 The total amount of energy required to kill her is

$$(60\text{ kg})\left(\frac{1\text{ J}}{0.24\text{ g}\cdot{}^\circ\text{C}}\right)\left(\frac{10^3\text{ g}}{1\text{ kg}}\right)(6^\circ\text{C}) = 1.5\times10^6\text{ J}$$

Solving the equation $P = \Delta E/\Delta t$ for $\Delta t = \Delta E/P$, we find that she can survive for 7500 s=120 min.

7-d10 (a) The energy stored in the gasoline is being changed into heat via frictional heating, and also probably into sound and into energy of water waves. Note that the kinetic energy of the propeller and the boat are not changing, so they are not involved in the energy transformation. (b) The crusing speed would be greater by a factor of the cube root of 2, or about a 26% increase.

7-g1 If she can run the 100 m dash in about 10 s, then she's going about 10 m/s. If her mass is 60 kg, then her kinetic energy is about 3000 J (one sig fig).

7-g2 As in the examples in ch. 1, we don't try to estimate the mass of the wing directly. Instead, we estimate linear dimensions and get mass from that. Let's approximate a fly's wing as a square 3 mm on a side. I don't know the thickness, but since flies' wings are translucent, it must be much less than the thickness of a piece of paper. I'm going to guess that the thickness is 0.1 mm. This gives a volume of about 1 mm^3, and if we assume the same density as water, we get a mass of 10^{-6} kg.

Let's say that a fly's wing beats 100 times a second, which would be of the right order of magnitude to match the fact that the buzzing is in the audible range of human hearing. Then the speed of a wing would be about $(3\text{ mm})/(10^{-2}\text{ s}) \approx 0.3$ m/s.

The result for the kinetic energy is about 10^{-7} J.

7-g3 As always (see sec. 1.4), we avoid estimating mass or volume directly and instead estimate linear dimensions. Let's say a blade of grass is 0.1 cm thick, 1 cm wide, and 10 cm tall. Then its volume is about 1 cm^3, and if it has the same density as water, then its mass is about 1 g, or 10^{-3} kg.

For the velocity, let's say it grows 1 cm in 10 days. This gives 10^{-8} m/s.

Putting these factors together, we have $KE = (1/2)mv^2 \sim 10^{-19}$ J.

7-g4 This is only an order-of-magnitude estimate, so let's make some simplifying approximations. We'll assume the surface area of the oceans is the same as the surface area of the planet ($4\pi R^2$), and that the density of ice is about the same as that of water, 1000 kg/m^3. Then the amount of energy required is

$$\begin{aligned}E &= (1000\text{ kg/m}^3)(10\text{ m})(4\pi R^2)(3\times10^3\text{ J/kg})\\ &\sim 10^{23}\text{ J}\end{aligned}$$

Dividing by 10 years (about 3×10^8 seconds), we find that the power is on the order of 1% of the sun's light output. This is a surprisingly small amount. The equilibrium is extremely delicate.

7-j1 Kinetic energy depends on how fast the ball is moving. It's moving fastest at the bottom, so that's where it has the maximum kinetic energy. Energy is conserved, so it's trading KE and gravitational energy back and forth, and it has the most gravitational energy when it has the least KE, which is at the top when it's momentarily at rest.

7-j2 Yes, if it's below the reference level you arbitrarily chose as $PE = 0$.

7-j3 (a) +, foot moves same direction as foot's force on ground
(b) +, force and motion are both forward
(c) −, force is backward, motion is forward
(d) −, force is up and motion is down

7-j4 (a) The definition of work is a (mechanical) transfer of energy. Changing the direction of the positive coordinate axis has no effect on the work. Energy is a scalar, so directions are irrelevant.
(b) Reversing the direction of the positive coordinate axis reverses the sign of both F and d, so $W = (-F)(-d) = Fd$ comes out the same.

7-j5 No. Work describes how energy was transferred by some process. It isn't a measurable property of a system.

7-j6 (a) Magnetic energy plus kinetic energy stays the same, so if it's speeding up, the KE is increasing and the magnetic energy must be decreasing. (b) By similar reasoning, magnetic energy must be increasing as they approach each other.

7-j7 (a) Example: As one child goes up on one side of a see-saw, another child on the other side comes down. (b) Example: A pool ball hits another pool ball, and transfers some KE.

7-j8 Kinetic energy doesn't depend on the direction of the motion, so since both pennies start out at the same speed, they both start out with the same kinetic energy. They also start out at the same height, so they have the same gravitational energy and the same total energy. Energy is conserved, so if they start out with a certain amount of energy, that's the same amount of energy they'll always have; their total energies will always be equal to each other. Although the impacts with the ground will happen at different times, we don't care about that. On impact they have the same gravitational energy and the same total energy, so they must have the same kinetic energy on impact as well. They hit the ground with equal kinetic energies and equal speeds.

7-j9 (a) False: the work done by friction depends on the distance traveled, and cannot be known from just the initial/final points.
(b) True: The object will change direction, but it will not speed up / slow down.
(c) False: This is true if the initial and final points are the same, but this doesn't have to be the case. For example, gravity is a conservative force, but it will do positive work on a rock that falls vertically downwards.
(d) False: Doubling the *distance* a (constant) force is applied will double the work. The relationship between force and time is explored with the concept of momentum.
(e) False: KE is positive, but $W_{\text{net}} = \Delta\text{KE}$ can be negative.

7-j10 At the atomic level, the helium's heat energy is the random motion of the helium atoms. Cylinder B's atoms are moving at the same average speed as A's, since they're at the same temperature, but B has twice as many atoms, so its total KE is twice as much.

7-j11 When the sweat evaporates, electrical energy is increasing, because as the sweat changes from a liquid to a gas, its molecules, which are attracting each other electrically, get farther away. By conservation of energy, some other form of energy must be decreasing: heat.

If you wipe the sweat off with a towel, you lose the cooling effect. The water evaporates after it's out of contact with you, so it doesn't cool you off. As a method of cooling yourself, it's no more effective than spitting!

7-j12 In a solid, the molecules are packed in place, and aren't free to twist around very much. In a liquid, they can rotate more freely. The microwaves exert the same force F on a molecule, regardless of whether it's in a solid or a liquid, but it's not just the force that matters, since $W = Fd$. In the solid, d is small, so little energy is transferred.

7-j13 As in the example of the water faucet, conservation of mass says that a narrower stream corresponds to a higher velocity. The water in the discharge pipe is therefore moving faster than the water in the drive pipe, in inverse proportion to its smaller cross-sectional area. A kilogram of water with a high velocity has a higher KE than a kilogram of water with a low velocity, and therefore it will be able to rise higher, converting that KE into a lot of PE. Note, however, that we're not really getting anything for free here. The amount of water that gets up to the top reservoir is a lot less than the amount of water that was allowed to flow into the big pipe. Most of the water is wasted into the ditch way down at the bottom. People use this device in areas where there's a free source of water, but no electricity to run a pump.

7-m1

$$\begin{aligned}
E_{total,i} &= E_{total,f} \\
PE_i + \text{heat}_i &= PE_f + KE_f + \text{heat}_f \\
\frac{1}{2}mv^2 &= PE_i - PE_f + \text{heat}_i - \text{heat}_f \\
&= -\Delta PE - \Delta\text{heat} \\
v &= \sqrt{2\left(\frac{-\Delta PE - \Delta\text{heat}}{m}\right)} \\
&= 6.4\ \text{m/s}
\end{aligned}$$

7-m2 Energy is power times time. The total energy required is 25 watts $*$ (31 days) $= 6.7 \times 10^7$ J. One kilowatt-hour is an energy equal to (1000 watts)(3600 s) $= 3.6 \times 10^6$ J. Therefore,

$$6.7 \times 10^7\ \text{J}\left(\frac{15\ \text{cents}}{3.6 \times 10^6\ \text{J}}\right) = \$2.80 \qquad (24.1)$$

(2 sig figs, because the power of 25 W had 2 sig figs).

7-m3 Power is energy divided by time, or in this case $P = mgh/\Delta t$. Plugging in the numbers given, $P = (44\ \text{kg})(9.8\ \text{m/s}^2)(11.0\ \text{m})/23\ \text{s} = 210$ Watts.

7-m4 Let's call the power of the motor P, the mass of the car M, and the height of the building H. If the car is moving upwards at constant velocity, then the forces acting on the car sum to zero, and upwards force on the car from the motor is Mg vertically upwards. If the motor does work MgH in a time T, then $P = MgH/T$, which gives $T = MgH/P$ as the time it takes the motor to lift the car. Plugging in numbers gives $T = (1150\ \text{kg})(9.8\ \text{m/s}^2)(25.0\ \text{m})/(5.26 * 746\ \text{W}) = 71.8$ s.

7-m5 Let H be the net distance that the roller coaster descends (in this problem, $H = (35 - 23)\ \text{m} = 12\ \text{m}$). If there were no friction, the roller coaster would pick up a kinetic energy equal to mgH, so the speed would be $\sqrt{2gH}$. Since energy is converted to heat by friction, we actually have $v < \sqrt{2gH}$. The additional information of the problem tells us that the apparent weight of the passenger (equal to the normal force from the seat) is $(2/3)mg$, implying that the net force in the vertical direction is $(1/3)mg$ vertically downward. Setting this equal to mv^2/R, solving for v, substituting into $v < \sqrt{2gH}$, and solving for R, we get $R < 6H = 72$ m. It makes sense that this is an upper bound on R. If there is more friction, we have to make the hill curve more sharply to make up for the smaller speed.

7-m6 (a) Gravity does a work $mgh = 44$ mJ on the piece of paper over the course of the fall, and the problem says that 37 mJ was dissipated (lost). Therefore, there is 7 mJ of kinetic energy when the paper reaches the ground.

(b) Using KE $= (1/2)mv^2$ and plugging in the KE found in part (a), we find $v = 1.76$ m/s.

7-m7 (a)

$$\begin{aligned}
PE_i &= PE_f + (\text{energy of boiling}) \\
PE_i &= PE_f + mE_b \\
\Delta PE_i &= -mE_b \\
F_{grav}\Delta y &= -mE_b \\
mg\Delta y &= -mE_b \\
\Delta y &= -\frac{E_b}{g}
\end{aligned}$$

(b)

$$\begin{aligned} \text{m} &=? \ \frac{\text{J/kg}}{\text{m/s}^2} \\ &=? \ \frac{\text{kg}\cdot\text{m}^2/\text{s}^2/\text{kg}}{\text{m/s}^2} \\ &=? \ \frac{\text{m}^2/\text{s}^2}{\text{m/s}^2} \\ &= \ \text{m} \end{aligned}$$

7-m8 The mechanical work done by the hand equals the area under the curve between A and B. I counted about 136 squares. Each square is 20 N tall and 0.05 m wide, so it represents an area of 1 N$\cdot$m = 1 J, and the total area is 135 J. Accounting for the loss of 30% of this energy, and solving for $v = \sqrt{2K/m}$, we find 85 m/s.

7-m9 (a) We have to heat it by 20 degrees, melt it, heat it by 100 degrees, boil it, and then heat it by another 37 degrees. The result is

$$\begin{aligned} &(20)(2.05\ \text{J}) \\ &+333\ \text{J} \\ &+(100)(4.19\ \text{J}) \\ &+2500\ \text{J} \\ &+(37)(2.01\ \text{J}) \\ &=3.4 \times 10^3\ \text{J} \qquad . \end{aligned}$$

Surprisingly, the majority of the energy is used in a single step of the process, the boiling.

(b) To minimize the amount required, we want the initial temperature of the water to be the maximum, 100°C, and similarly the ice should be at 0°C. At the end of the process, we'll just have one big sample of water at some intermediate temperature, and to minimize the amount of water, we want this temperature to be as low as possible, zero degrees. Conservation of energy gives

$$\begin{aligned} \Delta E &= 0 \\ 0 &= -m(100^\circ\text{C})(4.19\ \text{J/g/}^\circ\text{C}) + (1\ \text{g})(333\ \text{J/g}) \\ m &= 0.795\ \text{g} \end{aligned}$$

7-m10 (a) If the water lost potential energy and the energy was just destroyed, then energy wouldn't be conserved. The only reason Joule thought there would be a temperature effect was that he was already convinced that energy was conserved. By making surprising predictions and then proving them to be correct, he could convince people that his theory was right.

(b) A joule is defined as the amount of heat needed to raise 0.24 g of water by 1°C. If 0.24 g falls 50 meters, the amount of potential energy released is $(2.4\times10^{-4}\ \text{kg})(9.8\ \text{m/s}^2)(50\ \text{m}) = 0.1$ J. If all this energy is converted to heat, then it heats that particular amount of water by 0.1°C.

(c) We had to assume all the energy went into heating. There are three main effects I can think of. First, the water at the top and bottom of the falls might be moving, in which case there would be a third form of energy involved, kinetic energy. If the pool at the bottom is still, whereas the water above the top is moving rapidly before it goes over the edge, then there would be extra heating. On the other hand, waterfalls are noisy, so energy is released into sound waves at the bottom; this effect would tend to reduce the amount of heat production. The third effect is that water might evaporate in mid-air, further reducing the heat produced.

7-m11 (a) As Weiping raises the rock, the sign of the force and the sign of d are the same (both are up), so she does 1 joule of work. On the way down, her force is up and the motion is down, so she does -1 joule of work. The total is zero.

Bubba's force was in the same direction as the motion during both halves of his motion, since, unlike gravity, the force of friction reversed directions on the way back. Bubba did 2 J of work.

(b) Work is defined as a transfer of energy. Weiping added gravitational PE to the rock on the way up, but on the way back down, she took it back out. The rock ends up with the same gravitational PE it started with, so by the definition of work, we verify that her work on the rock is zero.

As Bubba slides the table, he's adding heat energy to it. When he slides the table back in the opposite direction, he's not refrigerating it! Therefore, based on the definition of work, we know that the total work he does is not zero. This agrees with the result from part a.

7-p1 Since $KE = (1/2)mv^2$, we have $v \propto 1/\sqrt{m}$, so the ratio of the rates at which the two gases leak out is $\sqrt{14.01/16.00} = 0.9357$, with the oxygen leaking out more slowly.

7-p2 (a) $KE = \frac{1}{2}mv^2$, so if the average KE is a fixed number, the average velocity is proportional to $m^{-1/2}$. The lighter He atom is moving $\sqrt{8} = 2.8$ times faster.

(b) He atoms are always speeding up and slowing down due to collisions, but because of their small mass, their average speed is very high, and a few unusual ones are moving very very fast — faster than escape velocity, meaning they leave the earth and never fall back.

7-p3 (a) The amount of KE to be converted into heat is the same in both cases, so the amount of work that has to be done is the same in both cases. Since work equals force times distance, and the work and force are the same, the stopping distance is the same.

(b) Since the stopping distances are equal, if you watched videos of the two boats stopping, the videos would look exactly the same except for a rescaling of the time. That is, the slower boat's motion would look like the faster one's, but in slow motion. Therefore the time for the slower boat to stop will be greater.

7-p4 (a) $K = \frac{1}{2}mv^2 = \frac{1}{2}ma^2t^2$.

(b) $P = dW/dt = dK/dt = ma^2t$.

(c) Four times more.

7-p5 If the car is to move at a constant velocity, the energy per unit time ("power") that the motor provides to the car must equal the energy per unit time taken out of the car by the drag force. The power delivered by a force $\mathbf{F}$ to an object moving at velocity $\mathbf{v}$ is $P = \mathbf{F}\cdot\mathbf{v}$. The drag force is proportional to v^2, and so the power from the drag force (which is negative) is proportional to v^3.

If you're able to double the power output of the motor, then you can reach a new speed such that the power associated with the drag force has doubled. This happens when you reach a final speed $v_f = (2)^{1/3}v_0 = (1.26)(55\text{ mph}) = 69\text{ mph}$.

7-s1 From the work-kinetic energy theorem, the work you do on the ball equals the change in kinetic energy of the ball. $\Delta\text{KE} = (1/2)m(3v)^2 - (1/2)mv^2 = 4mv^2$, and the work done on the ball is $W = Fx$, where x is the distance over which your foot is in contact with the ball. Setting these quantities equal gives $x = 4mv^2/F$.

7-s2 (a) Kinetic energy is equal to $(1/2)mv^2$. The two computers reach the ground with the same speed, so since the second computer has three times the mass of the first computer, the second computer lands with three times the kinetic energy of the first computer. Answer: $3K$.

(b) The speeds are the same for the two computers, so $v = \sqrt{2K/m}$.

7-s3 The energy required to pick up and throw one stone is $mgh + (1/2)mv^2$. In a time T, the girl can exert an energy PT. Therefore,

$$\begin{aligned} \#\text{ stones} &= (\text{total energy}) \;/\; (\text{energy per stone}) \\ \#\text{ stones} &= \frac{PT}{mgh + (1/2)mv^2} \end{aligned} \tag{24.2}$$

7-s4 (a) The distance traveled in time Δt is $d = v\Delta t$, so the work done is $\Delta E = Fd = Fv\Delta t$, and $P = \Delta E/\Delta t = Fv$.

(b) The vertical forces cancel, so $F_N = mg$, and $F_{s,max} = \mu_s F_N = \mu_s mg$. To burn rubber, the engine has to supply $P > F_{s,max}v = \mu_s mgv$. The critical speed is $v = P/(\mu_s mg)$.

(c) W/N = J/N·s = N·m/N·s = m/s.

(d) Physically, a more powerful engine should be able to burn rubber at even higher speeds; mathematically, increasing P increases v. Physically, better traction makes it harder to burn rubber; mathematically, increasing μ_s decreases v. Increasing m or g also has the effect of increasing traction, so the dependence on these two variables also makes sense, for similar reasons.

(e) (Plug in.)

(f) The power consumed by friction is $Fv \propto v^3$. At top speed, 100% of the engine's power is being consumed by fighting air friction. At the lower speed, this is decreased by $(40/176)^3 \approx .01$.

7-s5 (a) The work on the paper is $W = \mathbf{F} \cdot \Delta\mathbf{x} = +mgH$.

(b) Since KE $\propto v^2$, if the paper lands with 1/5th the speed, then it lands with 1/25th the kinetic energy. This means that 24/25ths of the work done by gravity was lost to the drag force. Also, the drag force tends to take energy out of the system, and so it does negative work on the paper. Combining these two facts gives $W_{\text{drag}} = -(24/25)mgH$.

7-s6 (a) Except for m, none of the variables has SI units that involve kg. Therefore the answer can't depend on m. Both θ and μ_k are unitless, so if we want to make an answer with units of meters per second, we can only do this using the variables g and L. The only way to make this happen is for the answer to be proportional to $\sqrt{gL}$.

(b) The block picks up an energy $mgL\sin\theta$ from gravitational potential energy and loses $f_kL = \mu_kNL = \mu_kmgL\cos\theta$ of energy due to friction. The overall energy gain goes to KE: $(1/2)mv^2 = mgL(\sin\theta - \mu_k\cos\theta)$, or

$$\begin{aligned} \text{KE} = (1/2)mv^2 &= mgL(\sin\theta - \mu_k\cos\theta) \\ v &= \sqrt{2gL(\sin\theta - \mu_k\cos\theta)} \end{aligned}$$

(c) On physical grounds, we expect the answer to be an increasing function of θ, g, and L, and a decreasing function of μ_k. The answer has these properties. The answer becomes unphysical when the quantity inside the square root is negative. This happens for $\theta > \tan^{-1}\mu_k$, and the physical significance of this must be that for angles this low, the block would stick rather than accelerating (even if we gave it an infinitesimal kick to get it going). (d) The speed of the object without friction can be obtained by plugging in $\mu_k = 0$ in the answer for the previous part. Therefore,

$$\sqrt{2gL(\sin\theta - \mu_k\cos\theta)} = \frac{1}{2}\sqrt{2gL\sin\theta}$$

Solving this for μ_k gives $\mu_k = (3/4)\tan\theta$.

7-s7 (a) The result is going to have SI units of meters, so any kilograms or seconds in the input variables will have to cancel out. The only way to make this happen is if the variables m, g, and k occur in the combination mg/k.

(b) The speed required at launch is determined by kinematics. The time for the ball to hit the ground is $t = \sqrt{2H/g}$, which means that the speed at launch needs to be $v = D/t = D\sqrt{g/(2H)}$. Setting $(1/2)mv^2 = (1/2)kx^2$ gives $x = D\sqrt{mg/(2kH)}$.

7-s8 Conservation of energy tells us that the initial energy equals the final energy. What times should we pick as initial and final? Our goal is to find out what height the bike has to start from, so the initial time should be when the bike is starting out, at the top of the ramp. The height is determined by the requirement that the bike not drop off the loop, even at the top, so to connect that to the other information, we should pick the final moment to be when the bike is at the top of the loop, at a height of $2r$.

First let's write conservation of energy in its most general form, and then make it more and more specific:

$$\begin{aligned} E_{total,i} &= E_{total,f} \\ PE_i + KE_i &= PE_f + KE_f \\ mgh + 0 &= mg(2r) + \frac{1}{2}mv^2 \\ h &= 2r + \frac{v^2}{2g} \end{aligned}$$

The height depends on the speed v the bike needs at the top of the loop. But how fast *does* the bike have to be moving when it's at the top? If it was just barely about to lose contact with the loop, then the only force on it would be gravity, so its downward acceleration would be $a = F_{grav}/m = (mg)/m = g$. For circular motion, the inward acceleration is v^2/r, so we have $g = v^2/r$, and $v = \sqrt{gr}$. Plugging this into the equation for h, we have

$$\begin{aligned} h &= 2r + \frac{v^2}{2g} \\ &= 2r + \frac{r}{2} \\ &= \frac{5}{2}r \end{aligned}$$

7-s9 (a) The minimum speed is the one at which the acceleration at the top of the loop is exactly the acceleration due to gravity, $v_{\text{min}}^2/r = g$. The extra-safe speed we want is $v = 2v_{\text{min}} = 2\sqrt{gr}$.

(b) Conservation of energy gives

$$mgy_i = mgy_f + \frac{1}{2}mv^2.$$

Using the result from part a and $y_i - y_f = h - 2r$, we find $h = 4r$.

7-s10 (a) The work done by gravity is $W_g = M_A gD$.

(b) The tension in the string can be determined from applying Newton's Second Law to both block A and block B:

$$\begin{aligned} T - \mu_k M_A g &= M_A a \\ M_B g - T &= M_B a \end{aligned} \qquad (24.3)$$

This gives $T = (1 + \mu_k)(M_A M_B g)/(M_A + M_B)$.

(c) The work done by tension is $W_T = -TD$:

$$W_T = \frac{-(1 + \mu_k)M_A M_B gD}{M_A + M_B} \qquad (24.4)$$

(d) From $W_{\text{net}} = W_g + W_T = (1/2)M_B v^2$,

$$v = \sqrt{\frac{2gD(M_B - \mu_k M_A)}{M_A + M_B}} \qquad (24.5)$$

Note there is an easier way to find the speed of the blocks, straight from conservation of energy. As the blocks move, they get energy from the gravitational work on block B (equal to $+M_B gD$), and friction on block A takes energy out of the system (friction does a work $-\mu_k M_A gD$). Adding these together gives the total KE of the system, also equal to $(1/2)(M_A + M_B)v^2$.

7-s11 (a) Each atom in the hoop travels the full circumference of the hoop in time T, so its speed is $2\pi r/T$. Plugging this into $KE = mv^2/2$ gives the result claimed.

(b) As the hoop rolls down the slope, the distance its center travels in one period equals the circumference. Therefore the hoop's center-of-mass speed is the same as the speed calculated in part a, $2\pi r/T$. That means that the hoop's KE due to center-of-mass motion is the same as its KE due to rotation. It's as though the hoop had double its normal inertia.

7-s12 (a) Let the distance that the spring is compressed be x. From conservation of energy, the gravitational PE turns fully into elastic PE. Therefore, $mg(H + x) = (1/2)kx^2$. Taking the positive root,

$$x = \frac{mg + \sqrt{(mg)^2 + 2mgkH}}{k} \qquad (24.6)$$

(b) From conservation of energy, $\text{PE}_{\text{grav}} = \text{KE}$. $mgH = (1/2)mv^2 \implies v = \sqrt{2gH}$.

(c) As stated in the hint, the box reaches maximum speed when the spring has been compressed an amount $x = mg/k$. Cons. of energy gives $mg(H + x) - (1/2)kx^2 = (1/2)mv_{\text{max}}^2$. Plugging in for x and solving for v_{max} gives

$$v_{\text{max}} = \sqrt{\frac{2kgH + mg^2}{k}} \qquad (24.7)$$

7-s13 Let θ be the angle by which he has progressed around the pipe. Conservation of energy gives

$$\begin{aligned} E_{total,i} &= E_{total,f} \\ PE_i &= PE_f + KE_f \\ 0 &= \Delta PE + KE_f \end{aligned}$$

$$0 = mgr(\cos\theta - 1) + \frac{1}{2}mv^2.$$

While he is still in contact with the pipe, the radial component of his acceleration is

$$a_r = \frac{v^2}{r},$$

and making use of the previous equation we find

$$a_r = 2g(1 - \cos\theta).$$

There are two forces on him, a normal force from the pipe and a downward gravitational force from the earth. At the moment when he loses contact with the pipe, the normal force is zero, so the radial component, $mg\cos\theta$, of the gravitational force must equal ma_r,

$$mg\cos\theta = 2mg(1 - \cos\theta),$$

which gives

$$\cos\theta = \frac{2}{3}.$$

The amount by which he has dropped is $r(1 - \cos\theta)$, which equals $r/3$ at this moment.

7-s14 First we need to relate the speed of the rollers to the speed of the slab. In the frame of reference of the rollers, the slab is traveling in one direction at a certain speed v, and the inclined plane is moving in the other direction at the same speed. Therefore back in the laboratory frame, the rollers have velocity v and the slab $2v$.

The problem can be solved either using Newton's laws or conservation of energy. Using Newton's laws is cumbersome, because it involves two normal forces and two static frictional forces, neither of which we know or want to know. Therefore let's use conservation of energy.

Although there are multiple rollers, the definitions of the total quantities m and I are such that it doesn't matter whether we sum over all the rollers or just pretend there is a single roller (along with a constraint that keeps the slab from tipping). Let the radius of the roller be r. It will clean things up if we define a unitless constant k so that $I = kmr^2$.

The kinetic energy of the roller has a part due to the roller's center-of-mass motion and another part due to its rotation. The former equals $(1/2)mv^2$. To find the latter, we use the fact that if the roller rotates through an angle $\mathrm{d}\theta$ in time $\mathrm{d}t$, we have $\omega = \mathrm{d}\theta/\,\mathrm{d}t$, while the distance traveled by the roller is $r\,\mathrm{d}\theta$. The c.m.'s velocity is therefore $v = r\omega$, giving $\omega = v/r$. The energy of the rotation is $(1/2)I\omega^2 = (k/2)mv^2$. The total kinetic energy is

$$K = \frac{1}{2}mv^2(1+k) + \frac{1}{2}M(2v)^2$$

This is independent of r, which we could have anticipated since there would have been no way to incorporate r into a relation of this form while making the units make sense.

Let s be the distance that the rollers have traveled at a certain point in the motion, which means that the slab has gone $2s$. The change in potential energy since the beginning of the motion (which we take to have been at rest) is

$$\Delta U = -(2M + m)gs\sin\theta.$$

To simplify these expressions, let

$$\mu = 4M + m(1+k)$$

and

$$\gamma = (2M + m)g\sin\theta.$$

Conservation of energy then reads as

$$\frac{1}{2}\mu v^2 - \gamma s = 0.$$

This is identical to the statement of conservation of energy for a free-falling mass with mass μ, and $mg = \gamma$, so the acceleration of the rollers is γ/μ, while that of the slab is $2\gamma/\mu$, or

$$\frac{a}{g\sin\theta} = \frac{M + \frac{1}{2}m}{M + \frac{1}{4}m(1+k)}.$$

This expression has the right limiting behavior when $m \rightarrow 0$, since then we have $a = g\sin\theta$, as if the slab was sliding frictionlessly. We obtain the opposite limit by taking $M \rightarrow 0$ and $k \rightarrow 0$ (i.e., the mass of each roller is concentrated at its center). Then $a = 2g\sin\theta$. The interpretation of the factor of 2 is that the rollers roll down the slope with acceleration $g\sin\theta$, and the massless slab comes along at twice the acceleration.

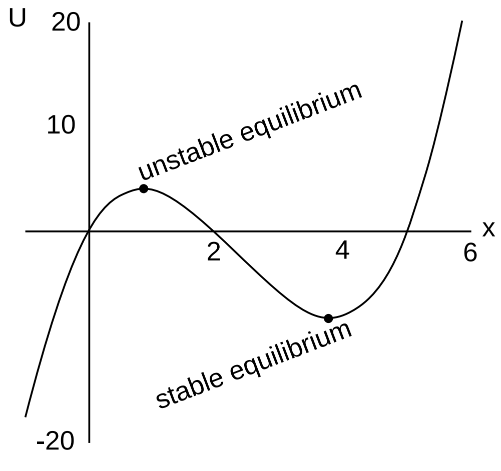
So far we haven't done any check on the dependence on k. For a sphere, $k = 2/5$, and if we let $M = 0$, our result is that $a = (10/7)g\sin\theta$ for the slab, or $a = (5/7)g\sin\theta$ for the roller. This matches with the well known result.

7-v1

$$
\begin{aligned}
E_i &= E_f \\
KE_i + PE_i &= KE_f + PE_f \\
0 + PE_i &= KE_f + PE_f \\
KE_f &= PE_i - PE_f \\
&= -\int_{r_i}^{r_f} kr^{-3}\,\mathrm{d}r \\
&= \frac{k}{2}\left(r_f^{-2} - r_i^{-2}\right)
\end{aligned}
$$

7-v2 (a) $F(x) = -dU/dx = 2\alpha x - 4\beta x^3 = 2x(\alpha - 2\beta x^2)$. This is zero for $x = \sqrt{\alpha/2\beta}$.
(b) $F(2\sqrt{\alpha/\beta}) = 2\alpha(2\sqrt{\alpha/\beta}) - 4\beta(2\sqrt{\alpha/\beta})^3 = -28\alpha^{3/2}\beta^{-1/2}$

7-v3 For the purposes of this solution, let $A = (1.00\ \mathrm{J/m}^3)$, $B = -(7.00\ \mathrm{J/m}^2)$, and $C = (10.0\ \mathrm{J/m})$, so that $U(x) = Ax^3 + Bx^2 + Cx$.
(a) $F(x) = -dU/dx = -3Ax^2 - 2Bx - C$.
(b) Conservative: if a potential energy function can be defined, then it is a conservative force.
(c) (See graph.)



(d) There is one critical point of $F(x)$ on $\mathcal{R}$ at $x_0 = -B/(3A) = (7/3)$ m. However, both endpoints of $\mathcal{R}$ have a larger value of $|F(x)|$, and the largest is at $x = +6.00$ m, where $|F(+6.00\ \mathrm{m})| = 34.0$ N.
(e) Looking at the graph of $U(x)$, we see that the particle is bound between $x_1 = +2.00$ m and $x_2 = +5.00$ m. The particle starts off at rest at x_1, feels a force and picks up speed to the right until reaching maximum speed at the critical point $x_c = +3.79$ m, and then slows down and comes to rest for an instant at $x = +5.00$ m before moving to the left. The sum of the kinetic plus potential energy is always zero, and so the maximum KE is the negative of the minimum U. Since $U(x_c) = -8.21$ J, the maximum KE is $+8.21$ J. With the knowledge that the particle has mass 1.00 kg, we can also say that the maximum speed achieved is 4.05 m/s.

7-v4 For the purposes of this solution, let $A = (1.00\mathrm{J/m}^4)$, $B = -(4.00\ \mathrm{J/m}^2)$, and $C = (1.00\ \mathrm{J/m})$, so that $U(x) = Ax^4 + Bx^2 + Cx$.
(a) The object will have maximum speed when it passes through the minimum of the $U(x)$ graph. By looking at the graph, it is clear that this occurs for the solution to $U'(x) = 0$ with a negative value of x. Numerically solving $U'(x) = 4Ax^3 + 2Bx + C = 0$ for this negative solution gives $x_1 = -1.47$ m.
(b) $\Delta\mathrm{KE} = -\Delta U = -(U(x_1) - U(x_0)) = -(-5.44 - 2.00)$ J $= 7.44$ J. Setting this equal to $(1/2)mv^2$ gives $v = 2.73$ m/s.
(c) The unstable equilibrium point is the root $x_2 = 0.126$ m. $\Delta\mathrm{KE} = -\Delta U = -(U(x_2) - U(x_0)) = -(0.063 - 2.00)$ J $= 1.94$ J. Setting this equal to $(1/2)mv^2$ gives $v = 1.39$ m/s.
(d) The lowest value of x that the particle reaches is when $U(x)$ again equals $+2.00$ m. This occurs at the location $x_3 = -2.21$ m.

7-v5 (a) The work done by the force is the area under the $F(x)$ graph. From $x = 0$ m to $x = 2$ m, the area is $10\ \mathrm{N\cdot m} = 10$ J.
(b) The area under the graph from $x = 2$ m to $x = 3$ m is 5 J.
(c) Power is energy/time. $P_{\mathrm{avg}} = (15\ \mathrm{J})/(0.5\ \mathrm{s}) = 30$ W.

7-v6 The object comes to rest when the force does a work $-(1/2)mv^2$, or in other words $-(1/2)mv^2 = \int_0^{x_{\max}}(-kx^2)dx$. This gives $x_{\max} = (3mv^2/2k)^{1/3}$.

7-v7 The force is the derivative of the potential energy with respect to distance. (Actually it's minus the derivative, but all that tells you is the direction of the force, and we know it's attractive anyway.)

$$\begin{aligned}\frac{\mathrm{d}PE}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(gr^{-1}e^{-r/a}\right)\\ &= g\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-1}e^{-r/a}\right)\\ &= g\left[r^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(e^{-r/a}\right)+e^{-r/a}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-1}\right)\right]\\ &= g\left[r^{-1}\left(-\frac{1}{a}\right)\left(e^{-r/a}\right)+e^{-r/a}\left(-r^{-2}\right)\right]\\ &= -gr^{-2}e^{-r/a}\left(\frac{r}{a}+1\right)\end{aligned}$$

7-v8

$$\begin{aligned}W &= \int_0^\infty F\,\mathrm{d}x\\ &= \int_0^\infty be^{-cx}\,\mathrm{d}x\\ &= b/c\end{aligned}$$

7-v9 (a) First we set up an equation that states conservation of energy:

$$E_{total,i} = E_{total,f}$$
$$KE_{1,i}+PE_{1,i}+KE_{2,i}+PE_{2,i} = KE_{1,f}+PE_{1,f}+KE_{2,f}+PE_{2,f}$$

Subtracting $PE_{1,i}$ and $PE_{2,i}$ from both sides gives:

$$KE_{1,i}+KE_{2,i} = KE_{1,f}+KE_{2,f}+\Delta PE_1+\Delta PE_2$$

Since the masses start from rest, we have:

$$0 = KE_{1,f}+KE_{2,f}+\Delta PE_1+\Delta PE_2$$

We no longer have any variables that occur in the equation with both i and f subscripts, so we can stop writing them without any risk of confusion.

$$\begin{aligned}0 &= KE_1+KE_2+\Delta PE_1+\Delta PE_2\\ 0 &= \frac{1}{2}Mv^2+\frac{1}{2}mv^2+Mg\Delta y_1+mg\Delta y_2\\ 0 &= \frac{1}{2}(M+m)v^2+Mg(-h)+mgh\\ v &= \sqrt{\frac{2(M-m)gh}{M+m}}\end{aligned}$$

(b) Comparing the result of part a with $v^2 = 2ah$ gives $a = (M-m)g/(M+m)$, which is the result found before using Newton's laws.

7-v10 As suggested in the hint, let $k = 1$, $m = 1$, $g = 1$, and also let the force of kinetic friction equal 1.

When the object is released at a, it travels a distance a while being acted on by a frictional force 1, so the work done by friction is a. This dissipates the entire initial potential energy $(1/2)ka^2 = (1/2)a^2$. This gives $a = 2$.

When the object is released at b, it travels to a position x on the far side of equilibrium at which the force of friction is canceled by the force of the spring. We're given that b is the maximum distance from which this can occur, so the force of friction must be at its maximum. Since we're given that $\mu_s = \mu_k$, this is the same as the force of kinetic friction, which is 1. The magnitude of the spring's force is $|kx| = x$, so $x = -1/$ In summary, the object travels from b to -1.

Applying conservation of energy in this case, we have $PE_i = PE_f + W_{friction}$, or $(1/2)b^2 = (1/2)1^2 + (b+1)\cdot 1$. This gives a quadratic equation, $(1/2)b^2 - b - 3/2 = 0$. The positive solution is $b = 3$.

The result is that $b/a = 3/2$.

7-v11 (a) Conservation of energy gives

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)+y = 0.$$

By the chain rule, $\dot{x} = (\mathrm{d}x/\,\mathrm{d}y)(\mathrm{d}y/\,\mathrm{d}t)$, and substituting this into the equation gives the result claimed.

(b) Under the approximation $y' \ll 1$, the equation of motion becomes

$$y = -\frac{1}{2}\dot{y}^2 y'^{-2}.$$

By the chain rule, $\dot{y} = y' \cdot 4t^3$. Solving this for y' and eliminating y' in the equation above, we end up with $y = -8x^{3/2}$.

(c) Clearly $x = 0$ (remaining at rest) is a solution to the equations of motion. Furthermore, the equations of motion have no explicit dependence on time, so for any function $x(t)$ that is a solution, so is any other function $x(t - t_0)$.

The only remaining issue (and it's a philosophically important one!) is to check that the solution satisfies the equation of motion at the moment t_0 when the bead decides to start moving. Both of the derivatives appearing in the equation of motion, y' and $\dot{y}$, are defined at this time and equal to zero. Plugging these in to the equation of motion gives an expression involving the square of the indeterminate form $\dot{y}/y'$. But if we go ahead and evaluate this indeterminate form, we get

$$4(t-t_0)^3/(-12x^{1/2}) = -\frac{(t-t_0)^3}{3(t-t_0)^{3/2}} = 0,$$

which does satisfy the equation of motion.

8-a1 Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancal out on the average. Cooling changes individual molecular momenta, but not the total.

8-a2 By conservation of momentum, the total momenta of the pieces after the explosion is the same as the momentum of the firework before the explosion. However, there is no law of conservation of kinetic energy, only a law of conservation of energy. The chemical energy in the gunpowder is converted into heat and kinetic energy when it explodes. All we can say about the kinetic energy of the pieces is that their total is greater than the kinetic energy before the explosion.

8-a3 The people plus the boat are, roughly speaking, a closed system, since the water does not exert much horizontal force on the boat. Momentum is conserved in a closed system. At the beginning, nothing is moving and there is zero momentum. If the two people's momenta didn't add up to zero, then the boat would have to move to conserve momentum. They should make sure their momenta add up to zero, i.e., they should move in opposite directions, and the heavier person has to move more slowly.

8-a4 Both energy and momentum are conserved, so neither can have changed after the collision. The kinetic energy has been converted into heat, sound, and the energy required to permanently deform the blobs. The momentum was zero before the collision (it canceled out), and it's zero afterward as well.

8-d1 $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$

8-d2 We start with

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} \qquad .$$

Since each object has the same value of m, m can be factored out of the sums, giving

$$x_{cm} = \frac{m\,(x_1 + x_2 + \ldots)}{m\,(1 + 1 + \ldots)} \qquad .$$

If you add up a series of 1's for each object, you just get the number of objects, N, so this equals

$$x_{cm} = \frac{x_1 + x_2 + \ldots}{N} \qquad ,$$

which is the average of all the x's.

8-d3 The two objects fall under the influence of gravity, and their accelerations are identical. Therefore, their speeds at the bottom of the building are identical. With $p = mv$, we can see that the object four times more massive will have four times the momentum, and the answer is $4p$.

8-d4 (a) The acceleration as a function of time is

$$a(t) = (A/m)e^{-t/\tau}$$

Integrating with respect to t, we can find $v(t)$:

$$\begin{aligned} v(t) &= \int_0^t a(t')dt' = \frac{A}{m}\int_0^t e^{-t'/\tau}dt' \\ &= \frac{A}{m\tau}(1 - e^{-t/\tau}) \end{aligned}$$

Therefore, $v(\tau) = (1 - 1/e)A/(m\tau)$.
(b) $\lim_{t\to\infty} v(t) = \frac{A}{m\tau}$.

8-d5 The change in momentum is $\Delta p = \int_0^T F(t)dt$. Also, since the object is at rest at $t = 0$, $\Delta p = p_f$ (which we'll just call $p(t = T)$):

$$\begin{aligned} p(T) &= \int_0^T At^2 dt \\ &= \frac{1}{3}AT^3 \end{aligned}$$

8-g1 (a) False: the (overall / "net") impulse delivered to an object is the change in momentum of the object. If the object was not initially at rest, then this is not the same as the final momentum of the object.
(b) True: this is the way we define averages. This is basically saying the same as saying that the average of a function over the interval from a to b is the area under the curve from a to b divided by $(b - a)$.
(c) False: Impulse is Force times Time.
(d) True: this is another way of stating the impulse-momentum theorem.
(e) False: momentum times time has dimensions mass × length, which is not the same as a force.

8-g2 No, there can easily be more than one solution, the basic reason being that conservation of energy and momentum gives four equations (a scalar equation plus three equations for the three components of momentum), which can be smaller than the number of unknowns.

As a simple counterexample, consider equal masses that come in at velocities $\pm u\hat{\mathbf{x}}$ along the positive and negative x axes and collide elastically at the origin. One solution that satisfies the conservation laws is if they both reverse their directions of motion and fly off along the x axes with velocities $\mp u\hat{\mathbf{x}}$. But other final states exist with the same energy and (zero) momentum, because we can pick a line through the origin at any angle and have them fly off along that line.

8-g3 A conservation law is about addition: it says that when you add up a certain thing, the total always stays the same. Funkosity would violate the additive nature of conservation laws, because a two-kilogram mass would have twice as much funkosity as a pair of one-kilogram masses moving at the same speed.

8-j1 The elegant way to do this is to switch to the center of mass frame. In that frame, the balls are initially moving toward each other at velocities $v/2$ and $-v/2$. After the collision, we can maximize the amount of heat released if we simply make the two masses stick together and stop. That means that *in the center of mass frame*, it's possible to convert all the kinetic energy, $2(1/2)m(v/2)^2 = (1/4)mv^2$, into heat and sound. Back in the original frame of reference, the balls are still moving after the collision, so not all the kinetical energy has been lost, but the amount of heat and sound is still the same amount we found in the c.m. frame: $(1/4)mv^2$, or half the initial kinetic energy of the single moving ball.

As an alternative, a brute force technique would be to require conservation of momentum and energy and crank out the result. Let the final velocities be b and c. Then

$$\begin{aligned} mv &= mb + mc \\ \frac{1}{2}mv^2 &= \frac{1}{2}mb^2 + \frac{1}{2}mc^2 + Q \qquad , \end{aligned}$$

where Q is the amount of heat and sound that comes out. Solving the first equation for $b = v - c$ and substituting into the second, we find $Q = m(vc - c^2)$. Using calculus to find the maximum, we set $0 = \mathrm{d}Q/\mathrm{d}c = mv - 2mc$, giving $c = v/2$, and then the rest of the result carries through as in the center-of-mass approach.

8-j2 (a) $KE = \frac{1}{2}mv^2$, so $v = \sqrt{2KE/m} = 130$ m/s.
(b) $p = mv = 1.3$ kg·m/s.
(c) same as part b (conservation of momentum)
(d) $KE = \frac{p^2}{2m} = 0.2$ J. The recoiling gun is harmless, because it would take more energy than this to break bones, sever tissues, etc.

8-j3 (a) The change of momentum of the tennis ball is equal to the impulse delivered to the tennis ball, which is the area underneath a force-vs-time graph. The area under the graph is $+(1/2)(100\text{ N})(20\text{ ms}) = +1.0\text{ kg}\cdot\text{m/s}$.
(b) If the ball was initially at rest, then $p_i = 0$

and $\Delta p = p_f - p_i = p_f = mv_f$. Plugging in $\Delta p = +1.0$ kg · m/s and $m = 58.5 \times 10^{-3}$ kg gives $v_f = +17$ m/s.
(c) If the ball was initially moving in the direction opposite that of the applied force, then $p_i = m(-25\text{ m/s})$. We already know from part (a) that, for this mass and this impulse, the change in velocity is +17 m/s. Therefore, the final velocity of the tennis ball is $v_f = v_i + \Delta v = (-25 + 17)\text{ m/s} = -8$ m/s.

8-j4 Horizontal momentum is conserved, since there are no external horizontal forces. Let's pick masses for the planes of 1 kg and 5 kg – although the real masses must be thousands of times bigger, it won't matter as long as the ratio is 5. Let the positive x axis point east and the positive y axis north, and measure angles counterclockwise from the x axis. The magnitudes of their initial momenta are $|p_L| = 300$ kg·mi/hr and $|p_J| = 750$ kg·mi/hr. The x components are

$$\begin{aligned} p_{L,x} &= 300\text{ kg·mi/hr} \qquad \text{and} \\ p_{J,x} &= (750\text{ kg · mi/hr})(\cos 225°) \\ &= -530\text{ kg·mi/hr} \qquad , \end{aligned}$$

giving

$$p_{total,x} = -230\text{ kg·mi/hr} \qquad .$$

Similarly,

$$\begin{aligned} p_{L,y} &= 0 \qquad , \\ p_{J,y} &= -530\text{ kg·mi/hr} \qquad \text{, and} \\ p_{total,y} &= -530\text{ kg·mi/hr} \qquad . \end{aligned}$$

Since momentum is conserved, their common direction of motion after the crash has to be the same as the direction of their total momentum vector before the crash, which was at an angle of

$$\tan^{-1}(-530/-230) = 67° \text{ or } 247° \qquad .$$

(Both angles are arctangents of -530/-230.) Since the x and y components are both negative, 247° is the correct angle. That's the same as 23° west of south.

8-j5 Let M_1 be the car initially traveling north with speed v_1, and let M_2 be the car initially traveling east with speed v_2. Choose a coordinate system where $+x$ is to the east, and $+y$ is to the north. The total momentum has x-component equal to $p_x = +M_1v_1$, and the total momentum has y-component equal to $p_y = +M_2v_2$. The total momentum $\mathbf{p} = (M_1 + M_2)\mathbf{v}_f = (M_1v_1)\hat{\mathbf{x}} + (M_2v_2)\hat{\mathbf{y}}$. Taking the absolute value of both sides, $(M_1 + M_2)v_f = \sqrt{(M_1v_1)^2 + (M_2v_2)^2}$. Plugging in numbers and solving for v_f gives $v_f = 15$ m/s.

8-j6 Draw the incoming masses on an xy-coordinate system such that, after the two masses collide at the origin, they are moving in the $+x$ direction. That is, one mass comes from the 2nd quadrant, and one comes from the 3rd quadrant, and each incoming mass as a velocity vector that makes an angle $\theta/2$ with respect to the horizontal. Let the initial velocity of each mass be v_0. From conservation of momentum in the x-direction, we have

$$\begin{aligned} 2[Mv_0\cos(\theta/2)] &= (2M)(v_0/3) \\ \theta &= 2\cos^{-1}(1/3) \approx 141° \end{aligned}$$

8-j7 (a) Since $KE \propto v^2$, $KE_1 = KE_0/4$, so $v_1^2 = v_0^2/4$, or $v_1 = v_0/2$.
(b)

$$\begin{aligned} \mathbf{p}_1 &= m_1\mathbf{v}_1 \\ &= m_1v_1(\cos 60°\hat{\mathbf{x}} - \sin 60°\hat{\mathbf{y}}) \\ &= m_1v_0[(1/4)\hat{\mathbf{x}} - (\sqrt{3}/4)\hat{\mathbf{y}}] \end{aligned}$$

Therefore, $p_{1,x} = m_1v_0/4$ and $p_{1,y} = -(\sqrt{3}/4)m_1v_0$.
(c) There was no initial momentum in the $\hat{\mathbf{y}}$ direction, so $p_{2,y} = +(\sqrt{3}/4)m_1v_0$. Also, the total momentum in the $\hat{\mathbf{x}}$ direction is $m_1v_0 = p_{1,x} + p_{2,x} = m_1v_0/4 + p_{2,x}$, and so $p_{2,x} = (3/4)m_1v_0$. The problem doesn't ask for this, but the angle that ball 2 makes with the horizontal is $\theta_2 = \tan^{-1}[(\sqrt{3}/4)/(3/4))] = 30°$.

(d)

$$\begin{aligned} v_2 &= |\mathbf{p}_2|/m_2 \\ &= (m_1/m_2)v_0\sqrt{(\sqrt{3}/4)^2 + (3/4)^2} \\ &= (\sqrt{3}/2)(m_1/m_2)v_0 \\ KE_2 &= (1/2)m_2 v_2^2 \\ &= (3/8)(m_1^2/m_2)v_0^2 \end{aligned}$$

$KE_2/KE_0 = (3/4)(m_1/m_2)$ is the fraction of the initial KE that ball 2 has after the collision.
(e) The ratio of final KE to initial KE is $(KE_1 + KE_2)/KE_0 = (3/4)(m_1/m_2) + 0.25$. The collision described is impossible (or, requires an "explosion" of stored elastic/chemical energy) if $m_1 > m_2$, is elastic if $m_1 = m_2$, and is inelastic if $m_1 < m_2$.

8-j8 The momentum transferred to the rocket by the exhaust pushing on it equals the area under the curve. Each rectangle represents a momentum of (0.5 MN) × (1 s) = 5×10^5 N·s, and there are about 54 rectangles under the curve, for a total Δp of $54 \times 5 \times 10^5$ N·s $= 2.7 \times 10^7$ N·s. It starts with zero momentum, so its final velocity equals the momentum transferred divided by its mass, which is 6.8×10^3 m/s.

8-j9 We are given that the KE is on the order of that of a well-thrown rock, which might be 10 J. This energy would be dissipated in your body as heat, and you would probably feel it. The momentum would be $p = \sqrt{2\,m\,KE}$, which comes out to be about 10^{-13} kg·m/s if we put in the mass of a proton. This is a tiny amount of momentum, and you definitely wouldn't notice it, i.e. you would not be knocked back perceptibly.

8-j10 Let's use M for the mass of the ball, U for its initial velocity, and V for its final velocity. Similarly, let m be the mass of the pin, and v its final velocity. Conservation of momentum and energy give

$$MU = MV + mv$$

and

$$\frac{1}{2}MU^2 = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 \qquad ,$$

two equations in the two unknowns V and v. We aren't given V and don't want to find it, so let's eliminate it. Solving the momentum equation for $V = U - (m/M)v$ and substituting into the second equation, we find

$$\begin{aligned} \frac{1}{2}MU^2 &= \frac{1}{2}M[U - (m/M)v]^2 + \frac{1}{2}mv^2 \\ U^2 &= [U - (m/M)v]^2 + \frac{m}{M}v^2 \\ 2\frac{m}{M}U &= \left(\frac{m}{M} + \frac{m^2}{M^2}\right)v \\ v &= \frac{2U}{1 + m/M} \\ &= 3.6 \text{ m/s} \qquad . \end{aligned}$$

8-m1 Let the velocity of the student after kicking the skateboard be v_f. From conversation of momentum, $(M + m)v = Mv_f$. Therefore, $v_f = v(M + m)/M$.

8-m2 If no external forces act on the system (in this problem, Rachel and Sara), then the momentum of the system is conserved. Here, since the two are initially at rest, the momentum of the system is zero, and it will remain at zero. Therefore, the center-of-mass of the system (which we'll call x_{CM}) will remain in the same spot.

$$\begin{aligned} x_{\text{CM}} &= \frac{M_R(-L/2) + M_S(+L/2)}{M_R + M_S} \\ &= L\left[\frac{M_S - M_R}{2(M_R + M_S)}\right] \end{aligned}$$

$M_R > M_S$, and so $x_{\text{CM}} < 0$. This makes sense, as the center-of-mass should be closer to the larger-mass person.

8-m3 Let m be the mass of the little puck and $M = 2.3m$ be the mass of the big one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector $p_x = Mv$ and $p_y = mv$, the direction of the vector is $\tan^{-1}(p_y/p_x) = 23°$ counterclockwise from the big puck's original direction of motion.

8-m4 The earth is 81 times closer to the center of mass. They both orbit this point once a lunar month, so the earth's speed is 81 times smaller. Kinetic energy is proportional to mv^2, so the earth's kinetic energy is $(81)(1/81)^2 = 1/81$ of the moon's.

8-m5 The simplest way to do this is by using the center-of-mass frame of reference. If the big object's mass is essentially infinite compared to the little one's ($M >> m$) then the center of mass of the whole system is always located at the big one's center of mass, so the center-of-mass frame is the frame fixed to the big object. (After the collision, the big object recoils a tiny bit from the collision, so you could say that it's no longer at rest in the c.m. frame, but because its mass is assumed to be infinite compared to the little one's mass, its recoil is negligible.)

In this frame, it's not the high-mass object with velocity v coming along and hitting the low-mass object, it's the other way around. The low-mass object, moving at velocity $-v$, hits the high-mass one. This is like hitting a brick wall: the recoil of the wall is negligible. The only thing moving before or after the collision is the small object, and since we're assuming no KE is converted into anything else, the small object must recoil at velocity $+v$, i.e., in the opposite direction but at the same speed. (Reasoning slightly more rigorously, it's not quite true that the big object doesn't recoil at all, but the momentum it absorbs equals $-2mv$, and its KE after the collision is $p^2/2M = 2m^2v^2/M$, which is extremely small because M is so big compared to m.)

In the center-of-mass frame, then, we've proved that the small object enters with velocity $-v$ and exits with $+v$. To convert back to the original frame of reference, we add v to all the velocities. In the original frame, the small object starts with velocity 0, and ends up with $2v$.

8-m6 (a) In a perfectly elastic 1D collision between two objects of the same mass, one of which initially at rest, all of the energy/momentum of the moving object goes to the object initially at rest, and the object that was initially moving comes to a rest. If you haven't heard this before, here is the argument: Let the initial velocity of the moving object be v as in the problem statement, let the final velocity of the initially-moving puck be v_1, and let the final velocity of the initially-stationary puck be v_2. Then, from conservation of momentum and energy,

$$mv = mv_1 + mv_2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

Divide both equations by m, and multiply the second equation by 2 (to clean thing up a bit). Then, $v_2 = v - v_1$ from the first equation, and plugging it into the (modified) second, we get

$$\begin{aligned} v^2 &= v_1^2 + (v - v_1)^2 \\ v^2 &= v^2 - 2vv_1 + 2v_1^2 \\ 0 &= 2v_1(v_1 - v) \end{aligned}$$

So, either $v_1 = v$, or $v_1 = 0$. The first situation occurs if there is no collision, but if we assume that one puck doesn't just magically pass through the other, then it must be the case that $v_1 = 0$. It follows immediately from the conservation of momentum equation above that $v_2 = v$.
(b) If the collision is perfectly inelastic, then conservation of momentum tells us:

$$mv = mv_f + mv_f$$

where v_f is the final speed of both pucks (the same for both, since "perfectly inelastic" means they stick together). Thus $v_f = v/2$.
(c) If both pucks have speed $v/2$, then each has one-fourth the KE of the moving puck before the collision. Thus half the total energy was lost during the collision.
(d) Here, we can set up conservation of momentum and conservation of energy equations as in part (a), with the caveat that we must only take 3/4 of the initial KE (since 1/4 was lost):

$$v = v_1 + v_2$$

$$\frac{3}{4}v^2 = v_1^2 + v_2^2$$

Plugging $v_1 = v - v_2$ into the second equation gives

$$\begin{aligned} \frac{3}{4}v^2 &= (v - v_2)^2 + v_2^2 \\ 0 &= 2v_2^2 - 2vv_2 + \frac{1}{4}v^2 \end{aligned}$$

Using the quadratic equation,

$$\begin{aligned} v_2 &= \frac{2v \pm \sqrt{4v^2 - 2v^2}}{4} \\ &= \frac{1}{4}(2 \pm \sqrt{2})v \end{aligned}$$

Again, assuming the puck that was moving before the collision doesn't "pass through" the puck initially at rest, then we must have $v_2 > v_1$, and so we must take the solution with the plus sign. It also makes sense that the solution for v_2 had better be somewhere between $v/2$ and v, since a partially-inelastic collision is somewhere between a perfectly-inelastic collision and an elastic collision. Therefore, the speeds of the two pucks after the collision are:

$$\begin{aligned} v_1 &= \left(\frac{2 - \sqrt{2}}{4}\right)v \\ v_2 &= \left(\frac{2 + \sqrt{2}}{4}\right)v \end{aligned}$$

8-m7 (a) By drawing a circular arc from where the block was originally hanging to the maximum height it reaches, you should be able to see that the maximum height reached is $H = L(1 - \cos\theta)$.
(b) We will have to do this in two parts: first, find the speed of the bullet/block system after the collision by using conservation of energy, and second, to find the initial speed of the bullet by using conservation of momentum. Note that you cannot just set the final potential energy of the bullet/block system equal to the initial kinetic energy of the bullet because some mechanical energy was lost during the collision.
Call the bullet/block system "BB." The final PE of BB is equal to the kinetic energy of BB right after the collision. Therefore,

$$\begin{aligned} KE_{\text{i,BB}} &= PE_{\text{final,BB}} \\ \frac{1}{2}(m + M)v^2 &= (m + M)gH \\ \frac{1}{2}v^2 &= gL(1 - \cos\theta) \\ v &= \sqrt{2gL(1 - \cos\theta)} \end{aligned}$$

Now, using conservation of momentum, we can solve for the initial speed of the bullet, v_0:

$$\begin{aligned} mv_0 &= (m + M)v \\ v_0 &= \frac{m + M}{m}\sqrt{2gL(1 - \cos\theta)} \end{aligned}$$

(c) $PE_{\text{final,BB}} = (m + M)gL(1 - \cos\theta)$ and $KE_{\text{bullet}} = mgL(1 - \cos\theta)$, so the ratio is $(m + M)/m$. For $M \gg m$, this is basically equal to M/m, which is a number much greater than 1. This means that almost all of the mechanical energy of the bullet is lost in such a collision.
8-m8 It's best to do part (c) before parts (a) and (b):
(c) Suppose that, initially, the truck is moving with velocity v and the car with velocity $-v$. Then, from conservation of momentum, $(2M)v + (M)(-v) = (2M + M)v_f$, where v_f is the final velocity of both vehicles after the inelastic collision. Solving for v_f, we have $v_f = v/3$.
(a) The change in momentum of the truck is $\Delta p_t = (2M)v_f - (2M)v = -(4/3)Mv$.
(b) The change in momentum of the car is $\Delta p_c = (M)v_f - (M)(-v) = +(4/3)Mv$.
(d) $\text{KE}_\text{i} = (1/2)(2M + M)v^2 = (3/2)Mv^2$; $\text{KE}_\text{f} = (1/2)(2M + M)v_f^2 = (1/9)\text{KE}_\text{i}$. Therefore, $8/9 = 89\%$ of the initial KE was lost as a result of the collision.
8-m9 (a) The maximum height of $H = 140$ m tells us how long fragment A takes to fall to the ground: $H = (1/2)gT^2$. Plugging in numbers gives $T = 5.35$ s. Looking at the horizontal direction, we have $v_A = D_A/T = (290\text{ m})/(5.35\text{ s}) = 54.3$ m/s.
(b) The horizontal momentum was zero just before the collision, so it's zero just after the explosion. This means $M_A v_A = M_B v_B$ (where v_A

and v_B are speeds, and not velocities). Thus, $v_B = (2/3)v_A = 36.2$ m/s.

(c) $\mathbf{J}_{\text{exp}} = \Delta\mathbf{p}_A = M_A v_f \hat{\mathbf{x}} = 108.6$ kg $\cdot$ m/s$(+\hat{\mathbf{x}})$. Note we've defined the horizontal direction where fragment A flies off in as the $+\hat{\mathbf{x}}$ direction.

(d) We've been ignoring gravity, but since we're told how long the explosion lasts, we can actually calculate the impulse delivered to fragment A over the time of the explosion. $\mathbf{J}_{\text{grav}} = (M_A g)t_{\text{exp}}(-\hat{\mathbf{y}}) = 0.29$ kg $\cdot$ m/s$(-\hat{\mathbf{y}})$ (defining the $\hat{\mathbf{y}}$ direction to be vertically upwards). Note this is negligible compared to the impulse due to the explosion, which is why we're justified in saying that the dominant contribution to $\Delta\mathbf{p}_A$ comes from the explosion.

(e) Since the center of mass remains at the launch point, $M_A D_A = M_B D_B$ (where each D is a distance, not a position, and so both are defined to be positive). Thus $D_B = (2/3)(290\text{ m}) = 193$ m.

8-m10 (a) Let the final velocities of M_1 and M_2 be v_1 and v_2, respectively. We have conservation of momentum and conservation of energy (no factor of 1/2 in the latter, since we can multiply the entire equation by 2):

$$M_1 u = M_1 v_1 + M_2 v_2$$

$$M_1 u^2 = M_1 v_1^2 + M_2 v_2^2$$

We can use the first equation to write $v_2 = (M_1 u - M_1 v_1)/M_2$, and then substitute this expression for v_2 into the second equation and solve for v_1:

$$\begin{aligned} M_1 u^2 &= M_1 v_1^2 + M_2 \left(\frac{M_1 u - M_1 v_1}{M_2}\right)^2 \\ u^2 &= v_1^2 + k\left(\frac{u - v_1}{k}\right)^2 \\ k u^2 &= k v_1^2 + (u^2 - 2uv_1 + v_1^2) \\ 0 &= (1+k)v_1^2 + (-2u)v_1 + (1-k)u^2 \end{aligned}$$

Note that in going from the first to the second line, we divided both sides of the equation by M_1, and replaced $M_2/M_1 \to k$. Using the quadratic equation, we have

$$\begin{aligned} v_1 &= \frac{2u \pm \sqrt{4u^2 - 4(1-k^2)u^2}}{2(1+k)} \\ v_1 &= \frac{2u \pm 2u\sqrt{1-(1-k^2)}}{2(1+k)} \\ v_1 &= u\left(\frac{1 \pm k}{1+k}\right) \end{aligned}$$

If we choose the plus sign, then $v_1 = u$, which gives $v_2 = 0$ and means that no collision occurred at all. Assuming that one ball did not just simply pass through the other, we must take the other solution, and so

$$v_1 = \left(\frac{1-k}{1+k}\right)u$$

(b) Recall $v_2 = (M_1 u - M_1 v_1)/M_2 = (u - v_1)/k$:

$$\begin{aligned} v_2 &= \frac{1}{k}u\left(1 - \frac{1-k}{1+k}\right) \\ v_2 &= \frac{1}{k}u\left(\frac{2k}{1+k}\right) \\ v_2 &= \left(\frac{2}{1+k}\right)u \end{aligned}$$

8-m11 There are two ways to do this, and we'll do both here. The first way is more straightforward, but the algebra is a little messier. The second way involves analyzing the problem in the center-of-mass frame.

(1) Let v_1 be the final velocity of the block that has slowed down, and let v_2 be the final velocity of the block that has sped up (i.e., the "boosted" block). Momentum and energy are conserved:

$$(2M)v_0 = Mv_1 + Mv_2$$

$$\frac{1}{2}(2M)v_0^2 + \frac{1}{2}kx^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}Mv_2^2$$

Cleaning these up a bit,

$$v_1 = 2v_0 - v_2$$

$$2v_0^2 + (k/M)x^2 = v_1^2 + v_2^2$$

Plug in the expression for v_1 into the second equation and solve for v_2 (in the second line, we define $C \equiv 2v_0^2 - (k/M)x^2$):

$$\begin{aligned} 2v_0^2 + (k/M)x^2 &= (2v_0 - v_2)^2 + v_2^2 \\ 0 &= 2v_2^2 - 4v_0 v_2 + C \\ 0 &= (1/2)v_2^2 - v_0 v_2 + C/4 \end{aligned}$$

Using the quadratic equation:

$$\begin{aligned} v_2 &= v_0 \pm \sqrt{v_0^2 - (1/2)C} \\ v_2 &= v_0 (1 \pm \sqrt{1 - (1 - (1/2)(k/M)(x/v_0)^2)}) \\ v_2 &= v_0 \left(1 \pm \frac{x}{v_0}\sqrt{\frac{k}{2M}}\right) \end{aligned}$$

Now, v_2 is the boosted block, and so it's the solution with the plus sign (v_1 is the same expression, but with the minus root).

(2) Suppose we are moving along with the blocks before the string breaks (with speed v_0 relative to the ground). After the string breaks, each block flies off in opposite directions with a speed we'll call u. From conservation of energy,

$$\begin{aligned} \frac{1}{2}kx^2 &= 2\left(\frac{1}{2}Mu^2\right) \\ x\sqrt{\frac{k}{2M}} &= u \end{aligned}$$

Then, from another observer at rest relative to the ground, the block which was boosted has speed $v + u$, which is exactly the same result obtained in (1). (although much more easily here!)

$$v_2 = v_0 + x\sqrt{\frac{k}{2M}}$$

8-m12 (a) Particle i had velocity v_i in the center-of-mass frame, and has velocity $v_i + u$ in the new frame. The total kinetic energy is

$$\frac{1}{2}m_1 \left(\mathbf{v}_1 + \mathbf{u}\right)^2 + \ldots,$$

where "…" indicates that the sum continues for all the particles. Rewriting this in terms of the vector dot product, we have

$$\frac{1}{2}m_1 \left(\mathbf{v}_1 + \mathbf{u}\right)\cdot\left(\mathbf{v}_1 + \mathbf{u}\right) + \ldots = \frac{1}{2}m_1 \left(\mathbf{v}_1 \cdot \mathbf{v}_1 + 2\mathbf{u}\cdot\mathbf{v}_1 + \mathbf{u}\cdot\mathbf{u}\right) + \ldots$$

When we add up all the terms like the first one, we get K_{cm}. Adding up all the terms like the third one, we get $M|\mathbf{u}|^2/2$. The terms like the second term cancel out:

$$m_1 \mathbf{u}\cdot\mathbf{v}_1 + \ldots = \mathbf{u}\cdot\left(m_1 \mathbf{v}_1 + \ldots\right),$$

where the sum in brackets equals the total momentum in the center-of-mass frame, which is zero by definition.

(b) Changing frames of reference doesn't change the distances between the particles, so the potential energies are all unaffected by the change of frames of reference. Suppose that in a given frame of reference, frame 1, energy is conserved in some process: the initial and final energies add up to be the same. First let's transform to the center-of-mass frame. The potential energies are unaffected by the transformation, and the total kinetic energy is simply reduced by the quantity $M|\mathbf{u}_1|^2/2$, where $\mathbf{u}_1$ is the velocity of frame 1 relative to the center of mass. Subtracting the same constant from the initial and final energies still leaves them equal. Now we transform to frame 2. Again, the effect is simply to change the initial and final energies by adding the same constant.

8-m13 (a) Let T be the tension in the rope at the corner of the table. For the bottom part of the rope with length x that hangs vertically, we let positive be down (in the direction of motion), so

$$\begin{aligned} F &= \frac{\mathrm{d}p}{\mathrm{d}t} \\ m_{bottom}g - T &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m_{bottom}\frac{\mathrm{d}x}{\mathrm{d}t}\right) \\ \frac{x}{L}mg - T &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{L}mg\frac{\mathrm{d}x}{\mathrm{d}t}\right) \\ xg - \frac{L}{m}T &= \frac{\mathrm{d}}{\mathrm{d}t}\left(x\frac{\mathrm{d}x}{\mathrm{d}t}\right) \\ xg - \frac{L}{m}T &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + x\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \quad . \end{aligned}$$

For the top part, with length $L - x$, we let positive be toward the edge of the table (again, in the direction of motion).

$$
\begin{aligned}
F &= \frac{\mathrm{d}p}{\mathrm{d}t} \\
T &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1-\frac{x}{L}\right)m\frac{\mathrm{d}x}{\mathrm{d}t}\right] \\
\frac{LT}{m} &= -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + (L-x)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \qquad .
\end{aligned}
$$

Adding the results of the two calculations, we have $xg = La$, which is the result that was claimed.

(b) For $x = 0$, nothing is hanging over the edge, so the upward normal force of the table completely cancels the force of gravity. For $x = L$, the whole rope is hanging, and there is no upward normal force from the table at all; the result becomes $a = g$. In between these two extreme cases, the upward normal force of the table only partially cancels gravity.

(c)

$$
\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{g}{L}x \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(be^{ct}\right) &= \frac{g}{L}be^{ct} \\
bc^2e^{ct} &= \frac{g}{L}be^{ct} \\
c^2 &= \frac{g}{L} \\
c &= \pm\sqrt{\frac{g}{L}}
\end{aligned}
$$

(d) This is true because the derivative of a sum is the sum of the derivatives.

(e) The most general solution is of the form

$$x = qe^{ct} + re^{-ct} \qquad ,$$

where q and r are constants with units of meters, and we'll say c represents the positive root $\sqrt{g/L}$. At $t = 0$, both exponentials equal one, and we have

$$x_\mathrm{o} = q + r$$

and

$$
\begin{aligned}
0 &= \frac{\mathrm{d}x}{\mathrm{d}t} \\
&= qce^{ct} - rce^{-ct} \\
&= (q-r)c \\
q &= r
\end{aligned}
$$

The solution is therefore $q = r = x_\mathrm{o}/2$, or

$$x = x_\mathrm{o}\left(e^{ct} + re^{-ct}\right)/2 \qquad ,$$

which if you like you can write using the hyperbolic cosine.

8-m14 (a) Suppose that at some time, the rocket has mass m remaining. Now an infinitesimal time passes, and the rocket's mass changes to $m+\mathrm{d}m$ (with $\mathrm{d}m < 0$) because it expels $-\mathrm{d}m$ worth of exhaust. This causes its velocity to increase by the infinitesimal amount $\mathrm{d}v$. We can now require conservation of momentum, and the algebra is reduced if we do this in the center of mass frame, in which the rocket is initially at rest, i.e., its velocity is increasing from 0 to $0+\mathrm{d}v$.

$$0 = (-\mathrm{d}m)(-u) + (m + \mathrm{d}m)\,\mathrm{d}v$$

Discarding the product of the two infinitesimals $\mathrm{d}m\,\mathrm{d}v$, and separating variables, this becomes

$$-\frac{\mathrm{d}m}{m} = \frac{\mathrm{d}v}{u} \qquad ,$$

and integration gives

$$-\ln m + c = \frac{v}{u} \qquad .$$

The interpretation of the minus sign is that as the mass gets smaller, the logarithmic term gets less negative, the left-hand side gets bigger, and the velocity goes up. If we now switch back to the original frame of reference, in which the velocity is not zero but v, the effect is only to add a constant onto v, and that can be absorbed into the constant of integration. Back in this frame, the only time when we have $v = 0$ is initially, when $m = m_i$, and this can be used to determine the constant of integration $c = \ln m_i$. For the final velocity, we have

$$v = u \ln \frac{m_f}{m_i}$$

(b) Solving for m_i/m_f, we have

$$\begin{aligned} m_i/m_f &= e^{v/u} \\ &= \exp(3\times10^7 \text{ m/s } / \text{ } 4000 \text{ m/s}) \\ &= 2\times10^{434} \qquad . \end{aligned}$$

This is a ridiculously huge amount of fuel, vastly more than the mass of the entire galaxy.

9-a1 If the rider speeds up the back wheel, then it's gaining angular momentum. By conservation of angular momentum, this requires that the bike's body "nose up." Slowing down the back wheel makes it "nose down." This allows the rider to control the orientation of the bike so that it makes a two-point landing.

9-a2 The angular momentum is conserved, and so

$$\begin{aligned} &M_e R_p v_p = M_e R_a v_a \\ &\qquad v_p = (R_a/R_p)v_a = (1.017/0.983)v = 1.035v \end{aligned}$$

9-a3 (a) Alice could be in a frame of reference in which Cathy isn't moving. For instance, if Alice and Cathy were in an airplane together, it would be natural for both of them to adopt the frame of reference of the plane. Bob could be using the frame of reference of the dirt.

(b) Your opinions about angular momentum will depend on what point in space you arbitrarily chose as the axis. Let's say Dong is sitting on some grass. Bob might be using the center of the spinning Earth is his axis, while Alice uses the spot Dong is sitting on.

9-a4 No. The angular momentum of a particle is $p_\perp r$, so if the particles differ in the distance, r, from the axis or in their direction of motion, they can have different angular momenta. Furthermore, they could even have the same $|\mathbf{r}|$ and $|\mathbf{p}|$, but if they were located in different directions relative to the axis, or their directions of motion were different, they could have different angular momenta. As an example of the effect of the direction, a particle has zero angular momentum if it is moving straight in towards the axis or directly away from it ($\theta = 0$ or 180°).

9-a5 Taking the vector cross product of these two vectors using the component form for the cross product, I got $-2\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + \hat{\mathbf{z}}$, with all the numbers in units of $\text{kg}\cdot\text{m}^2/\text{s}$. As a check on my result, I decided to use $|\mathbf{r}\times\mathbf{p}| = |\mathbf{r}||\mathbf{p}|\sin\theta_{rp}$. The left-hand side is $\sqrt{30} = 5.48$. The angle between the two vectors can be found from $\mathbf{r}\cdot\mathbf{p} = |\mathbf{r}||\mathbf{p}|\cos\theta_{rp}$, which gives $\theta = 2.40$ radians, resulting in $|\mathbf{r}||\mathbf{p}|\sin\theta = 5.49$, which checks out to within rounding errors.

9-d1 (a) $L = (4\pi/5)MR^2/T = 1.1\times10^{42}\ \text{kg}\cdot\text{m}^2/s$.

(b) Angular momentum is conserved because no outside torques are acting, so the angular momentum is the same.

(c) Solving the same equation for T gives $T = 150$ s.

9-d2 Let the mass of each atom be m, and the center-to-center distance between neighboring atoms b.

$$\begin{aligned} \frac{I_{cross}}{I_{line}} &= \frac{m\cdot0^2 + 4mb^2}{m\cdot0^2 + 2mb^2 + 2m(2b)^2} \\ &= \frac{2}{5} \end{aligned}$$

9-d3 Note that, when converting to SI units, we have $v_{\text{CM}} = 38.3$ m/s and $\omega = 262$ rad/s. Also, the moment of inertia of the baseball about its center-of-mass is $I = (2/5)mR^2 = 7.94\times10^{-5}\ \text{kg}\cdot\text{m}^2$.

$$\begin{aligned} \text{KE}_{\text{trans}} &= (1/2)mv_{\text{CM}}^2 = 107\text{ J} \\ \text{KE}_{\text{rot}} &= (1/2)I\omega^2 = 2.7\text{ J} \\ \text{KE}_{\text{rot}}/\text{KE}_{\text{total}} &= 0.025 = 2.5\% \end{aligned}$$

9-d4 (a) N r.p.m. corresponds to $2\pi N$ radians per minute, and dividing by 60 gives $\omega_i = (N\pi/30)$ rad/s.

(b) Let the merry-go-round have moment of inertia $I_{\text{mgr}} = (1/2)MR^2$, and Mary have moment of inertia $I_M = MR^2$. From conservation of angular momentum,

$$\begin{aligned} I_{\text{mgr}}\,\omega_i &= (I_{\text{mgr}} + I_M)\omega_f \\ \omega_f &= \left(\frac{(1/2)MR^2}{(1/2)MR^2 + MR^2}\right)\omega_0 \\ \omega_f &= \frac{1}{3}\omega_0 = \frac{\pi}{90}\,\text{rad/s} \end{aligned}$$

This is one-third the initial angular velocity, and corresponds to $N/3$ r.p.m.

(c) Mary should have an angular momentum equal in magnitude to that of the merry-go-round before Mary jumps on. Let the answer be v (the speed Mary must run for this to be true). Thus

$$\begin{aligned} MRv &= I_{\text{mgr}}\,\omega_i = (1/2)MR^2\omega_i \\ v &= R\omega_i/2 \end{aligned}$$

9-d5 (a) The total angular momentum of the two-disc system is conserved, and therefore

$$I_i\omega_i = I_f\omega_f$$

The "initial" and "final" values are those before and after the collision. The initial angular momentum comes only from the disc of mass $2M$, and the final angular momentum comes from all three discs, and so $I_i/I_f = 2/3$. Therefore,

$$\omega_f = (I_i/I_f)\omega_i = (2/3)\omega$$

(b) The initial and final KE's are

$$\begin{aligned} \text{KE}_i &= \frac{1}{2}I_i\omega^2 \\ \text{KE}_f &= \frac{1}{2}I_f\omega_f^2 \\ &= \frac{1}{2}\left(\frac{3}{2}I_i\right)\left(\frac{2\omega}{3}\right)^2 = \frac{2}{3}\left(\frac{1}{2}I_i\omega^2\right) = \frac{2}{3}\text{KE}_i \end{aligned}$$

It appears that $1/3 = 33\%$ of the initial KE was lost during the collision.

9-d6 (a) When the student is done rotating the bicycle wheel, the angular momentum of the wheel has reversed direction and become negative. Therefore, the answer is $-I\omega$.

(b) The total angular momentum is still $+I\omega$, which means that the rest of the system (other than the wheel spinning) must pick up and angular momentum equal to $+2I\omega$.

9-d7 Let T be the time required for one rotation. Then the velocity of a ball rolling without slipping is $v = 2\pi R/T$, and we have $L/p = (4\pi mR^2/5T)/(2\pi mR/T) = (2/5)R$.

9-d8 (a) The total moment of inertia is a sum of the contributions from the rod and the two point masses:

$$I_{\text{total}} = (1/12)ML^2 + 2M(L/2)^2 = (7/12)ML^2$$

(b) Again, the total moment of inertia is the sum of the contributions from the rod and the two spheres. However, to find the moment of inertia of one of the spheres, we need to use the parallel-axis theorem:

$$I_{\text{sphere}} = (2/5)Mr^2 + M(L/2)^2.$$

Therefore,

$$\begin{aligned} I_{\text{total}} &= (1/12)ML^2 + 2[(2/5)Mr^2 + ML^2/4] \\ &= (7/12)ML^2 + (4/5)Mr^2 \end{aligned}$$

(c) If $r = L/5$, then $I^{(b)}_{\text{total}} = (923/1500)ML^2$.

$$\left|\frac{I^{(a)}_{\text{total}} - I^{(b)}_{\text{total}}}{I^{(b)}_{\text{total}}}\right| = \frac{48}{923} \approx 5.2\%$$

9-d9 Let's call the mass of the sphere m, where $M = W/g$ (at the very end of the solution, we can replace M with W/g). The total kinetic energy of the solid sphere is

$$\begin{aligned} \text{KE}_{\text{total}} &= \text{KE}_{\text{trans}} + \text{KE}_{\text{rot}} \\ &= (1/2)Mv^2 + (1/2)I\omega^2 \\ &= (1/2)Mv^2 + (1/2)[(2/5)MR^2](v/R)^2 \\ &= (7/10)Mv^2 \end{aligned}$$

Note we could replace ω with v/R because the sphere is rolling without slipping. Setting this KE equal to the gravitational PE reached, we get

$$\begin{aligned} Mgh &= (7/10)Mv^2 \\ h &= \frac{7v^2}{10g} \end{aligned}$$

where h is the vertical height of the sphere when it comes to rest at the top of the incline. If we define L to be the length along the ramp that the sphere travels, then $\sin\theta = h/L$, and

$$L = \frac{7v^2}{10g\sin\theta}$$

9-d10 The total kinetic energy of the sphere is

$$\begin{aligned}\mathrm{KE}_{\mathrm{total}} &= \mathrm{KE}_{\mathrm{trans}} + \mathrm{KE}_{\mathrm{rot}}\\ &= (1/2)Mv^2 + (1/2)I\omega^2\\ &= (1/2)Mv^2 + (1/2)cMR^2\omega^2\end{aligned}$$

If the sphere rolls without slipping, then $\omega = v/R$, and

$$\mathrm{KE}_{\mathrm{tot}} = (1/2)(1+c)Mv^2$$

Keep in mind that $1/(1+c)$ of the total KE is translational, and $c/(1+c)$ of the total KE is rotational. From conservation of energy, we know that the total KE of the sphere when it reaches the bottom of the ramp is MgH. By combining these two facts, we know that at the bottom of the ramp

$$\begin{aligned}\mathrm{KE}_{\mathrm{trans}} &= \left(\frac{1}{1+c}\right)MgH\\ \mathrm{KE}_{\mathrm{rot}} &= \left(\frac{1}{1+c}\right)MgH\end{aligned}$$

As the sphere climbs up the ramp again, it experiences no friction, and therefore the rotational KE doesn't change. (there is no torque on the sphere about the center-of-mass!) Therefore, the height h reached is governed by conservation of energy, but only the translational KE is converted to gravitational PE. Therefore,

$$\begin{aligned}Mgh &= \left(\frac{1}{1+c}\right)MgH\\ h &= \left(\frac{1}{1+c}\right)H\end{aligned}$$

9-d11 Let's take the recommendation in the problem and do this just once for an object with $I = cMR^2$. If we draw a FBD for the object, there are two forces that have a non-zero component along the direction of motion (gravity and static friction), and there is only one force that produces the torque about the center-of-mass (static friction). These give the equations

$$\begin{aligned}Mg\sin\theta - f_s &= Ma\\ Rf_s &= I\alpha = (cMR^2)(a/R)\end{aligned}$$

By dividing the second equation by R and adding the two equations, we get

$$\begin{aligned}Mg\sin\theta &= aM(1+c)\\ a &= \left(\frac{1}{1+c}\right)g\sin\theta\end{aligned}$$

Taking $c_{\text{hoop}} = 1$, $c_{\text{solid sphere}} = 2/5$, and $c_{\text{solid cylinder}} = 1/2$, the answers are as follows:
(a) $a_{\text{hoop}} = (1/2)g\sin\theta$.
(b) $a_{\text{solid sphere}} = (5/7)g\sin\theta$.
(c) $a_{\text{solid cylinder}} = (2/3)g\sin\theta$.

9-d12 Newton's second law applied to the hanging mass gives

$$mg - T = ma,$$

where T is the tension in the string. The torque acting on the platform is TR, so

$$\alpha = TR/I.$$

The relationship between the linear acceleration of the string and the angular acceleration of the platform is

$$a = R\alpha.$$

These are three equations in the unknowns α, T, and I. Eliminating α and T gives

$$I = mR^2\left(\frac{g}{a} - 1\right).$$

9-d13 The moment of inertia is $I = \int r^2\,\mathrm{d}m$. Let the ring have total mass M and radius b. The proportionality

$$\frac{M}{2\pi} = \frac{\mathrm{d}m}{\mathrm{d}\theta}$$

gives a change of variable that results in

$$I = \frac{M}{2\pi} \int_0^{2\pi} r^2 \, \mathrm{d}\theta.$$

If we measure θ from the axis of rotation, then $r = b \sin\theta$, so this becomes

$$I = \frac{Mb^2}{2\pi} \int_0^{2\pi} \sin^2\theta \, \mathrm{d}\theta.$$

The integrand averages to 1/2 over the 2π range of integration, so the integral equals π. We therefore have $I = \frac{1}{2}Mb^2$. This is, as claimed, half the value for rotation about the symmetry axis.

9-d14 The moment of inertia of the rod for rotation about its center is $(1/12)m\ell^2$, where m is the mass of the rod. Using the parallel axis theorem, its moment of inertia about the fulcrum is $(1/12)m\ell^2 + mb^2$. The gravitational torque, with the fulcrum as the axis, is mgb. The resulting angular acceleration is $\alpha = \tau/I$. The tangential acceleration at the bug's position is $(\ell/2 + b)\alpha$, and setting this equal to g gives $b = \ell/6$.

9-d15 The moment of inertia for a uniform rod about one end is $(1/3)m\ell^2$. The torque due to gravity is $(1/2)mg\ell$. The resulting angular acceleration is $(3/2)g/\ell$. The tangential acceleration at the bug's position is $a = b\alpha$. Subtracting this from g gives

$$g_a = \left(1 - \frac{3b}{2\ell}\right) g.$$

It is perhaps a little surprising that this can be negative for $b > (2/3)\ell$, meaning that the bug will lose contact with the door unless its feet are sticky enough.

9-e1 (a) The result from problem 19 was $E = L^2/2I$. This suggests plotting E as a function of L^2, which should give a line with a slope of $1/2I$. Since my spreadsheet software was rounding L^2 off to zero, I actually scaled both variables by powers of 10 before plotting them, as indicated on the labels on the axes.

The plot is indeed a pretty good straight line, although some small deviations from linearity are apparent. The fact that the y-intercept is nonzero is an indication that the form of the data is not quite $E \propto L^2$, which would give $E = 0$ for $L = 0$. The actual data do have $E = 0$ at $L = 0$, since all energies were measured relative to the lowest one. But since the graph has some curvature to it, the best-fit line doesn't go through the origin.

The slope is .1834, which has to be multiplied by 10^{-14+68}, giving $I = 2.7 \times 10^{-54}$ kg$\cdot$m^2.

(b) The total volume of this nucleus is about 168 cubic femtometers, corresponding to a sphere with a radius of about 3 fm, and the moment of inertia of a sphere gives $I = (2/5)mr^2 \approx 1 \times 10^{-54}$ kg$\cdot$m^2. The actual moment of inertia of the nucleus is about three times this value, which isn't bad agreement at all, considering that we're trying to describe a quantum-mechanical system using classical physics.

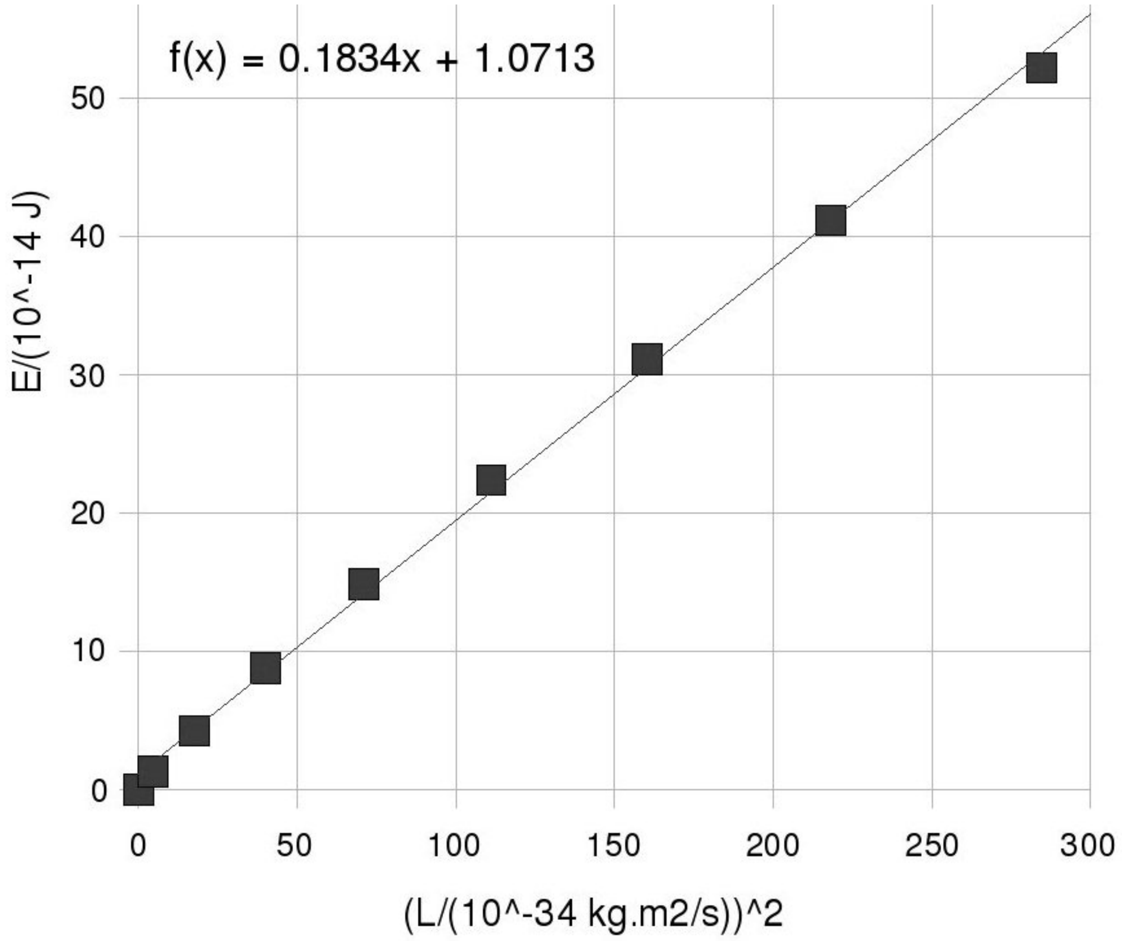
9-e2

$$\begin{aligned} I &= \int_0^\ell r^2 \, \mathrm{d}m \\ &= \int_0^\ell r^2 \frac{\mathrm{d}m}{\mathrm{d}r} \, \mathrm{d}r \\ &= \mu \int_0^\ell r^2 e^{r/\ell} \, \mathrm{d}r \end{aligned}$$

Although this looks like a definite integral because it has limits of integration, it's really more like an indefinite integral in the sense that the upper limit of integration is a variable. By switching to the unitless variable $u = r/\ell$, we can transform this into a definite integral that we'd be more likely to be able to find in a table of integrals, and this also has the advantage of making the calculus less messy. The result is

$$I = \mu\ell^3 \int_0^1 u^2 e^u \, \mathrm{d}u \qquad .$$

The units are all in the factor in front: (kg/m)(m^3) = kg$\cdot$m^2, which is correct for a moment of inertia. The integral can be done by integration by parts, by looking it up in a table,


f(x) = 0.1834x + 1.0713
E/(10^-14 J)
0
10
20
30
40
50
(L/(10^-34 kg.m2/s))^2
0
50
100
150
200
250
300

Problem 9-e1.

or using computer software.

$$\begin{aligned}I &= \mu\ell^3\ e^u(u^2-2u+2)\big|_0^1\\ &= (e-2)\mu l^3\end{aligned}$$

9-e3 (a) Since it's infinitesimally small, we can aproximate it as a rectangle. Its dimensions are $\mathrm{d}r$ and (using the definition of radian measure) $r\,\mathrm{d}\theta$, so the area is $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.
(b) Now we get an (approximately) rectangular box with height $\mathrm{d}z$. Multiplying by the height of the box gives a volume $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.
(c) The flavor of this problem is very similar to that of the calculation of the moment of inertia of a disk, $(1/2)Mr^2$, in section 4.2.5. In fact, we can save quite a bit of time by slicing the cone into disks and just integrating over all the disks. Each disk has infinitesimal height $\mathrm{d}z$ and mass $\mathrm{d}M$. It turns out to be convenient to pick a coordinate system in which $z = 0$ is the tip of the cone, and the cone extends to $z = h$ (i.e., if we want to visualize the positive z axis as being up, then the cone is upside-down). The advantage of this choice is that the surface of the cone is defined by the simple equation $r = (b/h)z$.

$$\begin{aligned}I &= \int \frac{1}{2}\,\mathrm{d}Mr^2\\ &= \int_0^h \frac{1}{2}(\rho A\,\mathrm{d}z)r^2\\ &= \int_0^h \frac{1}{2}\rho\pi r^2\,\mathrm{d}zr^2\\ &= \frac{1}{2}\rho\pi\left(\frac{b}{h}\right)^4\int_0^h z^4\,\mathrm{d}z\\ &= \frac{\pi}{10}\rho b^4h\end{aligned}$$

Our answer is in terms of ρ, but we're supposed to express it in terms of M. We can either look up the formula for the volume of a cone, $(1/3)\pi b^2h$, or evaluate it using a similar integral, omitting a factor of $(1/2)\rho r^2$ in the integrand. The final result is

$$I = \frac{3}{10}Mb^2 \quad .$$

It makes sense that the result depends only on b, not h, since changing h without changing the mass would not change how far any of the mass was from the axis. It also makes sense that the numerical factor of 3/10 in front is smaller than the factor of 1/2 for a disk; a cone of radius b has more of its mass closer to the axis than does a disk of radius b.

9-e4 This is exactly the same as the calculation of the moment of inertia of a cube in section 4.2.5, except for the limits of integration.

$$\begin{aligned}I &= \rho\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}\left(y^2+z^2\right)\mathrm{d}y\ \mathrm{d}z\ \mathrm{d}x\\ &= \rho a\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}\left(y^2+z^2\right)\mathrm{d}y\ \mathrm{d}z\\ &= \rho a\int_{-b/2}^{b/2}\int_{-c/2}^{c/2} y^2\,\mathrm{d}y\ \mathrm{d}z + \rho a\int_{-b/2}^{b/2}\int_{-c/2}^{c/2} z^2\,\mathrm{d}y\ \mathrm{d}z\\ &= \rho ab\int_{-c/2}^{c/2} y^2\,\mathrm{d}y\ + \rho ac\int_{-b/2}^{b/2} z^2\,\mathrm{d}z\\ &= \frac{1}{12}\rho abc^3 + \frac{1}{12}\rho acb^3\\ &= \frac{1}{12}\rho abc(b^2+c^2)\\ &= \frac{M}{12}(b^2+c^2)\end{aligned}$$

The result doesn't depend on a, and that makes sense, because changing a without changing the mass wouldn't change how far any of the mass was from the axis.

9-e5 (a)

$$\begin{aligned}(\mathbf{b}\times\mathbf{c})_y &= b_zc_x - c_zb_x\\ (\mathbf{b}\times\mathbf{c})_z &= b_xc_y - c_xb_y\\ (\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))_x &= a_x(b_zc_x - c_zb_x) - a_z(b_xc_y - c_xb_y)\\ &= b_x(a_yc_y + a_zc_z) - c_x(a_yb_y + a_zb_z)\end{aligned}$$

This can be put in the desired form by adding $a_xb_xc_x$ to the first term and subtracting it from the second.
(b) In the case where $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$, applying the rule found in part a to the integrand results in $\omega r^2 - x^2\omega = \omega(y^z + z^2)$.
(c) Tackled by brute force, this is messy but

straightforward. To get the result much more simply, let's start with a sphere of radius 1, whose moment of inertia is $(2/5)m$. Since this equals $\int(y^2+z^2)\,\mathrm{d}m$, by symmetry we must have $\int y^2\,\mathrm{d}m=\int z^2\,\mathrm{d}m=(1/5)m$. What happens if we stretch it out along each axis, increasing its length along the x axis by a factor of a, and similarly for the other two axes? In the integral $\int y^2\,\mathrm{d}m$, the stretching has had the effect of increasing the integrand by a factor of b^2, so it contributes $(1/5)mb^2$, and similarly for the z^2 integral. The results for the diagonal elements of the moment of inertia matrix are

$$\begin{aligned} I_{xx} &= \frac{1}{5}m(b^2+c^2) \qquad , \\ I_{yy} &= \frac{1}{5}m(a^2+c^2) \qquad \text{, and} \\ I_{zz} &= \frac{1}{5}m(a^2+b^2) \qquad . \end{aligned}$$

9-e6 (a) Let $\mathbf{r}=u\mathbf{A}+v\mathbf{B}$, where the interior of the triangle is defined by $u>0$, $v>0$, and $u+v<1$. The parallelogram-shaped region defined by the infinitesimal intervals $\mathrm{d}u$ and $\mathrm{d}v$ then has area $\mathrm{d}A=|\mathbf{A}\times\mathbf{B}|\,\mathrm{d}u\,\mathrm{d}v$. The position of the center of mass is then given by

$$\begin{aligned} \mathbf{r}_{cm} &= \frac{\int \mathbf{r}\,\mathrm{d}A}{\int \mathrm{d}A} \\ &= \frac{\int_0^1\int_0^{1-v}(u\mathbf{A}+v\mathbf{B})\,\mathrm{d}u\,\mathrm{d}v}{\int_0^1\int_0^{1-v}\mathrm{d}u\,\mathrm{d}v} \qquad , \end{aligned}$$

where we can treat the differential of area $\mathrm{d}A$ as if it were the differential of mass $\mathrm{d}m$ because the mass per unit area cancels out. The numerator and denominator can both be split up into two similar integrals, and straightforward calculus gives the result as claimed.

(b) Here we do need to explicitly introduce the mass per unit area $2m/|\mathbf{A}\times\mathbf{B}|$, but we get a cancellation between this factor of $|\mathbf{A}\times\mathbf{B}|$ and the one arising from the relation between $\mathrm{d}A$ and $\mathrm{d}u\,\mathrm{d}v$. The moment of inertia is

$$\begin{aligned} I &= 2m\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} r^2\,\mathrm{d}u\,\mathrm{d}v \\ &= 2m\left[\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} A^2u^2\,\mathrm{d}u\,\mathrm{d}v+\ldots+0\right] \qquad , \end{aligned}$$

where ... represents a similar term for vector $\mathbf{B}$, and 0 is a cross-term that vanishes. The integral is separable, and the result is as claimed.

(c) Let I_o be the moment of inertia calculated in part b, and let $h=(1/6)|\mathbf{A}+\mathbf{B}|$ be the distance between the quadrilateral's center of mass and the center of mass of one of the triangles into which it can be split up, which follows from part a. Then $2(I_\mathrm{o}+mh^2)=(1/6)m(A^2+B^2)$, and solving for I_o gives the result claimed.

9-e7 (a) Suppose that $\mathbf{u}$ is a solution. Then if $\boldsymbol{\delta}$ is any vector for which $\mathbf{a}\cdot\boldsymbol{\delta}=0$, then $\mathbf{u}+\boldsymbol{\delta}$ is also a solution. Thus if we fix the tail of $\mathbf{u}$ at the origin, its tip traces a plane perpendicular to $\mathbf{a}$. We also need to show that a solution exists. Since $\mathbf{a}$ is nonzero, one solution is $\mathbf{a}b/|\mathbf{a}|^2$.

(b) The demonstration of nonuniqueness is similar to the one for the dot product, except that now we want $\boldsymbol{\delta}$ *parallel* to $\mathbf{a}$. The solution set forms a line parallel to $\mathbf{a}$. Existence follows because given any line segment with nonzero length, we can form a (possibly degenerate) rectangle with a given area and having that segment as one of its sides.

(c) Because the plane and line described in parts a and b are perpendicular, they have a unique point of intersection.

9-g1 The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to 90° that we can pretend they're 90 degrees, considering the rough nature of the other assumptions and measurements. The result is (300 N)(2.5 cm) = (F)(8 cm), or $F = 90$ N.

9-g2 Yes. For example, the force vanishes if the particle's velocity is parallel to the field, so if the beam had been launched parallel to the field, it would have gone in a straight line rather than a circle. In general, any component of the velocity vector that is out of the plane perpendicular to the field will remain constant, so the motion can be helical.

9-g3 No, it just means that any torques that do act are adding up to produce zero *total* torque. For instance, a merry-go-round rotating with constant angular velocity has torques acting on it, but the motor's torque and the torque due to friction are just canceling out to produce zero total torque. Of course it is also *possible* that no torques at all are acting (e.g., the earth has no torques on it at all), but it would be incorrect to infer that the merry-go-round's motion at constant angular velocity implied that there were no torques on it.

9-g4 You can get the maximum torque by setting everything up with θ=90°, giving

$$\begin{aligned}\tau &= rF\sin\theta\\ &= rmg\\ &= 270\ \text{N}\cdot\text{m} \qquad .\end{aligned}$$

9-g5 From the constant angular-acceleration equations,

$$\omega_f = \omega_0 + \alpha t$$

Plugging in $\omega_0 = 90 \times 2\pi/60\,\text{rad/s} = 3\pi\,\text{rad/s}$ and $t = 30$ s, we find $|\alpha| = 0.31\,\text{rad/s}^2$. Plugging this into $\sum\tau = \tau_f = I\alpha$ gives $\tau_f = 0.05$ N $\cdot$ m.

9-g6 (a) In linear momentum, we had

$$\int F dt = \Delta p$$

With angular momentum, this becomes

$$\int \tau dt = \Delta L$$

In particular, from $t = 2$ s to $t = 5$ s,

$$\begin{aligned}\int_{2\text{s}}^{5\text{s}} (2\ \text{N}\cdot\text{m})dt = \Delta L = L_{5\text{s}} - L_{2\text{s}} = L_{5\text{s}}\\ 6\ \text{N}\cdot\text{m}\cdot\text{s} = I\omega_{5\text{s}} = I(90\,\text{rad/s})\\ I = (1/15)\ \text{kg}\cdot\text{m}^2\end{aligned}$$

(b) Using the same equation from part (a) (relating the integral of torque to the change in angular momentum), we see that the change in angular velocity from 5 s to 7 s is one-sixth the change in angular velocity from 2 s to 5 s. Therefore, the angular velocity is 105 s^{-1} at $t = 7$ s.

9-g7 The rotational analog of $W = F\Delta x$ is $W = \tau\Delta\theta$. Both τ and $\Delta\theta$ are the same for the second revolution as for the first, so the work done in the second revolution is the same.

9-g8 The total torque required is the torque needed to lift his arm plus the torque needed to lift the weight. Since his arm is not changing, she needs to tell him to set things up so that the torque needed to lift the weight is also the same as before. Torque is $rF\sin\theta$, and since he is decreasing r by a factor of 17/33, she should tell him to increase the mass by a factor of 33/17 so that the increased weight force will compensate for the decreased r. The new mass should be 2.1 kg × 33/17 = 4.1 kg.

9-g9 For the purpose of calculating gravitational torques, we can treat each branch as if its weight were concentrated at its center of mass. The longer branch's center of mass is twice as far from the trunk, so the r is doubled. The gravitational force is also twice as much, so $\tau = rF$ is four times greater.

9-j1 (a) $\tau = mg(\ell/2+b) - mg(\ell/2-b) = 2mgb$
(b) This is the same as the torque we would get if the full weight $2mg$ acted at the center of mass, which is at a distance b from the fulcrum.

9-j2 (a) The initial angular momentum comes only from the piece of clay and is equal to mdv_0. From conservation of angular momentum,

$$\begin{aligned}mdv_0 &= I_{\text{tot}}\omega_f\\ mdv_0 &= \left(I + md^2\right)\omega_f\\ \omega_f &= \frac{mdv_0}{I + md^2}\end{aligned}$$

(b) The initial and final KE's are

$$\begin{aligned} \text{KE}_i &= (1/2)mv_0^2 \\ \text{KE}_f &= (1/2)I_{\text{tot}}\omega_f^2 \\ &= \frac{1}{2}\frac{(mdv_0)^2}{I+md^2} \end{aligned}$$

The fraction of kinetic energy lost is

$$1-\frac{\text{KE}_f}{\text{KE}_i} = 1-\frac{md^2}{I+md^2} = \frac{I}{I+md^2}$$

(c) The kinetic energy of the door-plus-clay system is equal in magnitude to the work done by friction, $|W_{\text{fr}}| = \tau_{\text{fr}}\Delta\theta$. Therefore,

$$\tau_{\text{fr}} = \frac{1}{2\Delta\theta}\frac{(mdv_0)^2}{I+md^2}$$

9-j3 Astrid is right and Michelle is wrong. The force on the ball acts along the line of the string. This line is perpendicular to the ball's motion, so the string does no work on the ball, and the ball must therefore keep the same kinetic energy. After all, if the ball's kinetic energy was going to change, conservation of energy says that some other form of energy would have to change in the opposite direction, but the pole and the string aren't storing or releasing any energy.

The flaw in Michelle's reasoning is that the string's force on the ball is not straight toward the center of the pole, since the string is wrapping around the pole. The string makes a nonzero torque on the ball, and the ball's angular momentum changes. The ball is transferring angular momentum to the planet Earth.

9-j4 (a) There is no torque on the book, and so its angular momentum about the center of the circle is conserved. Therefore,

$$\begin{aligned} mRv_0 &= m(R/2)v_f \\ v_f &= 2v_0 \end{aligned}$$

(b) The work done (W) is equal to the change in KE (from the work-energy theorem). Therefore,

$$\begin{aligned} W &= \text{KE}_f - \text{KE}_i \\ &= (1/2)m(2v_0)^2 - (1/2)mv_0^2 = (3/2)mv_0^2 \\ &= 3\text{KE}_i = (3/4)\text{KE}_f \end{aligned}$$

9-j5 There is a kinetic frictional force on the ball, which reduces its momentum by Δp. Associated with this force there is also a torque. Just as force is the rate of change of momentum, torque is the rate of change of angular momentum, and since the ratio of torque to force is R, the ratio of ΔL to Δp is also R: $\Delta L = R\Delta p$. The ball has no angular momentum initially, so ΔL is the same as the final angular momentum L. We then have

$$\begin{aligned} \frac{p}{L} &= \frac{5}{2R} = \frac{p_{\text{o}}}{R\Delta p} - \frac{1}{R} \\ \frac{5}{2} &= \frac{p_{\text{o}}}{\Delta p} - 1 \\ \Delta p &= \frac{2}{7}p_{\text{o}} \qquad , \end{aligned}$$

so the ball loses 2/7 of its initial momentum, and is left with 5/7.

9-j6 Newton's second law is

$$a = \frac{mg-T}{m} \qquad ,$$

where T is the tension in the string. The yo-yo's angular momentum is the same as the angular momentum of a single cylinder of mass m and radius R rotating about its axis:

$$L = \frac{\pi mR^2}{P} \qquad ,$$

where to avoid confusion I've used P rather than T for the period. The period is related to the yo-yo's downward velocity by $v = 2\pi r/P$, so we have

$$L = \frac{mR^2v}{2r} \qquad .$$

The torque on the spindle is Tr, so

$$\begin{aligned} \tau &= \frac{\Delta L}{\Delta t} \\ Tr &= \frac{mR^2a}{2r} \\ T &= \frac{mR^2a}{2r^2} \end{aligned}$$

We now have two equations in the two unknowns T and a. One equation is Newton's second law, and the other is the equation just derived. Subsituting the latter into Newton's second law to eliminate T, we find

$$\begin{aligned} a &= g - \frac{R^2 a}{2r^2} \\ a &= \frac{g}{1 + R^2/2r^2} \end{aligned} \qquad .$$

Let's check this answer. The units work, because g is an acceleration, and the denominator is unitless. The denominator is greater than one, so $a < g$, which matches up with our real-world experience: a yo-yo doesn't drop like a rock. The acceleration gets smaller if R is bigger in relation to r, and that makes sense too, because increasing R makes it harder to spin (the same torque produces a slower change in the rate of rotation).

9-m1 Note that, since the pulley has a non-negligible moment-of-inertia, the two tensions need to be different. Call the tension connecting block A to the pulley T_A, and call the tension in the part of the string connecting block B to the pulley T_B. We know $T_B > T_A$ in order for the pulley to start rotating clockwise. A free-body diagram for the two blocks give the following two equations:

$$\begin{aligned} T_A &= M_A a \\ M_B g - T_B &= M_B a \end{aligned}$$

Additionally, the fact that the sum of the torques on the pulley equals $I\alpha$ lets us write the following for the pulley:

$$\begin{aligned} R(T_B - T_A) &= I\alpha = Ia/R \\ T_B - T_A &= Ia/R^2 = (1/2)M_C a \end{aligned}$$

By adding this last equation to the first two, the tensions cancel, and we're left with

$$\begin{aligned} M_B g &= (M_A + M_B + (1/2)M_C)a \\ a &= \frac{M_B g}{M_A + M_B + (1/2)M_C} \end{aligned}$$

9-m2 (a) This is a nice example of a question that can very nearly be answered based only on units. Since the three variables, m, b, and g, all have different units, they can't be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form

$$p = Am^j b^k g^l,$$

where A, j, k, and l are unitless constants. The result has to have units of kg·m/s. To get kilograms to the first power, we need

$$j = 1,$$

meters to the first power requires

$$k + l = 1,$$

and seconds to the power -1 implies

$$l = 1/2.$$

We find $j = 1$, $k = 1/2$, and $l = 1/2$, so the solution must be of the form

$$p = Am\sqrt{bg}.$$

Note that no physics was required!

Consideration of units, however, won't help us to find the unitless constant A. Let t be the time the rod takes to fall, so that $(1/2)gt^2 = b/2$. If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be 1/4, or 3/4, or 5/4, ..., but all the possibilities greater than 1/4 would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time $T = 4t$, and it has angular momentum $L = (\pi/6)mb^2/T$.

The momentum lost by the object striking the rod is p, and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum $bp/2$, which is also transferred to the rod. Equating this to the expression above for L, we find $p = (\pi/12)m\sqrt{bg}$.

(b) Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The foot of the rod is moving in a circle relative to the center of the rod, with speed $v = \pi b/T$, and acceleration $v^2/(b/2) = (\pi^2/8)g$. This acceleration is initially upward, and is greater in magnitude than g, so the foot of the rod will lift off without dragging. We could also worry about whether the foot of the rod would make contact with the floor again before the rod finishes up flat on its back. This is a question that can be settled by graphing, or simply by inspection of figure **??** on page **??**. The key here is that the two parts of the acceleration are both independent of m and b, so the result is univeral, and it does suffice to check a graph in a single example. In practical terms, this tells us something about how difficult the trick is to do. Because $\pi^2/8 = 1.23$ isn't much greater than unity, a hit that is just a little too weak (by a factor of $1.23^{1/2} = 1.11$) will cause a fairly obvious qualitative change in the results. This is easily observed if you try it a few times with a pencil.

9-m3 (a)

$$\begin{aligned} I &= \int r^2 \, \mathrm{d}m \\ &= (m/b^2) \int r^2 \, \mathrm{d}A \\ &= (m/b^2) \int_0^b \int_0^b \left(x^2 + y^2\right) \mathrm{d}x \ \mathrm{d}y \\ &= (2/3)mb^2 \end{aligned}$$

(b) Since there is no horizontal force, the center of mass drops along a vertical line. When the square has rotated through an angle θ, the height of the center of mass above the floor is $(b/\sqrt{2})\cos\theta$. Differentiating this, we find that at the final position, $\theta = 45°$, the speed of the center of mass is $v = b\omega/2$. Conservation of energy gives

$$mg\frac{b}{\sqrt{2}} = mg\frac{b}{2} + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \qquad .$$

The result is

$$\omega = \sqrt{\frac{12}{11}(\sqrt{2}-1)\frac{g}{b}}$$

9-p1

No. It is momentarily at rest, but not at equilibrium, because there is a force on it, the force of gravity, which is not canceled out by any other force. Another way of putting it is that it has $p = 0$ at that moment, but not $\mathrm{d}p/\,\mathrm{d}t = 0$, which is what equilibrium means.

9-p2 Basically we have a situation with two equations in two unknowns. There is an equation that says the total torque on the foot is zero, and another equation that says the total force on the foot is zero. The two unknowns are the tension in the calf muscle, T, and the force exerted by the shinbone, F_s. We can however simplify the solution by choosing the axis to be the place where the shinbone attaches to the foot. With this choice, the shinbone makes no torque on the foot, and the torque equation thus has only the tension in the calf muscle as an unknown. There are two torques on the foot, a clockwise torque exerted by the muscle and a counterclockwise torque from the floor. The floor's upward force equals the person's weight, so if we use plus signs for clockwise torques the torque equation is

$$Ta - Wb = 0 \qquad ,$$

which is easily solved to give

$$T = W\frac{b}{a} \qquad .$$

Letting positive forces be upward, the force equation is

$$T - F_s + W = 0 \qquad ,$$

so

$$\begin{aligned} F_s &= T + W \\ &= W\frac{b}{a} + W \\ &= \left(\frac{b}{a} + 1\right) W \qquad . \end{aligned}$$

9-p3 (a) We have two unknowns, F and T, so we're going to need to find two equations to solve for them. Setting the total vertical force on the ruler equal to zero gives one equation (+=up),

$$F - mg + T = 0 \qquad ,$$

and setting the total torque equal to zero gives a second one (axis on the right end, counterclockwise=+),

$$-bF + \frac{L}{2}mg = 0 \qquad .$$

Solving these, we find

$$\begin{aligned} F &= \frac{L}{2b}mg \\ T &= \left(1 - \frac{L}{2b}\right)mg \end{aligned}$$

(b) In the case of $b = L$, we have $F = T = mg/2$, which makes sense because the finger and the string are at the ends of the rod, each supporting half the rod's weight. For $b = L/2$, we get $T = 0$ and $F = mg$, because the rod is balanced on the finger, which means the string is slack.
(c) For $b < L/2$, the tension in the string becomes negative, which doesn't make sense — you can't push with a rope. In this situation, there is no possible equilibrium, and the rod just flips over in the counterclockwise direction.

9-p4

(a) The equilibrium occurs where the interaction energy has a local minimum. Setting $a = 1$ as suggested, we have

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}k\left(r^{-12} - 2r^{-6}\right) \\ &= k\left(-12r^{-13} + 12r^{-7}\right) \qquad . \end{aligned}$$

Setting this equal to zero,

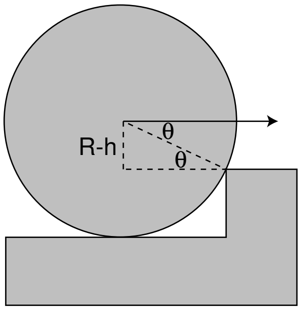
$$\begin{aligned} 12r^{-13} &= 12r^{-7} \\ r^{-13} &= r^{-7} \\ r &= 1 \qquad . \end{aligned}$$

Since $a = 1$, this is the same as saying $r = a$.
(b) Making a graph shows that this is a minimum of U rather than a maximum. (You could also use the second derivative test.)
(c) At $r = \infty$, the equation gives $U = 0$. At $r = a$, we have $U = k(1 - 2) = -k$. The difference in energy is k.

9-s1 (a) Let h be the height of the step, R the radius of the wheel, m its mass, and F the minimum force required. When the wheel is just about to lift off, the normal force of the ground under it goes to zero, so the only forces involved are gravity, F, and the force exerted by the corner of the step. If we're going to get it over the step, we have to make enough torque on it so that the total torque is clockwise. The minimum force corresponds to the case where the total torque is zero, so that any greater force would do the job. It's natural to consider the corner of the step as the axis, since that's the point around which the wheel will acutally rotate. With this choice, the only nonzero torques are the torque we're applying and the torque of gravity. Defining the angle θ shown in the figure, we have

$$\begin{aligned} F\sin\theta &= mg\sin(90° - \theta) \\ F\sin\theta &= mg\cos\theta \\ F &= \frac{mg}{\tan\theta} \\ &= mg\frac{\sqrt{R^2 - (R-h)^2}}{R-h} \\ &= mg\frac{\sqrt{1 - (1 - h/R)^2}}{1 - h/R} \end{aligned}$$

(b) For $h = R$, the denominator equals zero, and the force required becomes infinite. This makes sense, because in this situation, you're just trying to pull the wheel straight into the wall of the step.



9-s2 (a) Taking the point where the bar touches the ground to be the pivot point, there are two non-zero torques on the bar: one from the normal force, and one from the gravitational force. Take the ccw torque to be positive, and let $d = L\cos\theta$:

$$\begin{aligned} Mg(d/2) - T(d) &= 0 \\ T &= Mg/2 \end{aligned}$$

(b) From Newton's second law applied to the bar (vertical components):

$$\begin{aligned} T + N - Mg &= 0 \\ N &= Mg/2 \end{aligned}$$

(c) From Newton's second law applied to the bar (horizontal components):

$$f_s = 0$$

9-s3 (a) Let the axis be the hinge, and let the length of the bar br ℓ. The gravitational torque can be calculated by pretending that it acts at the bar's center of mass, with $r = \ell/2$. Cancellation of the torques gives $(1/2)\ell mg = \ell T\sin\theta$, and the result is $T = mg/(2\sin\theta)$.
(b) To minimize T, we want to maximize $\sin\theta$, which means letting θ be 90°. In other words, the cable is vertical.
(c) As θ approaches zero, the tension approaches infinity. No matter how hard you pull, you can't support the weight of the bar with a horizontal cable.
9-s4 (a) Choose our pivot point to be the location where the block is in contact with the ground. We require that the torques on the block about this pivot be zero. The torque from the rope is TL out of the page. To find the gravitational torque, draw a vertical line downwards from the center-of-mass of the block. With a picture and some trig, you can solve for the moment arm of this torque to be $L(\cos\theta - \sin\theta)$. Therefore,

$$\begin{aligned} TL &= mgL(\cos\theta - (1/2)\sin\theta) \\ T &= mg(\cos\theta - (1/2)\sin\theta) \end{aligned}$$

(b) From Newton's second law applied to the block (vertical components):

$$\begin{aligned} N - mg - T\sin\theta &= 0 \\ N &= mg + T\sin\theta \\ &= mg[1 + \sin\theta(\cos\theta - (1/2)\sin\theta)] \end{aligned}$$

(c) From Newton's second law applied to the block (horizontal components), we see that the static frictional force required is

$$f_s = T\cos\theta = mg\cos\theta(\cos\theta - (1/2)\sin\theta)$$

Also, we know that the maximum amount of static friction possible is

$$f_s^{\max} = \mu_s N = \mu_s mg[1+\sin\theta(\cos\theta - (1/2)\sin\theta)]$$

By requiring that $f_s \leq f_s^{\max}$,

$$\begin{aligned} \cos\theta(\cos\theta - \frac{1}{2}\sin\theta) &\leq \mu_s[1 + \sin\theta(\cos\theta - \frac{1}{2}\sin\theta)] \\ \mu_s &\geq \frac{\cos\theta(2\cos\theta - \sin\theta)}{2 + \sin\theta(2\cos\theta - \sin\theta)} \end{aligned}$$

For $\theta = 30°$, we have $\mu_s \geq 0.408$.
9-s5 We know that, without the weight, the meter stick is balanced at the 48.0 cm mark. Therefore, the center-of-mass of the meter stick is located here. When the weight is attached, the torque from the weight of the meter stick cancels with the torque from the hanging weight. Therefore,

$$\begin{aligned} (2.0\ \text{cm})M &= (30.0\ \text{cm})(4.0\ \text{g}) \\ M &= 6.0 \times 10^1\ \text{g} \end{aligned}$$

9-s6 (a) Look at the torques on the bar taking the left side to be the pivot. Taking the reading N_2 to be equal to Mg,

$$\begin{aligned} Mg(L/2) + (2Mg)x - Mg(L) = 0 \\ x = L/4 \end{aligned}$$

(b) From Newton's second law, we know that the upwards forces from the two scales must balance the weight of the bar and Julia, $3Mg$. Since $N_2 = Mg$, we know $N_1 = 2Mg$.

9-s7 (a) Taking the hinge to be the pivot point, we find (taking cw torques to be positive)

$$\begin{aligned} mgL/2 + Wx - T\sin\theta = 0 \\ T = \frac{mgL + 2Wx}{2\sin\theta} \end{aligned}$$

(b) The x-component of the force on the bar from the hinge cancels with the horizontal component of tension. Therefore,

$$F_x = +T\cos\theta = \frac{mgL + 2Wx}{2\tan\theta}$$

(c) The torque from the y-component of the force on the bar from the hinge cancels with the torques from the weight W and the gravitational torque on the bar. Therefore,

$$\begin{aligned} F_yL - mgL/2 - W(L - x) = 0 \\ F_y = mg/2 + W(1 - x/L) \end{aligned}$$

9-s8 (a) Let the cw torques be positive and the ccw torques be negative. Take the right side of the base to be the pivot point, and look at torques on the stoplight-system. Let's call the magnitude of the downwards force from the left screw $F_{l.s.}$

$$Mg\left(\frac{D}{2} - \frac{w}{2}\right) + Mg\left(D - \frac{w}{2}\right) - Mg\left(\frac{w}{2}\right) - F_{l.s.}w = 0$$

$$F_{l.s.} = \frac{3}{2}Mg\left(\frac{D}{w} - 1\right)$$

(b) Call the upwards normal force on the base equal to N. By looking at Newton's second law (vertical components) of forces acting on the stoplight-system, we get

$$\begin{aligned} N - F_{l.s.} - 3Mg = 0 \\ N = \frac{3}{2}Mg\left(\frac{D}{w} + 1\right) \end{aligned}$$

9-s9 (a) There are three forces on the ladder, F_1, F_2, and its own weight, which can be thought of as acting at its center for the purpose of calculating torques. If it's going to remain at rest, the total horizontal force, total vertical force, and total torque must all be zero. Taking the axis of rotation as the center of the ladder, the weight force makes no torque. If ϕ was less than θ, both torques would be of the same sign (clockwise), so they could not cancel.

(b) The three conditions for equilibrium are:

zero total horizontal force:

$$F_1\cos\phi - F_2 = 0 \qquad (1)$$

zero total vertical force:

$$F_1\sin\phi - mg = 0 \qquad (2)$$

zero total torque:

$$\frac{L}{2}F_2\sin\theta - \frac{L}{2}F_1\sin(\phi - \theta) = 0$$

or

$$F_2\sin\theta - F_1\sin(\phi - \theta) = 0 \qquad (3)$$

For our numerical check, let's arbitrarily choose $mg = 1.000$ N. L is irrelevant, since it canceled out in eq. (3). Eq. (2) gives $F_1 = 1.118$ N. Eq. (1) gives $F_2 = 0.500$ N. As advertised, eq. (3) checks out.

9-v1 In the previous problem, we got three equations, which involved several variables we want to get rid of: F_1, F_2 and m. It should be possible to solve three equations in three unknowns. We can simplify things right off the bat by realizing that eq. (2) is useless; it introduces another variable, m, which we don't know and don't want to know. So we're better off just ignoring it and considering this as two equations, (1) and (3), in two unknowns. The general strategy in solving simultaneous equations like this is to solve one of the equations for one of the unknowns, then substitute into the other equation(s) to eliminate it. Let's solve eq. (1) for $F_2 = F_1 \cos\phi$. Eliminating F_2 from eq. (3), we have

$$F_1 \cos\phi \sin\theta - F_1 \sin(\phi-\theta) = 0 \qquad ,$$

which we can now simplify by dividing by F_1, resulting in

$$\cos\phi \sin\theta - \sin(\phi-\theta) = 0 \qquad .$$

We've eliminated all the unknowns, and we now have a relationship involving only θ and ϕ, the things we're interested in. Using the trig identity as suggested, $\sin(\phi-\theta) = \sin\phi\cos\theta - \sin\theta\cos\phi$, so our equation becomes

$$\cos\phi \sin\theta - \sin\phi\cos\theta + \sin\theta\cos\phi = 0 \qquad .$$

Simplifying, we get $\tan\phi = 2\tan\theta$, so $\phi = \tan^{-1}(2\tan\theta)$. As claimed, m and L are irrelevant.

9-v2 Label the normal force from the wall N_w and the normal force from the ground N_g. The normal force from the ground on the ladder must support the weight of the person, and so $N_g = Mg$. Therefore, the maximum amount of static friction that the ground can provide is $f_s^{\max} = \mu_s N_g = \mu_s Mg$. The only forces in the horizontal direction are N_w and f_s, and so the maximum force that the wall will provide assuming static equilibrium is $N_w = f_s^{\max} = \mu_s Mg$. Label the maximum distance that the person can climb up the ladder (along the ladder) "x". By looking at the torques on the ladder about the point of contact with the ground,

$$\begin{aligned} Mgx\cos\theta &= N_w L \sin\theta = \mu_s MgL\sin\theta \\ x &= \mu_s L \tan\theta \end{aligned}$$

9-v3 What makes this a little hard is that not only do we not know the weights w and W of the ship and bottle, we also can't tell the exact positions of their centers of mass. Let the bottle's center of mass lie at a distance H to the right of the fulcrum, and let the ship's center of mass be h to the right of the ship's stern. Suppose that for our first measurement we slide the ship around until its stern is a distance b_1 to the right of the fulcrum. Let the axis be at the fulcrum, so that the (unknown) force from the fulcrum doesn't make a torque. Then setting the total torque equal to zero gives

$$(b_1+h)w + HW - S_1\ell = 0 \qquad ,$$

where S_1 is the scale's reading in this position, and ℓ is the scale's distance from the fulcrum. This is one equation in four unknowns (h, H, w, and W). It wouldn't do us any good to write down the equation that says that the total force is zero, because that would bring in another unknown: the force at the fulcrum.

It should now be fairly clear that a solution exists. Every time we slide the ship around to a new b, we generate a new equation, without introducing any new unknowns. It is therefore definitely possible to get a unique solution for all four unknowns simply by sliding the ship to locations b_1, b_2, b_3, and b_4.

Actually it turns out that only two locations are necessary. At a second location,

$$(b_2+h)w + HW - S_2\ell = 0 \qquad .$$

Subtracting the two equations gives $(b_1-b_2)w - (S_1-S_2)\ell = 0$, which has only the unknown w in it.

9-v4 (a) Let's use subscripts r for the rope, g for gravity, and n for the floor's normal force. In the case where the block topples, it rotates and its center of mass rises as it tips up and over. In the case where it slides without tipping, neither of these is supposed to happen, so the vertical forces must cancel, and the torques must cancel. Cancellation of the vertical forces gives $F_n = mg$. The torques are most easily calculated using $\tau = r_\perp F$, with the axis taken to be at the center of the box. Note that in the condition where the box is not tipping, the normal force is actually being distributed over the whole bottom surface of the block, so it doesn't actually have a single, well-defined value of $r_\perp$; however, as soon as the tipping starts, the only point of contact is at $r_\perp = a$. The torques are therefore $\tau_r = bF_r$ and $\tau_n = amg$. If the box is just about to tip, i.e., the force from the rope has its maximum value, then these torques cancel, and $F_r = (a/b)mg$. The dependence on a/b makes sense, because a squat box with a large value of a would be more stable, allowing F_r to be greater. (What we've actually calculated here is the value of F_r for which the box would just barely *start* to tip. Once the box started to tip, we could ask ourselves whether it might then manage to right itself again. The answer is no, because once it starts to tip, both of the $r_\perp$ values start to decrease, but their ratio is still a/b.)

(b) The force of friction is $F_f = \mu F_n = \mu mg$. (In the case of static friction, this is the maximum force, which is what we're interested in because we want to see if we can unstick the box and get it to slide without tipping. In the case of kinetic friction, this is simply the value of the frictional force.) We now have a torque from friction, which is in the same clockwise direction as the torque from the rope. Cancellation of the torques gives $bF_r + b\mu mg = amg$, yielding $F_r = (a/b - \mu)mg$.

(c) The answer to part b, unlike the one from part a, could come out negative under certain circumstances. The negative solution would be unphysical, and the interpretation would be that it would be impossible to accelerate the box without making it tip over. This happens for $b > a/\mu$. That makes sense. A big value of b means a tall box, which would be less stable. In the limit $\mu \to 0$, we get $b \to \infty$, meaning that there is no maximum value of b, recovering the behavior of the frictionless case.

9-v5 (a) Let the hinge be the axis, and set the total torque on one bar equal to zero. With this choice of axis, the normal force at the hinge makes zero torque, so we have

$$\frac{1}{2}\ell mg\cos\alpha = \ell T\sin(\beta-\alpha) \qquad .$$

This is a single equation in two unknowns, α and T, so we're not ready to solve anything yet.

The brute-force technique would be to set the total horizontal and vertical force on this bar equal to zero, and this would give us two more equations, while introducing one more unknown (the horizontal normal force at the hinge). We would then have three equations in three unknowns.

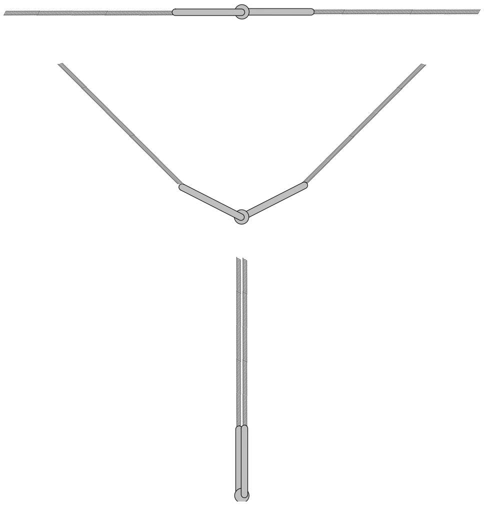
A trick that simplifies the calculation is to set the total vertical force on *both* bars equal to zero, in which case the normal force at the hinge becomes an internal force that we can ignore. We then have

$$2T\sin\beta = 2mg \qquad .$$

Solving the second equation for T and plugging in to the first equation, we get

$$\begin{aligned}\frac{1}{2}\cos\alpha &= \frac{\sin(\beta-\alpha)}{\sin\beta}\\ \frac{1}{2}\cos\alpha &= \frac{\sin\beta\cos\alpha-\cos\beta\sin\alpha}{\sin\beta}\\ \alpha &= \tan^{-1}\left(\frac{1}{2}\tan\beta\right) \qquad .\end{aligned}$$

(b) For $\beta = 0$, 45°, and 90°, we get $\alpha = 0$, 27°, and 90°, respectively. All three of these make sense physically. In particular, the intermediate case has $\alpha < \beta$, which looks intuitively right.



9-v6 (a) Let N be the normal force of the cylinder on each bar. The total force on the bars in the vertical direction must be zero, so $2N\cos\theta = 2mg$.

The total torque on each bar must also be zero. Let the axis be at the hinge. The torque due to gravity is $mg(L/2)\sin(90° - \theta) = mg(L/2)\cos\theta$. Constructing the right triangle with vertices at the hinge, the point of contact, and the center of the cylinder, we find that the distance from the hinge to the point of contact is $r\tan\theta$, so the torque of the cylinder on each hinge is $Nr\tan\theta$.

We now have the two equations $2N\cos\theta = 2mg$ and $mg(L/2)\cos\theta = Nr\tan\theta$ in the two unknowns θ and N. Eliminating N, we find the result claimed in the problem.

(b) For large values of r/L, $\tan\theta$ must approach zero, so θ approaches zero. This makes sense, because the cylinder is so huge that the hinge is essentially lying on a flat surface.

For small values of r/L, $\tan\theta$ must approach infinity, so θ approaches 90 degrees. This is intuitively appealing if we imagine the cylinder as a hair-thin wire: the hinge folds flat.

9-v7 (a) We work by induction. Clearly $L_1 = w/2$. Given L_n, we can find L_{n+1} by thinking of the stack of $n + 1$ as being composed of a single book at the bottom with n books on top of it. We don't want the n books to tip, so they should protrude beyond the bottom book by L_n. In addition, we don't want the entire stack of $n + 1$ to tip over the edge of the table, i.e., its center of mass should be above the table's edge. Letting $\delta = L_{n+1} - L_n$, the horizontal position of this center of mass relative to the table's edge is $n\delta + (\delta - w/2)/(n + 1) = 0$. This results in $\delta = w/2(n + 1)$, so

$$L_n = \frac{w}{2}\sum_{m=1}^{n}\frac{1}{m} \qquad .$$

(b) If we draw rectangles on graph paper to represent the sum occurring in part a, then the curve $y = 1/(x + 1)$ always lies below the rectangles, so $L_n \geq (w/2)\int_0^n \mathrm{d}x/(x + 1) = (w/2)\ln(n + 1)$. Plugging in numbers proves the result claimed for $L > 7w$.

9-v8 Let positive x be to the left, positive y out of the page, and opsitive z up. Let the axis be at the universal joint. The total torque is

$$\begin{aligned}&\frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) \times (-\hat{\mathbf{z}})\\ &+\left(\frac{1}{2}\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}\right) \times (a\hat{\mathbf{z}})\\ &+(\hat{\mathbf{x}} + \hat{\mathbf{z}}) \times (-b\hat{\mathbf{y}})\\ &+\left(\frac{1}{2}\hat{\mathbf{y}} + \hat{\mathbf{z}}\right) \times (-c\hat{\mathbf{x}}).\end{aligned}$$

Computing the cross products and setting this equal to zero gives

$$\begin{aligned}0 = &\left(-\frac{1}{2} + a + b\right)\hat{\mathbf{x}}\\ &+\left(\frac{1}{2} - \frac{1}{2}a - c\right)\hat{\mathbf{y}}\\ &+\left(-b + \frac{1}{2}c\right)\hat{\mathbf{z}}.\end{aligned}$$

This is a system of three equations in the three unknowns. The solution is $a = 1/3$, $b = 1/6$, and $c = 1/3$.

10-a1 The difference in pressure between the inside and the outside of the plane is $(3/4)P_0 = 7.6 \times 10^4$ Pa. Multiplying this by the area of the window, $A = 0.20\ \mathrm{m}^2$, we see that a force of 15 kN is required!

10-a2 (a) Let's call a the cross-sectional area of the (small) piston, and A the cross-sectional area of the (large) load platform. We need a force Mg at the platform end to lift the car, which corresponds to a gauge pressure of Mg/A. This is the pressure we will need to apply to the piston end. Setting $F/a = Mg/A$, we get $F = Mg(a/A) = Mg(d/D)^2$.

(b) The piston moves a distance H, where $Ha = LA$ (a and A being the areas as defined in the solution to part a). This is because the volume of water remains constant. Therefore, $H = L(A/a) = L(D/d)^2$.

(c) The work done by the force applied to the piston is $FH = [Mg(d/D)^2][L(D/d)^2] = MgL$, which is exactly the change in gravitational PE of the car.

10-d1 (a) The pressure in the straw can be reduced by a maximum amount equal to $\rho_w gh = 9800$ Pascals. Therefore, the minimum pressure you can get the air inside is $(1.013 \times 10^5 - 9800)$ Pa $= 9.2 \times 10^4$ Pa $= 0.90$ atm.
(b) Setting $\rho_w gH$ equal to atmospheric pressure, we get $H = 10.3$ m.

10-d2 The pressure is $P_0 + \rho_w gd$, where d is the depth of the water. This gives an absolute pressure of 3.1×10^7 Pascals, or 310 times atmospheric pressure!

10-d3 (a) $\rho gL = P_0$ gives $\rho = P_0/(gL)$.
(b) $\rho_{\text{Hg}} = 13600\ \text{kg/m}^3$.

10-d4 The pressure at the bottom of the column of water is equal to the pressure at the bottom of the column of the unknown liquid (otherwise, there would be a net force on the water underneath this height, and the water would move). The pressure at the bottom of each column is $P_0 + \rho gd$, where ρ is the density of the liquid, and d is the depth as measured from the surface (which is at atmospheric pressure, P_0). Therefore,

$$\begin{aligned}\rho_w h &= \rho_X H\\ \rho_X &= \rho_w (h/H)\end{aligned}$$

10-d5 (a) If the expression $1 + by$ is to make sense, then by has to be unitless, so b has units of m^{-1}. The input to the exponential function also has to be unitless, so k also has of m^{-1}. The only factor with units on the right-hand side is P_o, so P_o must have units of pressure, or Pa.
(b)

$$\begin{aligned}\mathrm{d}P &= \rho g\,\mathrm{d}y\\ \rho &= \frac{1}{g}\frac{\mathrm{d}P}{\mathrm{d}y}\\ &= \frac{P_o}{g}e^{-ky}(-k - kby + b)\end{aligned}$$

(c) The three terms inside the parentheses on the right all have units of m^{-1}, so it makes sense to add them, and the factor in parentheses has those units. The units of the result from b then look like

$$\begin{aligned}\frac{\text{kg}}{\text{m}^3} &= \frac{\text{Pa}}{\text{m/s}^2}\text{m}^{-1}\\ &= \frac{\text{N/m}^2}{\text{m}^2/\text{s}^2}\\ &= \frac{\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2}}{\text{m}^2/\text{s}^2},\end{aligned}$$

which checks out.

10-d6 (a)

$$\begin{aligned}P_{surface} - P_{center} &= \int_{r=0}^{b} \mathrm{d}P\\ &= -\int_{r=0}^{b} \rho g\,\mathrm{d}r\end{aligned}$$

The pressure at the surface is essentially zero, so

$$\begin{aligned}P_{center} &= \int_{r=0}^{b} \rho g\,\mathrm{d}r\\ &= \int_{r=0}^{b} \rho\frac{Gmr}{b^3}\,\mathrm{d}r\\ &= \left.\frac{G\rho mr^2}{2b^3}\right]_{r=0}^{b}\\ &= \frac{G\rho m}{2b}\\ &= \frac{3Gm^2}{8\pi b^4}\end{aligned}$$

(b) The units of Gm^2/b^2 are newtons, so the units of Gm^2/b^4 are N/m^2.
(c) 1.7×10^{11} Pa.
(d) Atmospheric pressure is 10^5 Pa, which is, as expected, about six orders of magnitude less than the answer to part c.

10-g1 (a) The strength of the gravitational field is irrelevant: it cancels out because both the weight of the sphere and the weight of the displaced liquid are equal to it. This means that there must be a relationship among the following four variables: the density of the liquid, the density of the sphere, the radius of the sphere, and the draft y. The two densities are the only variables that have kilograms in their SI units, so there is no way they could be related to the two lengths except if the ratio of the densities is the only thing that matters. Similarly, only the ratio of y to the radius can be related to the density ratio. We therefore have a relation of the following form:

$$\frac{\text{density of sphere}}{\text{density of liquid}} = f\left(\frac{\text{draft}}{\text{radius}}\right).$$

The appearance of the ratio on the right tells us that the physics is scale-invariant. That is, if we take a picture of a large sphere and a small sphere floating, the pictures will look the same except for a rescaling.

(b) Let the density of the liquid be 1 and let the radius of the sphere also be 1. Let v be the volume of the displaced liquid and ρ the density of the sphere. Archimedes' principle says

$$v = \frac{4\pi}{3}\rho.$$

All that remains is to find v in terms of y. To accomplish this, we can break the sphere down into horizontal disks and integrate over the disks. Let x be the radius of the disk (which is less than or equal to than the unit radius of the sphere).

$$\begin{aligned} v &= \int_{-1}^{y} \pi x^2 \cdot dy' \\ &= \pi \int_{-1}^{y} (1 - y'^2) \cdot dy' \\ &= \pi \left[y + 1 - \frac{1}{3}(y^3 + 1) \right] \end{aligned}$$

Solving for the density we have

$$\rho = \frac{1}{4}\left[-y^3 + 3y + 2\right].$$

As a check on the result, we find that $y = -1$ gives $\rho = 0$, and $y = 1$ gives $\rho = 1$.

10-g2 (a) Let V be the volume of the can and ρ_A be the density of the air. The ratio of the two forces is $F_B/F_g = \rho_A V/(m_{\text{can}} + \rho_{\text{He}} V) = 1.3$. Since this ratio is greater than one, the can floats.

10-g3 Suppose you have a maximum lift force of F. The fact that you can lift a mass M of lead underwater tells us

$$\begin{aligned} Mg &= F + V_{\text{disp}}\rho_w g \\ Mg &= F + Mg(\rho_w/\rho_{\text{Pb}}) \\ Mg &= F + \frac{Mg}{11.3} \\ F &= 0.912 Mg \end{aligned}$$

When you replace the lead with aluminum, the maximum mass you can lift is M', where

$$\begin{aligned} M'g &= F + M'g(\rho_w/\rho_{\text{Al}}) \\ M'g &= F + \frac{M'g}{2.8} \\ M' &= 1.56(F/g) \\ M' &= 1.42M \end{aligned}$$

10-g4 Suppose the volume of the block is V. The mass of the block is $M = (k\rho_w)V$, and the fact that the block is fully submerged with a downwards force F gives us the equation

$$F + Mg = \rho_w V g$$

Plugging in $V = M/(k\rho_w)$,

$$\begin{aligned} F + Mg &= \frac{Mg}{k} \\ M &= \left(\frac{k}{1-k}\right)\frac{F}{g} \end{aligned}$$

10-g5 (a) $V = M/\rho_w$ gives $V = 0.062\ \text{m}^3$.
(b) $F_{\text{buoy}} = V\rho_{\text{air}} g = 0.73$ N.
(c) Gina would weigh an additional 0.73 N, which translates to a mass increase of 0.074 kg.

10-g6 The information in the problem statement tells us that the buoyant force is $(2/5)mg$. Since the buoyant force is $V_{\text{obj}}\rho_w g$, and since we can write $V_{\text{obj}} = m/\rho_{\text{obj}}$, we know $\rho_w/\rho_{\text{obj}} = 2/5$, or the density of the object is $2.5\rho_w$. If the object is placed in Bromine, it will float and a fraction $2.5/3.1 = 0.81$ of it will be submerged.

10-k1 This is closely analogous to example 1 in section 1.1.1. Since all three channels have the same width and depth, they all have the same cross-sectional area. If the speed of the current was the same in all three channels, then we'd have nonconservation of mass, because the rate at which mass was leaving the fork would be double that at which it entered. We'd be creating water out of nothing. The speed of the currents leaving the fork can't both be as high as the speed of the current entering. Based on the symmetry of the situation, it's probably a good approximation to expect that each of the two channels below the fork will have a current with half the speed seen above the fork.

10-k2 (a) The volume-flow-rate (volume per unit time) at this section of the river is $A_1v_1 = 2200\ \text{m}^3/\text{s}$.

(b) The mass-flow-rate (mass per unit time) at this section of the river is $\rho_w A_1 v_1 = 2.2\times10^6$ kg/s.

(c) Water is incompressible and satisfies conservation of volume-flow-rate. Therefore,

$$\begin{aligned} A_1v_1 &= A_2v_2 \\ v_2 &= 0.77\ \text{m/s} \end{aligned}$$

10-k3 The cross-sectional area is proportional to the square of the radius, so by continuity we have $a^2u = b^2v$, or $v = (a/b)^2u$.

10-k4 From Bernoulli's Equation,

$$\begin{aligned} P_0 &= P + (1/2)\rho v^2 \\ v &= \sqrt{\frac{2(P_0 - 0.75P_0)}{\rho_w}} = 7.1\ \text{m/s} \end{aligned}$$

11-a1 Newton's law of gravity depends on the inverse square of the distance, so if the two planets' masses had been equal, then the factor of $0.83/0.059 = 14$ in distance would have caused the force on planet c to be $14^2 = 2.0\times10^2$ times weaker. However, planet c's mass is 3.0 times greater, so the force on it is only smaller by a factor of $2.0\times10^2/3.0 = 65$.

11-a2 Kepler's law of periods says $T \propto R^{3/2}$, and solving for R gives $R \propto T^{2/3}$. Converting this into a statement about ratios gives $R_1/R_2 = (T_1/T_2)^{2/3} = 2.9$, or 3 if we round off to one sig fig.

11-d1 Newton's law of gravity is $F = GMm/r^2$. Both G and the astronaut's mass m are the same in the two situations, so $F \propto Mr^{-2}$. In terms of ratios, this is

$$\frac{F_c}{F_e} = \frac{M_c}{M_e}\left(\frac{r_c}{r_e}\right)^{-2} .$$

The result is 11 N.

11-d2 (a) The asteroid's mass depends on the cube of its radius, and for a given mass the surface gravity depends on r^{-2}. The result is that surface gravity is directly proportional to radius. Half the gravity means half the radius, or one eighth the mass. (b) To agree with a, Earth's mass would have to be 1/8 Jupiter's. We assumed spherical shapes and equal density. Both planets are at least roughly spherical, so the only way out of the contradiction is if Jupiter's density is significantly less than Earth's.

11-d3 (a) Since he's on the earth's surface, either $F = mg$ or $F = GMm/r^2$ will work. $mg = 590$ N.

(b) Same as in part a (Newton's third law).

(c) $F = Gm_{Laurie}m_{Roy}/r^2 = 1.2\times10^{-7}$ N.

(d) Her distance from the sun is essentially the same as the center-to-center distance between the sun and the earth, 1.5×10^{11} m, so $F = Gm_{sun}m_{Laurie}/r^2 = 0.38$ N.

11-d4 (a)

$$\begin{aligned} \frac{g_U}{g_E} &= \frac{m_U/r_U^2}{m_E/r_E^2} \\ &= \left(\frac{m_U}{m_E}\right)\left(\frac{r_E}{r_U}\right)^2 \\ &= 0.91 \end{aligned}$$

(b) It's surprising that Uranus's gravitational field is weaker than Earth's, since Uranus is a giant planet. However, "giant" means two things: large in size and large in mass. The large size means that if you were at the visible surface of Uranus (the cloud-tops), you'd be very far away from most of its mass. The exact result of this competition between mass and distance depends on how dense the planet it. Uranus is far less dense than Earth.

11-d5 It's tempting to say that the gravity on the space station is zero, since we see the astronauts floating around on TV. But that can't be true, because Newton's law of gravity implies $g \propto 1/r^2$, which could only be zero if r were infinite. (By the shell theorem, this holds as long as we're on or above the surface of the Earth.) The astronauts just seem weightless because they're orbiting the earth in the same orbit as the space station that surrounds them, so they move along with it. Let r be the radius of the Earth, which, by the shell theorem, is also the distance that appears in Newton's law of gravity for people living on the Earth's surface. Let primed variables refer to the space station, so that in particular r' is its distance from the center of the earth.

$$\begin{aligned} g' &= g\left(\frac{r}{r'}\right)^2 \\ &= g\left(\frac{r}{r+h}\right)^2 \\ &= g\left(\frac{1}{1+h/r}\right)^2 \qquad , \end{aligned}$$

where h is the space station's height above the earth's surface. Since h/r is small, the quantity in parentheses won't be too different from one, and g' won't be very different from g. Looking up r, we find g' to be about 8.8 m/s^2.

11-d6 (a) From conservation of energy, where i is the moment of release and f is the moment just before they collide,

$$\begin{aligned} \text{KE}_i + \text{PE}_i &= \text{KE}_f + \text{PE}_f \\ 0 - \frac{Gm^2}{4R} &= 2\left(\frac{1}{2}mv_f^2\right) - \frac{Gm^2}{2R} \\ v_f &= \sqrt{\frac{Gm}{4R}} \end{aligned}$$

(b) Assuming constant acceleration, $v_f = aT$, where v_f is given above, and $a = F_g/m = Gm/(4R)^2$. Solving this for T,

$$T = \frac{(4R)^2}{Gm}\sqrt{\frac{Gm}{4R}} = \sqrt{\frac{64R^3}{Gm}}$$

(c) Plugging in numbers gives $T = 10^4$ s (3 hours).

11-d7 The acceleration is $a = \omega^2 r = (2\pi/T)^2 r = 4.7 \times 10^{-5}$ m/s^2. If the astronaut stands on the inside of the outer bulkhead, the force pushing up on her feet is what produces her circular motion. This force is $F = ma = 2.8 \times 10^{-3}$ N. If she adopts the rotating frame, she interprets this "upward" force as canceling the "downward" fictituous force of apparent gravity. This fictitious force is roughly equivalent to the weight of a raisin.

11-d8 The relevant data are in the back of the book. Shortcut: for parts a, d and e, it's accurate enough to assume the distance between the moon and the sun equals the distance from the earth to the sun.
(a) sun on moon: -4.35×10^{20} N;
(b) earth on moon: 1.98×10^{20} N;
(c) sun on earth: -3.52×10^{22} N;
(d) total force on sun: $|\text{part a}| + |\text{part c}| = 3.56 \times 10^{22}$ N;
(e) total force on moon: $\text{a} + |\text{b}| = -2.37 \times 10^{20}$ N;
(f) total force on earth: $-|\text{b}| - |\text{c}| = -3.54 \times 10^{22}$ N

11-d9 The earth's force on the moon points up and to the right, and has magnitude 1.99×10^{20} N, as calculated in problem #2. The sun's distance from the moon is slightly different than in problem #2, but not enough to cause a significant change in the force, which points to the left, and has magnitude 4.36×10^{20} N.

For vector addition, we need to find the x and y components. Let x be to the right, and let y be up.

$$\begin{aligned} F_{sun,x} &= -4.36 \times 10^{20} \text{ N} \\ F_{earth,x} &= (1.99 \times 10^{20} \text{ N})\cos 45° \\ F_{total,x} &= -2.95 \times 10^{20} \text{ N} \\ F_{sun,y} &= 0 \\ F_{earth,y} &= (1.99 \times 10^{20} \text{ N})\sin 45° \\ F_{total,y} &= 1.41 \times 10^{20} \text{ N} \end{aligned}$$

The magnitude of the total force is $\sqrt{F_{total,x}^2 + F_{total,y}^2} = 3.27 \times 10^{20}$ N.

11-d10 Let the positive x axis be toward the sun, and the positive y axis be toward the top of the figure. The sun's force then has a positive x component and a zero y component. Using Newton's law of gravity, we find

$$F_{s,x} = 0.096 \text{ N}$$
$$F_{s,y} = 0 \qquad .$$

For Jupiter's force,

$$F_{J,x} = (GMm/r^2)\cos 95.2529^\circ = -1.015 \text{ N}$$
$$F_{J,y} = (GMm/r^2)\sin 95.2529^\circ = 11.044 \text{ N}$$

The sun's force is about a thousand times smaller than Jupiter's. That makes sense, because although the sun's mass is greater than Jupiter's, the sun is also hundreds of times farther away, and the $1/r^2$ in Newton's law of gravity means that it depends very strongly on distance. As a further check, we also observe that Jupiter's force has a small negative x component and a large positive y; this makes sense geometrically.

The total force has components

$$F_x = F_{s,x} + F_{J,x} = -0.919 \text{ N}$$
$$F_y = F_{s,y} + F_{J,y} = 11.044 \text{ N}$$

Its magnitude is 11.082 N, which is just slightly less than the 11.091 N magnitude of Jupiter's force alone. It makes sense that it's less. If the two forces had been in the same direction, the magnitude of the total would have been more than the magnitude of the individual forces. If they'd been perpendicular, the total would have been given by the Pythagorean theorem applied to the individual forces, and it would have been very slightly greater than Jupiter's force. In reality, the angle between the two forces is not 90 degrees, it's somewhat more than 90 degrees. This causes some cancellation in the x components.

11-g1 Any fractional change in r results in double that amount of fractional change in $1/r^2$. For example, raising r by 1% causes $1/r^2$ to go down by very nearly 2%. A 27-day orbit is 1/13.5 of a year, so the fractional change in $1/r^2$ is

$$2 \times \frac{(4/13.5) \text{ cm}}{3.84 \times 10^5 \text{ km}} \times \frac{1 \text{ km}}{10^5 \text{ cm}} = 1.5 \times 10^{-11}$$

11-g2 First we take Newton's law of gravity, $F = Gm_1m_2/r^2$, and throw out the constant factor of Gm_1m_2, turning it into the proportionality $F \propto 1/r^2$, or $r \propto 1/\sqrt{F}$. Changing this into a statement about ratios, we have $r_1/r_2 = \sqrt{F_2/F_1} = 10$. At the earth's surface, r equals the earth's radius. To make it ten times bigger, we have to put the rocket nine earth radii above the surface.

11-g3 Your path to Mars is half of an elliptical orbit. Kepler's third law says that that the time required for an elliptical orbit is proportional to its long axis to the power 3/2. It is convenient to work the problem in units of astronomical units (a.u.) and years, where 1 a.u. is the average distance from the earth to the sun. The long axis of our ellipse equals the sum of the radii of the earth's and Mars' orbits, which is 1.00 a.u.+1.53 a.u.=2.53 a.u. Comparing with the earth's orbit, which takes one year, we have

$$\frac{\text{time for half of your orbit}}{\text{time for half the earth's orbit}} = \left(\frac{\text{long axis of your orbit}}{\text{diameter of earth's orbit}}\right)^{3/2}$$

The quantities in the denominators are 0.5 years and 2.0 a.u., and solving for the unknown we find that the outward leg of the trip takes 0.70 years.

11-g4 Looking up the masses of the earth and moon in the back of the book, we find that the earth is 81.2 times more massive than the moon. The cancellation point must therefore lie much closer to the moon than to the earth. Not 81.2 times closer, though, because gravity depends on the square of the distance. The distance to the moon only has to be $\sqrt{81.2} = 9.01$ times smaller than the distance to the earth, i.e., almost exactly 9/10 of the way from the earth to the moon.

11-j1 Not if they are well above the planet's atmosphere, where there is no air friction. They do not need any force other than the planet's gravity to keep them in orbit, any more than our moon needs a power plant to keep it going around the earth. As Kepler figured out, the only possible orbits are circles, parabolas, ellipses and hyperbolas. No spirals! If, however, they were low enough to get some air friction, it would be possible for their orbit to decay in a spiral.

The people who wrote these plots were probably reasoning based on a kind of intuitive Aristotelianism: the incorrect belief that things "naturally" slow down if there is no forward force on them.

In the special case where the orbit is a circle (not an ellipse), we can also relate this to what we learned in chapter 9 about circular motion. Only an inward force is required in order to cause circular motion, and in this situation the inward force is being supplied by the planet's gravity. If the ship is initially moving at the right speed to be in a circular orbit, there is no reason why it would slow down, and "the right speed" means the speed at which the planet's gravity is just the right amount of inward force to make the ship go in a circle.

11-j2 Let M be the mass of the mysterious object, and m the mass of one of the stars orbiting around it. The relevant equations are Newton's second law,

$$a = \frac{F}{m} \qquad ,$$

Newton's law of gravity,

$$F = \frac{GMm}{r^2} \qquad ,$$

and the circular-motion equation

$$a = \frac{v^2}{r} \qquad .$$

The acceleration a isn't given, and we don't want to know it either, so let's eliminate it by setting the first and third equations equal to each other. We're now down to two equations:

$$\begin{aligned} \frac{F}{m} &= \frac{v^2}{r} \\ F &= \frac{GMm}{r^2} \end{aligned}$$

The force is also something we don't know and don't care about, so we'll get rid of it by substituting the second equation into the first:

$$\begin{aligned} \frac{GMm/r^2}{m} &= \frac{v^2}{r} \\ \frac{GM}{r^2} &= \frac{v^2}{r} \\ M &= \frac{v^2 r}{G} \\ &\sim 10^{36}\ \text{kg} \qquad , \end{aligned}$$

which is something like a million times the mass of the sun!

11-j3 (a) We're given the period, T, and we know that the condition for a circular orbit is

$$v = \sqrt{gr} \qquad .$$

The gravitational field is

$$g = GM/r^2 \qquad .$$

To get the given period, the speed of the satellite must be related to the radius of its orbit by $v =$ circumference$/T$, or

$$v = \frac{2\pi r}{T} \qquad .$$

We have three equations in three unknowns, v, g, and r. We don't care about v or g, so let's eliminate them first. Plugging the first equation into the third one, we eliminate v. The result is two equations in two unknowns:

$$\begin{aligned} g &= GM/r^2 \\ \sqrt{gr} &= \frac{2\pi r}{T} \end{aligned}$$

Now we substitute the first equation into the second one, giving one equation in one unknown:

$$\sqrt{\frac{GM}{r}} = \frac{2\pi r}{T}$$

Solving this for r, the result is

$$\begin{aligned} r &= \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} \\ &= 4.2 \times 10^4\ \text{km} \end{aligned}$$

This is the distance from the center of the earth, not the altitude. Subtracting the radius of the earth, we get an altitude of 3.6×10^4 km.

(b) The gravitational field at this altitude can either be calculated directly from $g = GM/r^2$ or, more easily, by using the inverse-square nature of the field and computing it by comparison with the field at the earth's surface:

$$\begin{aligned}\frac{g}{g_{surface}} &= \left(\frac{\text{radius of earth}}{\text{radius of orbit}}\right)^2 \\ &= \left(\frac{6.4 \times 10^3 \text{ km}}{4.2 \times 10^4 \text{ km}}\right)^2 \\ &= 0.023 \\ &= 2.3\%\end{aligned}$$

11-j4 (a) Let M be the mass of the whole planet, and m the mass of some object on the surface. Then

$$\begin{aligned}g &= F/m - v^2/r \\ &= (GMm/r^2)/m - (2\pi r/T)^2/r \\ &= GM/r^2 - 4\pi^2 r/T^2 \qquad .\end{aligned}$$

The mass, m, of the object has canceled out, which is good because otherwise we would not be able to solve the problem. Eliminating $M = \rho V = 4/3\pi r^3\rho$, we get

$$g = \frac{4}{3}\pi G r\rho - \frac{4\pi^2 r}{T^2} \qquad .$$

(b) Plugging in, the second term is 3.4×10^{-2} m/s^2. We know that the first term is 9.8 m/s^2, so the fractional change is 0.35%.

(c) $g = 0 = (4/3)\pi G r\rho - 4\pi^2 r/T^2$, so $T = \sqrt{\frac{3\pi}{G\rho}}$, which does not depend on r.

(d) Plug in.

(e) Solving for ρ, $\rho = 3\pi/GT^2 = 1.4 \times 10^{11}$ kg/m^3 for $T = 1$ s.

(f) Plugging $\rho = 10^{17}$ kg/m^3 in to the equation in part c, we get $T \approx 1$ ms. If they went faster, they'd fly apart.

11-j5 Newton's second law gives

$$F = m_D a_D,$$

where F is Ida's force on Dactyl. Using Newton's universal law of gravity, F= $Gm_I m_D/r^2$,and the equation $a = v^2/r$ for circular motion, we find

$$Gm_I m_D/r^2 = m_D v^2/r.$$

Dactyl's mass cancels out, giving

$$Gm_I/r^2 = v^2/r.$$

Dactyl's velocity equals the circumference of its orbit divided by the time for one orbit: $v = 2\pi r/T$. Inserting this in the above equation and solving for m_I, we find

$$m_I = \frac{4\pi^2 r^3}{GT^2},$$

so Ida's density is

$$\begin{aligned}\rho &= m_I/V \\ &= \frac{4\pi^2 r^3}{GVT^2}.\end{aligned}$$

11-j6 (a) Combining Newton's law of gravity $F = GMm/r^2$ with Newton's second law $a = F/m$ gives $a = GM/r^2$, and if the motion is to be circular, then this has to equal v^2/r, so

$$\frac{v^2}{r} = \frac{GM}{r^2} \qquad .$$

We then have $v = \sqrt{GM/r}$, and since the volume of a sphere is $(4/3)\pi r^3$, we have $M = (4/3)\pi r^3\rho$ and

$$v = r\sqrt{(4/3)\pi G\rho} \qquad .$$

(b) Solving for r and plugging in numbers gives about 3 km.

11-j7 (a) Differentiating twice, we find

$$\begin{aligned}v &= kpt^{p-1} \\ a &= kp(p-1)t^{p-2}\end{aligned}$$

This acceleration is given by Newton's second law:

$$\begin{aligned}a &= F/m \\ &= GMr^{-2}\end{aligned}$$

To relate the two expressions for the acceleration, we need to get them both in terms of the same variable, say r:

$$t = (r/k)^{1/p}$$
$$a = kp(p-1)(r/k)^{\frac{p-2}{p}}$$
$$= GMr^{-2}$$

If the powers of r are to be the same, then $(p-2)/p = -2$, or $p = 2/3$. Setting the constants out in front equal to each other, we find $k = (9GM/2)^{1/3}$.

(b) It approaches zero. If you shoot the bullet straight up at exactly escape velocity, then it slows down more and more, but never quite stops and falls back down.

(c) This is a little tricky, because we have an equation for v, but it's in terms of time. We want the initial velocity, i.e., the velocity at the earth's surface, so we need to find the time t_o at which the bullet's distance from the center of the earth is r_o, the same as the earth's radius. The initial time is *not* zero! We already have the relation between t and r, so with a little algebra we find

$$t_\text{o} = (r_\text{o}/k)^{3/2}$$
$$v_\text{o} = \frac{2k^{3/2}}{3r_\text{o}^{1/2}}$$
$$= 1.1 \times 10^4 \text{ m/s} \qquad .$$

11-j8 (a) The analysis of forces is qualitatively the same in both cases. There is an upward force normal from the scale on the astronaut, and a downward gravitational force from the planet. At the north pole, the astronaut is not accelerating. The total force is zero, and the weight (mg_X) is equal to the normal force read off the scale (600.0 N). At the equator, the astronaut is moving in a great circle about the center of the planet, and therefore the total force acting on them must be down (towards the center of the circle). The weight (equal to mg_X) is no different than at the north pole because the planet is perfectly spherical; however, the normal force (i.e., the reading on the scale) is less, so that the total force is down.

(b) The mass of the astronaut is $m = (980.0 \text{ N})/(9.80 \text{ m/s}^2) = 100$ kg. Therefore, the acceleration due to gravity on planet X is $g_X = (600.0 \text{ N})/(100 \text{ kg}) = 6.00 \text{ m/s}^2$. Also, we're told in the statement of the problem that the distance from the north pole to the equator is 20,000 km, which is one-fourth of the circumference of the planet. Therefore, $R_X = 4(20,000 \text{ km})/(2\pi) = 1.27 \times 10^7$ m. We can use this with a general expression for g in terms of the mass of a spherical body to solve for M_X:

$$g_X = \frac{GM_X}{R_X^2}$$

This gives $M_X = 1.45 \times 10^{25}$ kg.

(c) From Kepler's law of periods (with $r = R$ since the satellite is very close to the surface),

$$T = \sqrt{\frac{(2\pi)^2R^3}{GM}} = 9150 \text{ s.}$$

(d) We have yet to use the fact that the astronaut weighs 500.0 N on the equator; this fact can be used to find the angular velocity of the planet:

$$600.0 \text{ N} - 500.0 \text{ N} = 100.0 \text{ N} = mR_X\omega^2.$$

Plugging in $m = 100.0$ kg for the astronaut, we get $\omega = 2.8 \times 10^{-4} \text{ s}^{-1}$, which corresponds to a period $T_X = 2\pi/\omega = 22,400$ s. Therefore, a day on planet X is longer than the time it takes a satellite to orbit near the planet's surface (despite the relatively large angular velocity of the planet).

11-j9 (a) The information from the problem about the satellite can me used to find the mass of the planet, using Kepler's law of periods:

$$T_Z = \sqrt{\frac{(2\pi)^2r^3}{GM_Z}} M_Z = \sqrt{\frac{(2\pi)^2r^3}{GT^2}} = 4.06 \times 10^{24} \text{ kg}$$

Alternatively, you can figure out that the satellite moves at speed $v = (2\pi r)/T = 5820$ m/s, and use

$$F_g = \frac{mv^2}{r} \implies \frac{GM_Xm}{r^2} = \frac{mv^2}{r}$$

$$M_X = \frac{rv^2}{G} = 4.06 \times 10^{24}\ \text{kg}$$

(b) $GM_Z/R_Z^2 = g_Z = 8.00\ \text{m/s}^2$

$$R_Z = \sqrt{\frac{(6.67 \times 10^{-11})(4.06 \times 10^{24})}{8.00}}\ \text{m} = 5.82 \times 10^6\ \text{m}$$

(c) KE $= (1/2)mv^2 = 3.38 \times 10^8$ J.
PE $= -GMm/r = 6.77 \times 10^8$ J.
PE/KE $= -2$ (this is always true for bound orbits with $U \propto -1/r$).
(d) Using Kepler's law of periods,

$$r = \left(\frac{GMT^2}{(2\pi)^2}\right)^{1/3} = 2.74 \times 10^8\ \text{m}.$$

11-m1 The sun's gravitational field is

$$g = \frac{GM}{r^2} \qquad ,$$

where M is the sun's mass. For the Earth's circular orbit, we have

$$v = \sqrt{gr} = \frac{2\pi r}{T} \qquad ,$$

where r is one astronomical unit, and T is one year. Eliminating g and solving for G, we find

$$G = \frac{4\pi^2 r^3}{MT^2} \qquad ,$$

which equals $4\pi^2$ in this system of units, since r, M, and T are all defined as 1.

11-m2 (a) Yes. The two masses are treated in a symmetric manner in the equation. It doesn't matter which one you label 1 and which one you label 2. (b) The force on the greater mass would be greater, but not greater in proportion to its mass. For example, if the more massive object had four times the mass, the force on it would only be double. According to $a = F/m$, a heavier object would therefore fall with a smaller acceleration. (c) The raindrop's mass is less by a factor of 10000, but the force on it would be smaller by a factor of only 100, so its acceleration would be 100 times greater, i.e. 1000 m/s^2. Raindrops falling from any significant height would be as deadly as bullets! (d) Both halves would suddenly start accelerating more rapidly, and the lighter one would pull away from the heavier one. (e) Instead of raising both masses to the 1/2 power, we'd have to raise them to some power greater than 1, e.g. make the force proportional to the product of the cubes of the masses.

11-m3 (a) The cloud's mass M is proportional to r^3. The gravitational force acting on an atom at the cloud's outskirts is proportional to M/r^2, or to r. The acceleration is therefore proportional to r to the first power. (b) Although the acceleration is not constant, we can approximate it as being constant for the purposes of comparing one cloud with another of a different size. For constant acceleration, the distance traveled is proportional to at^2, so the time required to collapse to some fraction of the cloud's original size is proportional to $\sqrt{r/a}$. But since a is proportional to r, the result is independent of r.

11-m4 S should have axial symmetry, for if not, then some mass could be moved from a location farther from the line of the vector $\mathbf{g}_S$ to a location closer to that line, and also closer to the point of maximum field strength. This would increase both the magnitude of its contribution to the field and the component of that contribution along the line.

Now that we've established axial symmetry, consider a particular thin ring of mass lying at the surface of S and in a plane perpendicular to the axis. This has some mass $\mathrm{d}m$. Take the point of maximum field to be the origin of a system of spherical coordinates, in which r is the distance from the origin and θ is the angle of the $\mathbf{r}$ vector away from the axis of symmetry. The ring is a curve of constant r and θ. Its contribution to the total field is $(G\,\mathrm{d}m/r^2)\cos\theta$. Since S is the optimal shape, moving this ring to some other point on the surface cannot improve the result. Therefore the surface has an equation of the form $r^2 \propto \cos\theta$. This turns out to be an egg shape with its flatter end at the point of maximum field.

11-m5 (a) $E_\text{tot} = 0 = \text{KE} + \text{PE}$:

$$\begin{aligned} 0 &= \frac{1}{2}mc^2 - \frac{GMm}{R_s} \\ R_s &= \frac{2GM}{c^2} \end{aligned}$$

(b) $M_\text{Earth} \approx 6\times 10^{24}$ kg and $M_\text{Sun} \approx 2\times 10^{30}$ kg give Schwarzschild radii 9 mm and 3 km, respectively.

11-m6 (a) Based on units, we must have $g = kG\lambda/y$, where k is a unitless universal constant. (b) For the actual calculation, we have

$$\begin{aligned} g &= \int \text{d}g_y \\ &= G\int \frac{\text{d}m}{r^2}\cos\theta, \end{aligned}$$

where θ is the angle between the perpendicular and the $\mathbf{r}$ vector. Then $\text{d}m = \lambda\,\text{d}x$, $\cos\theta = y/r$, and $r = \sqrt{x^2+y^2}$, so

$$\begin{aligned} g &= G\int_{-\infty}^{\infty} \frac{\lambda\,\text{d}x}{x^2+y^2}\cdot\frac{b}{\sqrt{x^2+y^2}} \\ &= G\lambda y\int_{-\infty}^{\infty}(x^2+y^2)^{-3/2}\,\text{d}x. \end{aligned}$$

Even though this has limits of integration, this is an indefinite integral because it contains the variable y. It's nicer to clean this up by doing a change of variable to the unitless quantity $u = x/y$, giving

$$g = \frac{G\lambda}{y}\int_{-\infty}^{\infty}(u^2+1)^{-3/2}\,\text{d}u.$$

The definite integral is the sort of thing that sane people these days will do using computer software. It equals 2. The result for the field is

$$g = \frac{2G\lambda}{y}.$$

11-m7 For the limit $g_1 = \lim_{y\to 0}\lim_{b\to 0} g$, we know that the inside limit is proportional to $1/y$ from the result of the previous problem. Therefore $g_1 = \infty$.

In the case of $g_2 = \lim_{b\to 0}\lim_{y\to 0} g$, the inside limit is zero by symmetry for all nonzero values of b. Thus $g_2 = 0$.

We find that the two ways of defining the field on the filament do not agree, and therefore there is no unambiguous way of saying what is the truth of the matter. The notion of the field at a point on such a filament is an idealization, and the idealization does not cleanly map onto actual observables.

11-m8 Let the filament be an arc of the unit circle, and let P be the point $(1,0)$. Then the distance r between P and the point at angle θ on the unit circle is given by $r^2 = (1-\cos\theta)^2 + \sin^2\theta = 2(1-\cos\theta)$. The infinitesimal arc from θ to $\theta+\text{d}\theta$ has mass proportional to $\text{d}\theta$ and exerts a force $\text{d}F$ on the mass element at P that is proportional to $\text{d}\theta r^{-2}$. The x component of this force is $\sin(\theta/2)\,\text{d}F$. The resulting force in the x direction is, ignoring positive constant factors,

$$\begin{aligned} F_x &= \int_{\theta_1}^{\theta_2} -\sin(\theta/2)\,\text{d}F \\ &= \int_{\theta_1}^{\theta_2} -\frac{\sin(\theta/2)\,\text{d}\theta}{1-\cos\theta}. \end{aligned}$$

Near $\theta = 0$, small-angle approximations give $\sin(\theta/2) \approx \theta/2$, and $1-\cos\theta \approx \theta^2/2$, so the integrand is proportional to $1/\theta$. Therefore the improper integral diverges logarithmically at $\theta = 0$. The integrand is negative everywhere, so the integral diverges to $F_x = -\infty$, i.e., the force acting on the element of mass at P is infinite and to the left (in the direction of concavity).

Based on this result, it would appear that such a filament would behave as if it had infinite stiffness with respect to being bent away from straightness. This result, however, is not necessarily physically realistic, for the reasons explored in the precednig problem.

11-p1 Above you there is a small part of the shell, comprising only a tiny fraction of the earth's mass. This part pulls you up, while the whole remainder of the shell pulls you down. However, the part above you is extremely close, so it makes sense that its force on you would be far out of proportion to its small mass.

11-p2 Let Newton's law of gravity have an exponent r^{-2+c}, where $c \neq 0$. Let's continue to define A using the exponent s^{-2}. For the case calculated explicitly in the book, where m is outside the shell, the integral becomes

$$\begin{aligned} F &= \left(\frac{1}{4b}\right) A \int_{s-b}^{s+b} \frac{\mathrm{d}r}{r^{1-c}} \left(r + \frac{s^2-b^2}{r}\right) \\ &= \left(\frac{1}{4b}\right) A r^c \left[\frac{r}{1+c} + \frac{s^2-b^2}{-1+c} r^{-1}\right]_{s-b}^{s+b} \end{aligned}$$

The factor of Ar^c is what this entire expression is supposed to boil down to according to the shell theorem. To prove that the theorem fails, it suffices to show that it fails for some specific choice of s and b. Let's take $s = 2b$. Then the expression in square brackets comes out to be $2b[1/(1+c) + 1/(1-c)]$. For the shell theorem to hold, we would need this to equal $4b$, but if we require this and solve for c, we find that the only solution is $c = 0$.

11-p3 In the original derivation for the r^{-2} version of the force law, we have

$$F = \frac{A}{4b} \int_{s-b}^{s+b} \frac{r\,\mathrm{d}r}{r^2} \left(r + \frac{s^2-b^2}{r}\right) \qquad ,$$

where $A = GMm/s^2$ and the r^2 in the denominator comes from the force law. To handle the r^1 case, we just change that exponent, obtaining the integral

$$F = \frac{A}{4b} \int_{s-b}^{s+b} r\,\mathrm{d}r \cdot r \left(r + \frac{s^2-b^2}{r}\right) \qquad .$$

Evaluating the integral gives $F = (A/16b)I$, where $I = Q^4 - P^4 + 2PQ(Q^2 - P^2)$, $Q = s + b$, and $P = s - b$. Expanding I by brute force makes a bit of a mess. We can simplify things a litle by observing that flipping the sign of b corresponds to interchanging the limits of integration, which should flip the sign of the result; therefore all even powers of b must cancel, and we can skip evaluating them. The result turns out to be $I = 16s^3b$, which gives $F = A$ as claimed.

11-p4 As the probe enters the interior of B, the shell theorem says that the probe feels zero force from B, but feels the same force from A that it would have felt if A's mass had been concentrated at its center. From this we can predict that the probe will curve to the right and speed up. Furthermore, since the probe's initial velocity is small, we can tell that this segment of the probe's orbit will be an ellipse, not a parabola or hyperbola.

The probe will then enter the region where the two clouds' interiors overlap. In this region, the shell theorem tells us that the probe feels no force from either cloud. It moves in a straight line at constant speed.

It will then emerge into the region that is interior to A but exterior to B. It now feels a force from B but none from A. The force from B is now probably more or less directly backward, which will decelerate the probe somewhat, but without more detailed data there is not much more we can say beyond this point.

11-p5 (a) If we divide the earth up into a central sphere that's deeper than P and an outer part that's shallower than P, then the shell theorem tells us that the outer part doesn't contribute anything to the field at P, and that the contribution of the inner part is the same as if it had been concentrated at the center. Because mass is proportional to volume, and volume is proportional to the cube of the linear dimensions, the mass of the inner part is 1/8 the mass of the earth. If this was the only factor, then we would have $g_P/g_S = 1/8$. But P is also closer to the center than S by a factor of 2, and since Newton's law of gravity depends on $1/r^2$, this brings the result back up by a factor of 4. The result is $g_P/g_S = 1/2$.

(b) Generalizing the result of part a, we have $g/g_S = r/r_S$. At $r = 0$, this produces the correct result, $g = 0$, because the field must cancel out by symmetry at the earth's center. At $r = r_S$, it gives $g = g_S$, which also makes sense.

11-p6 By the shell theorem, the mantle gives no contribution, and we can take the entire mass of the core (including both its layers) to be concentrated at the center. Let m be the mass of the core and r its outer radius.

$$\begin{aligned}\frac{g_G}{g_s} &= \frac{m}{M}\cdot\left(\frac{R}{r}\right)^2\\ &= (0.307)(1/0.55)^2\\ &= 1.01\end{aligned}$$

Surprisingly, it's almost the same as the field at the surface.

11-s1 To apply conservation of energy, we start by writing down the law in its most generic form,

$$E_{total,i} = E_{total,f} \qquad ,$$

and we then make it more specific by putting in the actual forms of energy involved,

$$KE_i + PE_i = KE_f + PE_f \qquad ,$$

and then substituting in the correct expressions for the kinetic and potential energies,

$$\frac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f} \qquad ,$$

Now we have to decide what moments in time will be initial and final. Initial should clearly be the beginning of the motion, when the object is at distance r, so in terms of the variables used in posing the question, we have $r_i = r$ and $v_i = v$, the escape velocity, giving

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f} \qquad .$$

But what should we choose as the final moment in time? We're interested in the case where the object just barely has enough energy to keep from falling back. If the object had less than this amount of energy, it would fly away from the planet, slow down, and finally stop at some distance before falling back down. The greater the energy, the greater the distance at which it stops. When the energy is just barely enough (the velocity is exactly equal to escape velocity), we can think of this distance as being infinite, or at least extremely large. Thus, $r_f = \infty$ and $v_f = 0$. Dividing by a very large number gives a very small result, so dividing by infinity gives zero. That means that both terms on the right are zero. In other words, the total energy is zero.

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0 \qquad ,$$

It may sound strange to say that this object, moving at a gazillion miles an hour, has zero energy, but that's because the object has a huge negative potential energy to start with, which is canceled by its huge initial kinetic energy.

Solving for the escape velocity v, we have

$$v = \sqrt{\frac{2GM}{r}} \qquad .$$

The direction of the velocity doesn't matter, because the only requirement was for the object to have a certain amount of energy, and energy is a scalar. This is counterintuitive: most people would expect that it would be easier to make the object escape if it was shot straight up. One way to think about it is that if the object is launched at an angle just barely above the horizontal, it has the disadvantage that at the beginning, it's hardly getting any farther away from the earth, but it has the advantage that the force of gravity is almost perpendicular to its direction of motion, so gravity is doing almost no (negative) work on it, and is therefore hardly slowing it down.

The mass of the object has canceled out, which is why we can speak of a single escape velocity, rather than different escape velocities for different objects.

No, it is not valid to think of the object as escaping because it is moving too fast for gravity to act on it. The force of gravity only depends on the masses and the distance between them, not on their motion.

11-s2

$$\begin{aligned}E_{total,i} &= E_{total,f}\\ PE_i &= KE_f + PE_f\end{aligned}$$

We can use the equation $PE = -GMm/r$ for the potential energy, and this equation is defined so that $PE = 0$ at $r = \infty$, so actually PE_i is zero. The potential energy is really the sum of the potential energies for the interaction of the comet with the two stars. Writing v for the final velocity, we have

$$0 = \frac{1}{2}mv^2 - \frac{Gm_1m}{b/2} - \frac{Gm_2m}{b/2}$$
$$v = \sqrt{\frac{4G(m_1+m_2)}{b}}$$

12-a1 0.07 to 0.10 s $\left(\frac{1}{15\text{ Hz}}\text{ to}\frac{1}{10\text{ Hz}}\right)$

12-a2 Sixteen periods take about 8.8 ms, so one period is 0.55 ms. The frequency is 1/0.55 ms=1.82 kHz. The angular frequency is $(1.82\text{ kHz})(2\pi) = 1.1\times10^4$ rad/s.

12-d1

The 2π is unitless. The rest of the right-hand side has units of $[\text{kg}/(\text{N/m})]^{1/2} = [\text{kg}\cdot\text{m}/(\text{kg}\cdot\text{m/s}^2)]^{-1/2} = \text{s}$, which matches the units on the left.

12-d2 (a) Pendulum 2 has then same total force on it when it is at the same angle, but when the two pendula are at the same angle, pendulum 2 is at twice the distance from equilibrium. Pendulum 2's $F-x$ graph will therefore look like pendulum 1's, but stretched out twice as wide. Since k is the slope of this graph, pendulum 2's k is half as much.
(b) $T = 2\pi\sqrt{m/k}$, so if pendulum 2's k is half as much, its period is greater by a factor of $\sqrt{2}$.

12-d3 The spring constant is found by differentiating the force and evaluating the result at the equilibrium position x_o.

$$\begin{aligned}F &= ax^{-\beta}\\ k &= -\left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x_\text{o}}\\ &= a\beta x_\text{o}^{-\beta-1}\big|_{x_\text{o}}\end{aligned}$$

We can find x_o from the condition that the forces cancel.

$$\begin{aligned}mg &= ax_\text{o}^{-\beta}\\ x_\text{o} &= (mg/a)^{-1/\beta}\\ k &= a\beta(mg/a)^{1+1/\beta}\\ \omega &= \sqrt{k/m}\\ &= \left[(a/m)\beta(mg/a)^{1+1/\beta}\right]^{1/2}\\ &= \beta^{1/2}g^{1/2+1/2\beta}\left(\frac{m}{a}\right)^{1/2\beta}\end{aligned}$$

12-d4 We have

$$x = A\sin(\omega t+\delta),$$

and differentiating this gives

$$v = A\omega\cos(\omega t+\delta).$$

The total energy is

$$\begin{aligned}PE+KE &= \frac{1}{2}kx^2+\frac{1}{2}mv^2\\ &= \frac{1}{2}A^2(k\sin^2\ldots+m\omega^2\cos^2\ldots),\end{aligned}$$

and, using $\omega = \sqrt{k/m}$, this becomes

$$\begin{aligned}PE+KE &= \frac{1}{2}A^2(k\sin^2\ldots+k\cos^2\ldots)\\ &= \frac{1}{2}A^2k,\end{aligned}$$

which is a constant, as required by conservation of energy.

12-d5 (a) The volume of the displaced water is $h_\text{o}b^2$, so its mass is $h_\text{o}b^2\rho$, and its weight is $h_\text{o}b^2\rho g$. Setting this equal to mg gives $h_\text{o} = m/b^2\rho$.
(b) A calculation similar to the one in part a gives $hb^2\rho g$ for the weight of the displaced water. Letting positive forces be upward, we have $F = hb^2\rho g - mg$.
(c) The equation $F = hb^2\rho g - mg$ is a line of slope $b^2\rho g$ when F is graphed against h, so the

"spring constant" is $k = b^2\rho g$. The period is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} \\ &= 2\pi\sqrt{\frac{m}{b^2\rho g}} \\ &= 2\pi\sqrt{\frac{h_\text{o}}{g}} \end{aligned}$$

12-d6 $T \propto k^{-1/2} \propto g^{-1/2}$, so $g \propto T^{-2}$. Therefore a fractional change in T causes twice the fractional change in g. For example, a 1% increase in T will cause a 2% decrease in the inferred value of g. In symbols, $\Delta g/g = 2\Delta T/T$. The ΔT that would cause a $\Delta g = .024\ \text{m/s}^2$ is about 1 ms.

12-d7 The result from problem 33 was $h = mg/k$. The constant-acceleration equation $h = (1/2)gT_{fall}^2$ results in

$$T_{fall} = \sqrt{\frac{2h}{g}} \qquad .$$

The total interaction energy is $(1/2)ky^2 + mgy$, and its second derivative is k, just as it would have been without the gravitational term. The interpretation is that hanging the mass vertically from the spring merely shifts the equilibrium position, without changing the period of the oscillations, which is still

$$T_{osc} = 2\pi\sqrt{\frac{m}{k}} \qquad .$$

The ratio of the two times is

$$\begin{aligned} \frac{T_{osc}}{T_{fall}} &= 2\pi\sqrt{\frac{m}{k}\cdot\frac{g}{2h}} \\ &= 2\pi\sqrt{\frac{m}{k}\cdot\frac{g}{2(mg/k)}} \\ &= \pi\sqrt{2} \\ &\approx 4.45 \qquad . \end{aligned}$$

Is this reasonable? In the case where the mass is falling, the distance h traveled is the same as the distance that it would have traveled in 1/4 of a cycle of oscillation. Roughly, then, we might expect the result to be about 4, and that is about what it is. The reason it's a little more than 4 is that in the free-fall case, the force acting on the mass is constant, whereas in the oscillating case, the force starts out being just as strong as gravity, but weakens and becomes zero at the end of a quarter cycle. Since the force in the oscillating case is always less than or equal to the force in the free-falling case, this lengthens the time for the oscillation, making the ratio slightly higher than the crude estimate of 4.

12-d8 (a) The angular frequency can be read directly off the position-vs-time function; it's the number multiplying t. Thus, $\omega = 25\ \text{s}^{-1}$. Also, $f = \omega/(2\pi) = 4.0$ cycles/s and $T = 2\pi/\omega = 0.25$ s.

(b) $v(t) = dx(t)/dt$:

$$\begin{aligned} v(t) &= -(35\ \text{cm})(25\ \text{s}^{-1})\sin[(25\ \text{s}^{-1})t + \pi] \\ v(0) &= -(35\ \text{cm})(25\ \text{s}^{-1})\sin[\pi] = 0 \end{aligned}$$

The initial velocity is zero. This makes sense because $x(0) = A\cos(\pi) = -A$, which is a turning point of the motion. The maximum speed of the object is $(35\ \text{cm})(25\ \text{s}^{-1}) = 8.8$ m/s.

(c) $a(t) = dv(t)/dt$:

$$\begin{aligned} a(t) &= -(35\ \text{cm})(25\ \text{s}^{-1})^2\cos[(25\ \text{s}^{-1})t + \pi] \\ a(0) &= -(35\ \text{cm})(25\ \text{s}^{-1})^2\cos[\pi] = 220\ \text{m/s}^2 \end{aligned}$$

This is also the maximum acceleration of the object. (d) At $t = 1.00$ s,

$$\begin{aligned} x(1.00\ \text{s}) &= (35\ \text{cm})\cos[(25\ \text{s}^{-1})(1.00\ \text{s}) + \pi] \\ &= -35\ \text{cm} \\ v(1.00\ \text{s}) &= -(8.75\ \text{m/s})\sin[(25\ \text{s}^{-1})(1.00\ \text{s}) + \pi] \\ &= -1.2\ \text{m/s} \end{aligned}$$

Note that, at $t = 1.00$ s, the object is very close to the turning point, but not exactly (which is why the position rounds to $-35\ \text{cm} \approx -A$ despite $v \neq 0$).

12-d9 (a) The period of oscillations can be read off the graph: the time from peak-to-peak

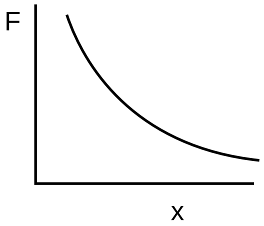
is 0.80 s, which also gives us $\omega = 2\pi/T = 5\pi/2 \approx 7.85\ \text{s}^{-1}$. Using $\omega = \sqrt{k/m}$ and k from the problem statement, we know the mass of the glider is $m = 2.4$ kg.
(b) $a(t) = -\omega^2 x(t)$, which means $a_{\text{max}} = A\omega^2$. Plugging in 12.0 m/s^2 for a_{max} and ω from part (a), we get $A = 0.19$ m.
(c) Since PE $\propto x^2$, when the glider is at $x = \pm A/3$, then the PE is one-ninth its maximum value (and the maximum value is the total energy). Also, the KE must be eight-ninths of the total energy.

$$\begin{aligned} E_{\text{tot}} &= (1/2)kA^2 = 2.71\ \text{J} \\ \text{PE} &= (1/9)E_{\text{tot}} = 0.30\ \text{J} \\ \text{KE} &= (8/9)E_{\text{tot}} = 2.41\ \text{J} \end{aligned}$$

(d) We know that, at $t = 0.10$ s, the acceleration is maximal and positive (which corresponds to $x = -A$). Thus

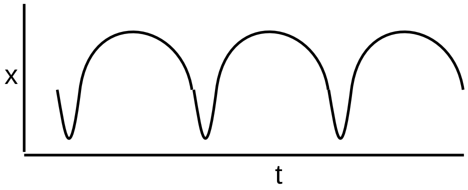
$$\begin{aligned} x(0.10\ \text{s}) = -A &= A\cos(\omega t + \phi) \\ -1 &= \cos[(7.85\ \text{s}^{-1})(0.10\ \text{s}) + \phi] \\ \phi &= 135^\circ = \frac{3\pi}{4} \end{aligned}$$

12-g1 The $F - x$ graph will look like this:



(Either graph a few points, or use your general math experience.) The graph in the previous problem only covered a small portion of this range, so it looked linear.

We only get simple harmonic motion (sinusoidal motion) when the $F - x$ graph is linear, so for large vibrations, where the $F - x$ graph can no longer be considered approximately linear, there will not be simple harmonic motion. Using the hint, we find that the motion must look something like this:



12-g2 This is similar to the spring-and-lever example in section 16.2. Let x be the vertical position of one of the kids relative to equilibrium. The distortion of each spring is $(b/a)x$, so the total energy in the two springs is $PE = (2)(1/2)k[(b/a)x]^2$. The second derivative of this quantity gives the effective spring constant $k' = 2k(b/a)^2$. The moving mass is $m' = 2m$, so $\omega = \sqrt{k'/m'} = (b/a)\sqrt{k/m}$. (We have canceling factors of 2 due to the two kids and the two springs.) The resulting period is $T = 2\pi/\omega = 2\pi(a/b)\sqrt{m/k}$.
12-g3 Let L be the amount by which the spring has been stretched or compressed with respect to equilibrium, so that its energy is $U_{spring} = (1/2)kL^2$. Let y be the height of the mass, and for convenience, let's define $y = 0$ to be the height at which the spring is relaxed, so that $L = 0$ coincides with $y = 0$. If the mass goes down by a certain amount, the spring is stretched by half that amount, so $L = y/2$. (We don't care about positive or negative signs of L, since only L^2 occurs in the spring's energy.) The total interaction energy of the system is then

$$U = mgy + \frac{1}{8}ky^2 \qquad .$$

The effective spring constant experienced by the mass is not really k, but rather

$$\begin{aligned} k' &= \frac{d^2}{dy^2}U \\ &= \frac{d^2}{dy^2}\left(mgy + \frac{1}{8}ky^2\right) \\ &= \frac{1}{4}k \end{aligned}$$

In other words, this arrangement makes the spring appear to be four times less stiff than if the weight had just been hung directly from it. The period is $T = 2\pi\sqrt{m/k'} = 4\pi\sqrt{m/k}$.

12-g4 (a)

$$\begin{aligned}U &= 2U_{one\ spring}\\ &= (2)(\frac{1}{2}k(L-b)^2)\\ &= k(L-b)^2\\ &= k(\sqrt{b^2+y^2}-b)^2\end{aligned}$$

(b) The effective spring constant is the second derivative of this quantity with respect to y, evaluated at $y=0$.

$$\begin{aligned}\frac{d^2U}{dy^2} &= \frac{d^2}{dy^2}k\left(\sqrt{b^2+y^2}-b\right)^2\\ &= \frac{d}{dy}\left[2k\left(\sqrt{b^2+y^2}-b\right)\frac{1}{2}\left(b^2+y^2\right)^{-1/2}(2y)\right]\\ &= \frac{d}{dy}2ky\left[1-b(b^2+y^2)^{-1/2}\right]\\ &= 2k\left[1-b(b^2+y^2)^{-1/2}\right]+2ky\left[\ldots\right] \qquad ,\end{aligned}$$

where the ...indicates stuff we don't need to evaluate, because it won't contribute to the derivative evaluated at $y=0$. The result is zero, which implies that the period of small oscillations is infinite. This is a loophole in the text's statement that small oscillations are *always* simple harmonic. If you were to start this system doing large oscillations, and then let them die out slowly, their period would get longer and longer.

12-g5 (a) The definition of k for motion in one dimension is $k = d^2U/dx^2|_0$. Translating the linear quantities x and k into the angular ones θ and κ, this becomes $\kappa = d^2U/d\theta^2|_0 = Bm$.

(b) Translating k and m to κ and I, $\omega = \sqrt{k/m}$ becomes $\omega = \sqrt{\kappa/I} = \sqrt{Bm/I}$.

12-j1 I did this by modifying the program from section 4.6.

```
import math
def f(k,b,x):
  return -k*x-b*x*x*x
def period(m,k,b,amplitude,n):
  print("A=",amplitude,"  b=",b)
  x = amplitude
  v = 0
  w0 = math.sqrt(k/m) # frequency it would have if 
  t0 = 2.*3.1416/w0  # ...and the corresponding per
  t_max = t0*10.     # guess the longest time it co
  dt = t_max/n       # Divide t into n equal parts.
  t = 0.
  for i in range(n):
    t = t+dt
    dx = v*dt
    x = x+dx         # Change x.
    a = f(k,b,x)/m
    dv = a*dt
    v = v+dv
    if v>0.:
      return 2.*t
  return -999.       # If we get here, t_max was to
```

Let's try it:

```
1  $ python -i simulate-anharmonic-force.py
2  >>> print(period(1.,1.,0.,1.,1000))
3  A= 1.0   b= 0.0
4  6.283200000000008
5  >>> print(period(1.,1.,0.01,1.,1000))
6  A= 1.0   b= 0.01
7  6.283200000000008
8  >>> print(period(1.,1.,0.01,10.,1000))
9  A= 10.0   b= 0.01
10 4.775232000000003
```

On lines 2-5, we run the program with $b=0$, and get the same result of exactly 2π that we had in section 2.5. On lines 6-9, we try a small value of b, and we observe that there is a slight decrease in the period, which makes sense, because the spring is stronger. On lines 10-12, a bigger amplitude produces a much smaller period. This is not the simple (and counterintuitive) behavior we observed when the spring's energy only had an x^2 term. In the pure x^2 case, the spring's greater energy at large amplitudes was exactly enough to compensate for the greater distance the mass needed to travel, causing it to cover that distance in the same time. The x^4 function, however, has a more severe dependence on

x: it starts off flat, but then blows up like crazy when x gets big. The increased stiffness at large amplitudes does more than just compensate for the extra distance of travel, it actually makes the mass travel the *greater* distance in a *smaller* time.

12-k1 (a) From Conservation of Energy, where i is the point of release (from rest) and f is the equilibrium point (the point where the bob moves with maximum speed),

$$\begin{aligned} \mathrm{KE}_i + \mathrm{PE}_i &= \mathrm{KE}_f + \mathrm{PE}_f \\ 0 + mgL(1-\cos\theta) &= \frac{1}{2}mv^2 + 0 \\ v &= \sqrt{2gL(1-\cos\theta)} \end{aligned}$$

(b) $A = L\theta$ (arclength formula), and $\omega = \sqrt{g/L}$. Therefore,

$$|v_{\max}| = A\omega = L\theta\sqrt{g/L} = \theta\sqrt{gL}$$

Now let's look at the exact expression from part (a) in the limit $\theta \ll 1$. For small θ, $\cos\theta \approx 1 - (1/2)\theta^2$. Plugging this into the exact expression,

$$v = \sqrt{2gL[1-(1-\theta^2/2)]} = \sqrt{2gL[\theta^2/2]} = \theta\sqrt{gL}$$

This agrees with the result obtained before.

12-k2 A pendulum has period $T = 2\pi\sqrt{L/g}$, which means $g \propto L/T^2$. If L doubles and T quadruples, then g goes down by a factor of 8. Therefore, $g_{\mathrm{W}} = (1/8)g_{\mathrm{Earth}} = 1.2\ \mathrm{m/s^2}$.

12-k3 For a physical pendulum,

$$\omega = \sqrt{\frac{mgd}{I_{\mathrm{pivot}}}},$$

where d is the distance from the pivot to the center-of-mass, and I_{pivot} is the moment of inertia of the object about the pivot. In this problem, $d = L/2$, and $I_{\mathrm{pivot}} = (1/3)mL^2$ (the moment of inertia of a rod about one end). Plugging these into the above equation gives $\omega = \sqrt{3g/(2L)}$, or $T = 2\pi\sqrt{2L/(3g)}$.

12-k4 For a physical pendulum,

$$\omega = \sqrt{\frac{mgd}{I_{\mathrm{pivot}}}},$$

where d is the distance from the pivot to the center-of-mass, and I_{pivot} is the moment of inertia of the object about the pivot. In this problem, $d = L$, and $I_{\mathrm{pivot}} = (2/5)m(xL)^2 + mL^2 = mL^2[1 + (2/5)x^2]$ (from the parallel-axis theorem). Plugging these in,

$$\omega = \sqrt{\frac{g}{L}\left(\frac{1}{1+(2/5)x^2}\right)}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}\sqrt{1+\frac{2}{5}x^2}$$

Therefore, $f(x) = \sqrt{1+(2/5)x^2}$. Indeed, $f(0) = 1$, which must occur because the answer must reduce to the case of a simple pendulum in the absence of any size of the pendulum bob.

12-m1 Power is proportional to energy, and energy is proportional to A^2, so $A \propto \sqrt{P}$. The ratio of the amplitudes is $\sqrt{50/20} = 1.6$.

12-m2 (a) Here we can take $x(t) = \pm A\cos(\omega t)$. Plugging in $t = T/3 = (2\pi)/(3\omega)$, we get $x = \mp A/2$. Since PE $\propto x^2$, the PE is one-fourth its maximum value, and therefore $0.75 = 75\%$ of the total energy at this time is kinetic.

(b) Here we can take $x(t) = \pm A\sin(\omega t)$. Plugging in $t = T/3 = (2\pi)/(3\omega)$, we get $x = \pm(\sqrt{3}/2)A$. Since PE $\propto x^2$, the PE is three-fourths its maximum value, and therefore 0.25 = 25% of the total energy at this time is kinetic.

12-m3 The maximum speed of the object is $v_{\max} = A\omega$. When the object is at a distance $A/4$ from equilibrium, then 1/16th of the total energy is potential, and 15/16th of the total energy is kinetic.

$$\begin{aligned} \mathrm{KE} &= \frac{15}{16}\mathrm{E_{tot}} \\ \frac{1}{2}mv^2 &= \frac{15}{16}\left(\frac{1}{2}mv^2_{\max}\right) \\ v &= \frac{\sqrt{15}}{4}v_{\max} = \frac{\sqrt{15}}{4}A\omega \end{aligned}$$

12-m4 (a) The information given in the problem statement can be used to find the spring constant. The elastic PE stored in the spring is $(1/2)kx^2$; if PE = 1.00 J when $|x| = 0.110$ m, then $k = 165$ N/m. With k, we can relate the mass of the object to the period of motion (which we require to equal 1.00 s):

$$T = 2\pi/\omega = 2\pi\sqrt{M/k}$$
$$M = k\left(\frac{T}{2\pi}\right)^2 = 4.19 \text{ kg}$$

(b) If we say that the object is moving in the negative x direction (which we'll call the left) and passing through the equilibrium point at $t = 0$, then we can model the position of the object with the equation

$$x(t) = -A\sin(\omega t) \tag{24.8}$$

where we know $A = 0.0500$ m and $\omega = 2\pi/T = 2\pi\,\text{s}^{-1}$. Also, since 0.35 s is between one-fourth of a full period and one-half of a full period, we know that the object is somewhere between $x = -A$ and $x = 0$ and that it is moving to the right. Furthermore,

$$x(0.35 \text{ s}) = -A\sin[\omega(0.35 \text{ s})] = -4.0 \text{ cm} \tag{24.9}$$

Also, the problem doesn't ask for this, but we can find the speed as well:

$$v(t) = -A\omega\cos[\omega t]$$
$$v(0.35 \text{ s}) = +0.18 \text{ m/s}$$

(c) Using the position and velocity found in part (b),

$$\text{KE} = \frac{1}{2}Mv^2 = 71 \text{ mJ}$$
$$\text{PE} = \frac{1}{2}kx^2 = 135 \text{ mJ}$$

(d) From Hooke's Law,

$$F = -kx = -(165 \text{ N/m})(0.0300 \text{ m}) = -4.95 \text{ N}$$

The force is 4.95 N and to the left. Note the fact that the mass is moving to the right doesn't matter when looking at the force from the spring; only the position does.

12-p1 (a) If W/E is the fraction of energy lost to heat per cycle, then the fraction remaining after one is $(1 - W/E)$. After two cycles, the fraction remaining is $(1 - W/E)^2$, and so on. After n cycles, the fraction remaining is $(1 - W/E)^n$.
(b) The definition of Q amounts to $(1-W/E)^Q = 1/535$, or

$$(1 - W/E)^Q = e^{-2\pi} \qquad .$$

(c) Taking natural logs of both sides,

$$Q\ln(1 - W/E) = -2\pi \qquad .$$

Since W/E is small, we can use the approximation $\ln(1 + x) \approx x$, so $QW/E = 2\pi$, or $1/Q \approx (1/2\pi)W/E$.

12-p2 (a) The definition of Q says that when $n = Q$, the energy falls off by a factor of $e^{2\pi}$. Since energy is proportional to the square of the amplitude, this corresponds to a factor of $Z = e^{\pi}$ in amplitude. Now we need to generalize this to the case where the number of oscillations we actually observe isn't actually Q. If the amplitude drop in n oscillations is Z, then the drop in one oscillation is $f^{1/n}$. The drop in Q oscillations is then $(Z^{1/n})^Q$. Setting this equal to e^{π} and solving for Q, we find $Q = n\pi/\ln Z$.
(b) The figure is pretty small, so this will be a fairly rough estimate. After $n = 2$ cycles of oscillation, it looks like the amplitude falls off by a factor of $Z = 2$. Applying the result from part a, we have $Q \approx 9$.

12-p3 (a) The total energy is proportional to $A(t)^2$. After the first cycle, the amplitude has reduced to 0.90 times its original value, and so the total energy has reduced to $0.90^2 = 0.81$ times its original value. Therefore, 19% of the energy was lost during the first cycle.
(b) The total energy will be reduced to $0.90^4 = 0.66$ times its original value. This corresponds to the maximum displacement being 0.81 times its initial value, so the object will be at $x = 0.81A$.

(c) Using $A(t) = A\exp[-bt/(2m)]$,

$$0.90A = A\exp\left[\frac{-b}{2m}T'\right] = A\exp\left[\frac{-C\sqrt{km}}{2m}T'\right]$$

$$0.90 = \exp\left[\frac{-C}{2}\sqrt{\frac{k}{m}}\frac{2\pi}{\omega'}\right] \approx \exp\left[\frac{-C}{2}\sqrt{\frac{k}{m}}\frac{2\pi}{\omega}\right]$$

$$0.90 = \exp\left[\frac{-C}{2}\sqrt{\frac{k}{m}}\sqrt{\frac{m}{k}}(2\pi)\right] = \exp\left[-C\pi\right]$$

$$C = 0.034$$

(d) We'll first solve this for general $C \ll 1$ and plug in our value of C at the end:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\omega' = \sqrt{\frac{k}{m}}\left(1 - \frac{b^2}{4m^2}\frac{1}{k/m}\right)^{1/2}$$

$$\omega' = \omega\left(1 - \frac{b^2}{4mk}\right)^{1/2} \approx \omega\left(1 - \frac{b^2}{8mk}\right)$$

Plugging in $b = C\sqrt{km}$, this gives $\omega' = \omega(1 - C^2/8)$. Since $T \propto 1/\omega$, this also implies

$$T' = T\left(1 - \frac{1}{8}C^2\right)^{-1} \approx T\left(1 + \frac{1}{8}C^2\right)$$

Plugging in $C = 0.034$, we find $T' = (1.00014)T$, or the period increases by only 0.014%!

12-s1 $\Delta f = f_o/Q = 100$ Hz, so since the curve is approximately symmetric, the range will be ±50 Hz, i.e., 250-350 Hz.

12-s2 To find the minimum of the quantity inside the square root, we set its derivative equal to zero,

$$0 = \frac{\mathrm{d}}{\mathrm{d}f}\left(4\pi^2 m^2\left(f^2 - f_o^2\right)^2 + b^2 f^2\right)$$

$$0 = 8\pi^2 m^2\left(f^2 - f_o^2\right)(2f) + 2b^2 f$$

One solution is $f = 0$, but that's normally a minimum. We're interested in solutions with $f \neq 0$, so we can divide by $2f$ to simplify:

$$0 = 8\pi^2 m^2\left(f^2 - f_o^2\right) + b^2$$

$$f = \sqrt{f_o^2 - \frac{b^2}{8\pi^2 m^2}}$$

12-s3 $Q = 1/\sqrt{2}$

12-s4 (a) The force is not constant, so we have to do the integral

$$\frac{1}{2}W_{damp} = \int_{-A}^{A} F\,\mathrm{d}x$$

$$= \int_{-A}^{A} -bv\,\mathrm{d}x$$

To find v as a function of x, we can use conservation of energy:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Solving this for v gives

$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} \qquad ,$$

so we now have

$$\frac{1}{2}W_{damp} = -\int_{-A}^{A} b\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}\ \mathrm{d}x$$

$$= -b\sqrt{\frac{k}{m}}\int_{-A}^{A}\sqrt{\left(A^2 - x^2\right)}\ \mathrm{d}x$$

The definite integral can be done by a change of variable and a trig substitution, or by looking it up in a table of integrals, or, most simply of all, by recognizing the integrand as the equation of a semicircle of radius A, so that the area under the curve equals half the area of a circle, $\pi A^2/2$.

$$\frac{1}{2}W_{damp} = -\frac{\pi}{2}A^2 b\sqrt{\frac{k}{m}}$$

We wish to get this in terms of f, which equals $(1/2\pi)\sqrt{k/m}$, so

$$W_{damp} = -2\pi^2 A^2 bf$$

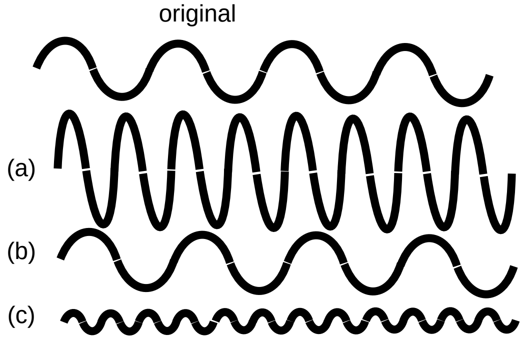
(b,c,d) The energy is $kA^2/2$, so the rest follows immediately.

12-s5 Let's not worry about units, since this is just a comparison with the approximate answers. Setting all those variables equal to 1 for convenience, the expression for the amplitude becomes

$$A = \left[(\omega^2 - 1)^2 + \omega^2/36\right]^{-1/2}$$

By trial and error, I found that the maximum amplitude was 6.02, at $\omega = 0.993$. The FWHM is going to be the difference between the two frequencies where the square of the amplitude equals $(6.02^2)/2$. By trial and error, I found that this occurred at frequencies of 0.906 and 1.074, giving a FWHM of 0.168, which is pretty close to 1/6.

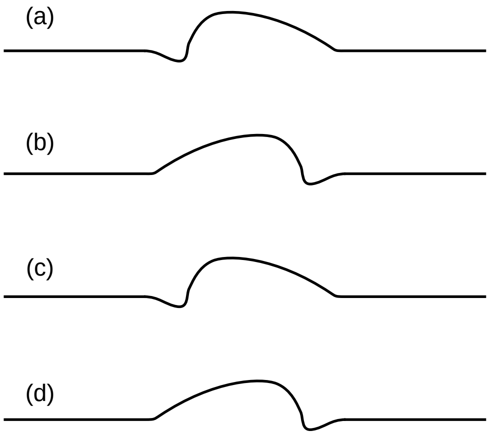
13-a1 Since it's a graph of the wave as a function of position, what it shows is amplitude and wavelength. (a) The amplitude is doubled, and $\lambda = v/f$, so the wavelength is cut in half. (b) Since $v = f\lambda$, doubling both velocity and frequency leaves the wavelength unchanged. This is exactly the same as the orginal wave. (c) The information about the velocity is irrelevant. We're given the wavelength and amplitude directly, so that's all we need to show. (The change in velocity implies an additional change in frequency for the given wavelength, but the wavelength is 1/3 the original, not 1/6.)



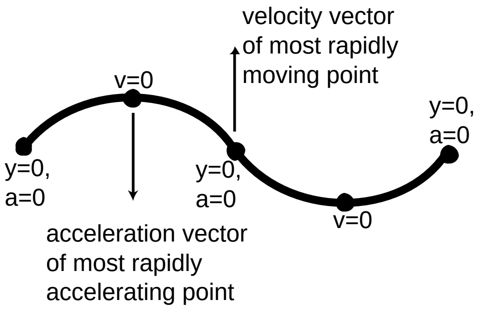
13-a2 (a) The scale on the time axis is not comparable to the scale of the distance axis, so there is no reason why your answer has to be the same width on the page. As the pulse passes through a certain point traveling to the right, the water will first go down a little, then up, so the graph should look similar, but reversed.

(b) As the pulse travels to the left, the water will first do the big, slow rise, then the small, short dip. The graph is not reversed compared to the original.

(c) and (d) are the same idea – the roles of question and answer are just reversed.



13-a3 The points with $v = 0$ are the ones that have gone as far out to one side as they'll go, and are now ready to come back. The points that are whipping across the center line are the ones that are going the fastest, and they also have zero acceleration because their v vs t graph is at a peak, where the tangent line has zero slope.


velocity vector of most rapidly moving point
v=0
y=0, a=0
y=0, a=0
y=0, a=0
v=0
acceleration vector of most rapidly accelerating point

13-a4 (a) In the expression

$$\lambda' = \left(1 - \frac{v_s}{v}\right)\lambda \qquad ,$$

we eliminate wavelength λ and replace it with v/f:

$$\frac{v}{f'} = \left(1 - \frac{v_s}{v}\right)\frac{v}{f} \qquad ,$$

We simplify by dividing out the factor of v on both sides, and inverting both sides:

$$f' = \frac{f}{1 - \frac{v_s}{v}}$$

(b) The ratio v_s/v is unitless, so it makes sense to subtract it from a unitless 1. Dividing a frequency by a unitless number gives a frequency.

(c) If $v_s = 0$, then $f' = f$, which makes sense. For v_s greater than zero (but less than v), the denominator is less than one, so $f' > f$, indicating an upward frequency shift, which makes sense if positive v_s indicates motion toward the observer.

13-a5

There are a couple of things that make it difficult to devise an accurate experiment: (1) it's hard to produce waves in a controlled way, and (2) it's hard to measure the height of the superposing waves in real time.

Possible solution: Find a machine that's capable of making spreading ripples of a consistent height, e.g., a wind-up toy car whose wheels go around. This takes care of problem 1. You could put two such cars in the water, and see if the peaks formed by constructive interference become twice as tall at a point equidistant from the two cars, compared to their height when only one car was turned on.

The toy cars also help to solve problem 2, because the waves are steady. You can put a ruler in the water, and watch the steady up-and-down vibration of the surface, measuring the height of the wave. This could be made easier by taking a video on a cell phone and then clicking through it frame by frame.

13-a6 $T = 1/f = 3.82$ ms, and $\lambda = v/f = (340\text{ m/s})/f = 1.3$ m.

13-a7 1% of the speed of sound, or about 3 m/s

13-a8 When a gas gets hot, it expands. For instance, that's why a hot-air balloon works: the hot air is less dense than the surrounding cold air. A hotter gas is less dense, and its lower inertia makes it easier for it to vibrate rapidly. Liquids and solids don't contract or expand nearly as much with changes in temperature, so the speed of sound in liquids and solids doesn't change much with temperature. (The role played by tension in waves on a string is played by pressure in sound waves. Pressure is basically irrelevant here. The pressure inside a solid or liquid doesn't vary with temperature, and the pressure in a gas only varies with temperature if the gas is confined to a fixed volume. This is not the case for sound waves in open air; the air pressure is essentially the same on a cold day as on a hot day.)

13-a9 (a) To keep the stuff inside the cosine the same, we'd have to decrease x while t increased. The wave is moving in the negative x direction.

(b) At a particular point, the quantity $0.45\pi t$ has to change by 2π for one cycle to go by, so $T = 2\pi/0.45\pi = 4.4$, which we assume represents 4.4 seconds, since we're told that everything is set up for SI units. The frequency is $f = 1/T = 0.23$ Hz. The relationship between x and the wavelength is the same as the relationship between t and the period, so $\lambda = 2\pi/0.73\pi = 2.7$ m.

(c) We could do this by plugging in to $v = f\lambda$, but let's use a different approach. If x changes by Δx while enough time goes by for the inside of the cosine to stay the same, then $0.73\pi\Delta x + 0.45\pi\Delta t = 0$. The velocity is $\Delta x/\Delta t = -0.45/0.73 = 0.62$ m/s.

(d) The velocity of the point on the string is

$$\begin{aligned} y' &= -3.5\sin(0.73\pi x + 0.45\pi t + 0.37\pi)(0.45\pi) \\ &= -4.9\sin(0.73\pi x + 0.45\pi t + 0.37\pi) \quad , \end{aligned}$$

which varies between -4.9 m/s and $+4.9$ m/s. This is completely different from the wave's velocity of 0.62 m/s, and is in the perpendicular direction.

13-a10 (a) $T = \mu\omega^2 r^2$

13-a11 (b) $g/2$

13-d1 Due to the difference in speed, there will be partial reflection of the sunlight.

13-d2 (a) Since the result only depends on the ratios of speeds, we can drop the powers of 10:

$$R = \frac{2.2 - 3.0}{2.2 + 3.0} = -0.15$$

(b) Energy is proportional to the square of amplitude, so the fraction of the energy reflected is $(0.15)^2 = 0.022$.

13-d3 Check: The actual length of a flute is about 66 cm.

13-d4 (a) These are all different standing wave patterns that fit within the tube. Notice how all the frequencies are multiples of 150 Hz. (Since we're getting all the multiples, not just the odd multiples, the saxophone apparently acts like a symmetric air column, and we can visualize the series of wave patterns as the one in which we fit some number of half-wavelengths in the length of the tube.)

(b) Higher frequencies correspond to shorter wavelengths. You can fit more than one half-wavelength in the tube, but you can't put less than one half-wavelength in, because the wave wouldn't be doing the right things at the ends.

13-d5 (a)

C:	261.6	523.2	784.8	1046.4	1308.0
G:	392.0	784.0	1176.0	1568.0	1960.0

The 784.8 and 784.0 Hz frequencies differ by much less than 1%, and the 1046.4 and 1176.0 Hz frequencies differ by 12%. Neither combination is really in the 1-10% range that causes dissonance.

(b)

C:	261.6	523.2	784.8	1046.4	1308.0
Bb:	466.2	932.4	1398.6	1864.8	2331.0

The 1308.0 and 1398.6 Hz frequencies differ by about 7%, which is clearly going to make them dissonant.

13-d6 (a) These are both reflections back into a faster medium. I don't even remember off the top of my head whether such a reflection is inverting or not, but even if it is inverting, two inversions in a row leave the wave with the same phase it had before. Since $d = 0$, the distance traveled by the two types of waves is the same, so their phase is the same when they emerge, and the interference is constructive.

(b) To get constructive interference again, we need the extra distance $2d$ traveled by the twice-reflected wave to equal one complete wavelength. In other words, $d = \lambda/2$.

(c) Each cycle means the top plate has dropped by a distance $\lambda/2$, so the diameter of the filament is $n\lambda/2$.

13-d7 (a) $f = 4\alpha/(1+\alpha)^2$ (b) $v_2 = \sqrt{v_1 v_3}$

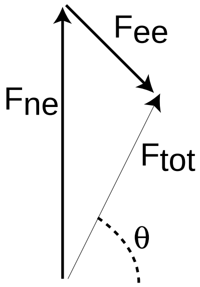
13-d8 (a) We have $R = (\alpha - 1)/(\alpha + 1)$. Inverting α gives $R = (1/\alpha - 1)/(1/\alpha + 1) = (\alpha - 1)/(1 + \alpha)$, which means we've simply negated R. The interpretation is that the energy of the reflected pulse is unchanged, but the reflection's inverting or uninverting character is flipped.

(b) Based on part a, we expect that there will be two values of α that will do the job, and that they will be inverses of each other. Solving for α in terms of R gives $\alpha = (1 + R)/(1 - R)$. Taking $R = 1/2$ gives $\alpha = 3$, and without having to do the arithmetic for the case of the inverting reflection, we can tell that it will happen for $\alpha = 1/3$.

14-a1 The positive and negative charges attract each other.

14-a2 Let's use positive numbers for forces to the right, and negative numbers for forces to the left. There are two forces on the right-hand electron: a repulsive force from the other electron, and an attractive force from the nucleus. These will be represented as positive and negative numbers, respectively. The charges of the electrons are $-e$, and the charge of the nucleus is $+2e$, so the electron-electron force is $+ke^2/(0.2\text{ nm})^2$, and the nucleus-electron force is $-k(e)(2e)/(0.1\text{ nm})^2$, for a total force of -4×10^{-8} N.

14-a3 You should always start this type of vector addition problem by drawing a rough diagram first so that you can check that your results make sense at every step.



The nucleus's force on the bottom electron, F_{ne}, is attractive, so it points up. The other electron's force on the bottom electron, F_{ee}, is repulsive, so it points down and to the right at a 45-degree angle. We'll state the final result using the angle θ defined in the figure. The magnitude of F_{ne} is given by Coulomb's law, $|F_{ne}| = k(e)(2e)/(0.1\text{ nm})^2 = 4.6 \times 10^{-8}$ N. The magnitude of F_{ee} is smaller by a factor of four: a factor of two because of the smaller charge, and another factor of two because its distance from the bottom electron is greater by a factor of $\sqrt{2}$. Thus, $|F_{ee}| = 1.1 \times 10^{-8}$ N. To do vector addition, we need to add components. Let positive x components be to the right, positive y components up. $\mathbf{F}_{ne}$ has components $F_{ne,x} = 0$ and $F_{ne,y} = 4.6 \times 10^{-8}$ N. $\mathbf{F}_{ee}$ has components

$$F_{ee,x} = +(1.1 \times 10^{-8}\text{ N})(\cos 45°) \qquad \text{and}$$
$$F_{ee,y} = -(1.1 \times 10^{-8}\text{ N})(\sin 45°) \qquad .$$

At this point, you can check the + and − signs of the components against the figure. The total force has components

$$F_{tot,x} = F_{ne,x} + F_{ee,x} = 7.8 \times 10^{-9}\text{ N} \qquad \text{and}$$
$$F_{tot,y} = F_{ne,y} + F_{ee,y} = 3.8 \times 10^{-8}\text{ N} \qquad .$$

Again, you can check the + and - signs against the figure. The magnitude of the total force is $\sqrt{F_{tot,x}^2 + F_{tot,y}^2} = 4 \times 10^{-8}$ N (one sig fig). The angle θ equals $\tan^{-1}(F_{tot,y}/F_{tot,x}) = 80°$ (one sig fig).

14-a4 (a) Your body is mostly water, and water is mosly oxygen by weight, so for the purposes of this estimate, let's pretend that your body just consists of oxygen atoms. The mass of an oxygen atom is about 16 times the mass of a proton, which is 1.7×10^{-27} kg. If your hand has a mass of about 0.2 kg, then the number of oxygen atoms in a hand is about 0.2 kg/$(16 \times 1.7 \times 10^{-27}$ kg). The number of electrons is eight times that, $(8) \times (0.2\text{ kg})/(16 \times 1.7 \times 10^{-27}\text{ kg}) = 6 \times 10^{25}$, or call it 10^{26} electrons since its only an estimate.

(b) Coulomb's law gives $F = k(10^{26}e)(10^{26}e)/(0.1\text{ m})^2 = 2 \times 10^{26}$ N, or call it 10^{26} N.

(c) Each hand has zero net charge, and the negative charge is distributed within each hand in almost the same way as the positive charge. Another way of looking at it is that there are repulsive forces acting from one set of electrons on the other set of electrons and from one set of protons on the other set of protons, but there are attractive forces from each set of electrons acting on the protons of the opposite hand.

(d) A force of 10^{26} N is enough to pulverize a planet, so the cancellation is obviously very nearly perfect; even a very tiny percentage difference in charge would lead to a huge force. If the fractional difference was ϵ, then the total charge on each hand would be ϵq, where $q = 10^{26}e$ as estimated above. The total force would be $k\epsilon^2 q^2/r^2 = F$. You could easily notice a force of as little as 0.1 N, so $\epsilon \lesssim (r/q)\sqrt{F/k} \sim 10^{-14}$.

14-a5 There are 82 protons in a lead nucleus. Since the lead nucleus is spherical, the electric force from the 81 other protons is the same as that of an equal amount of charge (81 times the charge of a proton) concentrated at the nucleus' center. The force is

$$k(q_{proton}) \times (81 q_{proton})/(6.5 \times 10^{-15}\text{ m})^2 = 440\text{ N} \qquad ,$$

which is less than the attractive force of 8 kN.

14-a6 Electricity is an inverse square force, so if r is smaller by a factor of 206.77, the force will be greater by a factor of $206.77^2 = 42754$.

14-a7 Let q stand for the absolute values of the charges, which are equal to each other. We can find the force either by using Newton's law of gravity or by using the kinematical equation $a = v^2/r$ for the moon's acceleration. Choosing the first approach, we have

$$k\frac{q^2}{r^2} = G\frac{m_1 m_2}{r^2} \qquad ,$$

which we solve for q:

$$\begin{aligned} q &= \sqrt{\frac{G m_1 m_2}{k}} \\ &= 5.70 \times 10^{13}\text{ C} \end{aligned}$$

This may sound like a big number, but it's actually not that much charge at all, on the same order of magnitude as the amount of charge in a human-scale object. It's just that in real life, the charges balance out perfectly, giving zero total charge on each body. Note that the reason the charge comes out fairly small, despite the large masses, is that G is much smaller than k, i.e. gravity is in some sense a much weaker force than electricity.

14-a8 The lithium is located nice and symmetrically in the lattice, so if there was no defect, the total force on it would be zero. Removing a Cl^- to make the hole removes a certain amount of force, so the total force without the Cl^- is equal to minus the force the Cl^- would have made, which we can find from Coulomb's law. The distance between the lithium and the hole is $\sqrt{(2.5\times 0.3\ \text{nm})^2+(0.5\times 0.3\ \text{nm})^2} = 0.77$ nm, so the magnitude of the force is $ke^2/(0.77\ \text{nm})^2 = 3.9\times 10^{-10}$ N, or call it 4×10^{-10} N. The Li^+ would have been attracted by the Cl^-, so removing that attractive force leaves a total force that points directly away from the hole.

14-a9 (a) Let each fixed charge be Q, and let the charge of the ball be q. When the ball is at position x, its distance from one of the fixed charges is $\sqrt{b^2+x^2}$, so the magnitude of each force on it is $|\mathbf{F}| = kQq/(b^2+x^2)$. There are two such forces, acting at different angles, so we have to add them using vector addition. Their y components cancel, and their x components are the same, so for the total force on the ball, we have

$$\begin{aligned} F &= 2F_x \\ &= \frac{2kQq}{b^2+x^2}\cdot\cos\theta \\ &= \frac{2kQq}{b^2+x^2}\cdot\frac{x}{\sqrt{b^2+x^2}} \\ &= 2kQq\left(b^2+x^2\right)^{-3/2}x \end{aligned}$$

(b) By $a = F/m$, the maximum acceleration occurs at the same point as the maximum force. To find the maximum force, we have to set the derivative of the force equal to zero and solve for x. The constants out in front don't affect the location of the maximum, so we can ignore them.

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(b^2+x^2\right)^{-3/2}x\right] \\ &= -\frac{3}{2}\left(b^2+x^2\right)^{-5/2}(2x)x+\left(b^2+x^2\right)^{-3/2} \end{aligned}$$

We can simplify by multiplying both sides by $\left(b^2+x^2\right)^{5/2}$.

$$\begin{aligned} 0 &= -3x^2+b^2+x^2 \\ x &= \frac{b}{\sqrt{2}} \end{aligned}$$

14-a10 There can be no stable equilibrium in which the string is slack, since one could always move the movable charge along a path at a constant distance from the fixed charge, bringing it to a position on the circumference at which its gravitational energy would be lower.

The result can only depend on the unitless quantity $\beta = kq/mg\ell^2$. To make the writing easier, we take $\ell = 1$. The distance between the charges is $r = 2\sin\theta/2$, and for this reason it will be convenient to define the variable $p = \sin\theta/2$, which equals 0 at the bottom of the circle and 1 at the top.

For $\beta < 4$ there is an equilibrium in which the movable charge is directly above the fixed one, with the string slack. These are unstable with respect to lateral motion, and not very interesting.

For large β there is an equilibrium at $\theta = \pi$, but we need to investigate when this is stable.

Using the identity $1-\cos\theta = 2\sin^2\theta/2$, we find for the potential energy

$$U = \frac{1}{2}\beta p^{-1}+2p^2,$$

where the first term is electrical and the second gravitational. Although it is θ that is physically relevant, it is easier to do as much of the calculus as possible with p as the variable, since the expressions are simpler. Expressed as a function of p, the energy is always concave up, so any point with $\mathrm{d}U/\,\mathrm{d}p = 0$ will be a stable equilibrium. In addition, we have $\mathrm{d}U/\,\mathrm{d}\theta = 0$ automatically at $\theta = \pi$, since $\mathrm{d}p/\,\mathrm{d}\theta = 0$ there.

We have $\mathrm{d}U/\,\mathrm{d}p = -(1/2)\beta p^{-2} + 4p$, which equals zero for $p = (\beta/8)^{1/3}$. As remarked earlier, this is always stable. However, it only exists for $\beta \le 8$, since we must have $p \le 1$.

To evaluate the stability of the upright equilibrium, we evaluate $\mathrm{d}^2 U/\,\mathrm{d}\theta^2$ at $\theta = \pi$, which gives $\beta/8 - 1$. This is positive if $\beta > 8$. Thus although levitation straight up, with the cord under tension, is possible for $4 < \beta < 8$, this equilibrium is not stable against lateral motion for these values of β.

In summary, we find that for $\beta \le 8$ the stable equilibrium is at

$$\sin\frac{\theta}{2} = \left(\frac{\beta}{8}\right)^{1/3} \quad ,$$

while for $\beta \ge 8$ the stable and only equilibrium is the one in which the cord is taut and upright.

14-d1 (a) $F = ma$ and $F = qE$, and since the field is uniform, the force and acceleration will be constant. For motion with constant acceleration, we have $v = at$, and eliminating the unknown F gives $v = qEt/m$.

(b) You can conclude that they have the same ratio of q/m, but you can't tell whether they have the same q and the same m. One could have twice the charge and twice the mass compared to the other, for instance.

14-d2 Let the square's sides be of length a. The field at the center is the vector sum of the fields that would have been produced individually by the three charges. Each of these individual fields is kq/r^2, where $r_1 = a/\sqrt{2}$ for the two charges q_1, and $r_2 = a/2$ for q_2. Vector addition can be done by adding components. Let x be horizontal and y vertical. The y components cancel by symmetry. The sum of the x components is

$$E_x = kq_1 r_1^{-2}\cos 45° + kq_1 r_1^{-2}\cos 45° - kq_2 r_2^{-2} \quad .$$

Substituting $\cos 45° = 1/\sqrt{2}$ and setting this whole expression equal to zero, we find $q_2/q_1 = 1/\sqrt{2}$.

14-d3 (a) The energy per unit volume is

$$\frac{E}{V} = \frac{1}{8\pi k}|\mathbf{E}|^2 \quad .$$

(E is energy, $\mathbf{E}$ is electric field.) At the moment when the lightning strikes, the field equals E_c. The volume of the air in which the field exists is $L^2 h$.

$$E = \frac{1}{8\pi k}E_c^2 L^2 h$$

(b) A greater h gives a greater energy, so it's more dangerous if it's higher.

(c) Let's use 1000 m for both L and h. Plugging in, the result is about 10^{10} J.

14-d4 (a) Plugging in to the equation for the electric field energy, we have $U_{one} = 4.43 \times 10^{-12}$ J.

(b) The two fields are perpendicular to each other, and add like vectors, so the result is given by the Pythagorean theorem, $E_{together} = \sqrt{2}(1\ \mathrm{V/m}) = 1.41\ \mathrm{V/m}$.

(c) $E_{together}^2$ is exactly twice E_{one}^2, so the energy is the same when they're together as when they're apart. There is no energy consumed or released when you slide them together. (You would feel torques, but no net repulsion or attraction.)

14-g1 (a) For simplicity, let's put everything on the x axis, so the dipole moment has only an x component. If we put -1 C at $x = 0$ and $+1$ C at $x = 1$ m, then the dipole moment is $D_x = 1\ \mathrm{C\cdot m}$. Now let's do a different coordinate system, whose origin is located 7 m to the left of the first one's. In this coordinate system, the charges are located at $x = 7$ m and 8 m. The dipole moment is the same, however, $(8 \times 1 + 7 \times -1) = 1\ \mathrm{C\cdot m}$. For this particular pair of charges, the dipole moment doesn't seem to depend on the choice of coordinates.

(b) If we have charges $+q$ and $-q$ located at $\mathbf{r}_+$ and $\mathbf{r}_-$, then the dipole moment is $\mathbf{D} = q(\mathbf{r}_+ - \mathbf{r}_-)$. This only depends on the vector $\mathbf{r}_+ - \mathbf{r}_-$, which points from the negative charge to the positive one. Shifting the origin has no effect on $\mathbf{r}_+ - \mathbf{r}_-$, so it has no effect on the dipole moment, either.

(c) We conjecture that the choice of origin doesn't matter as long as the total charge is zero. The most elegant way to prove this is to imagine that we can build the given charge distribution out of tiny point charges, each with charge $+q$ or $-q$, where q is very small. (Mathematically, q can even be less than the fundamental charge, e.) If the total charge is zero, then we have an equal number of $+q$ and $-q$ charges, which we can arbitrarily pair up like a roomful of married couples. The total dipole moment equals the sum of the dipole moments of the couples, and since none of the couples' dipole moments depend on the choice of origin, neither does the sum.

14-g2 They're the same. When we have a charge distribution with zero total charge, its dipole moment is unchanged when we move it around without rotating it, since the dipole moment is defined by the potential energy, and the potential energy is unchanged when we do this. These two distributions therefore have the same dipole moment, since we can make one into the other by sliding one of the $+-$ pairs.

14-g3 A simple example is $+--+$. The total charge is zero, and the dipole moment cancels by symmetry. This is known as an electric *quadrupole*.

14-g4 (a) They're both 1.000 C·m.

(b)

$$\begin{aligned} E_A &= -\frac{k|-1.000\ \text{C}|}{(10.000\ \text{m})^2} + \frac{k|1.000\ \text{C}|}{(9.000\ \text{m})^2} \\ &= 2.11 \times 10^7\ \text{N/C} \\ E_B &= -\frac{k|-2.000\ \text{C}|}{(10.000\ \text{m})^2} + \frac{k|2.000\ \text{C}|}{(9.500\ \text{m})^2} \\ &= 1.94 \times 10^7\ \text{N/C} \end{aligned}$$

They come out almost the same, but not quite. In general, the farther we are from a dipole, the more accurate it is to say that its field depends only on its dipole moment. Here, our distance from the dipoles was 10 or 20 times greater than their sizes, so it was a pretty good approximation.

14-g5 Our point lies at a distance R from the center of the dipole, and its distance from either charge is $\sqrt{R^2+\ell^2/4}$. The magnitude of the field contributed by one of the charges is therefore $E_{one} = kq/\left(R^2+\ell^2/4\right)$. The radial (in-out) components of the two fields cancel, but their z components reinforce, so $E = 2E_{one,z} = 2E_{one}\sin\phi$, where the small angle between the midplane and the individual field vector is $\phi = \tan^{-1}[(\ell/2)/R] \approx \ell/2R$. The dipole's field is therefore

$$\begin{aligned} E_z &= 2E_{one}\sin\phi \\ &\approx 2E_{one}\phi \\ &= \frac{2E_{one}\ell}{2R} \\ &= \frac{kq\ell}{R\left(R^2+\ell^2/4\right)} \\ &\approx \frac{kD}{R^3} \end{aligned}$$

14-j1

$$\begin{aligned} m &= \int \mathrm{d}m \\ &= \int \frac{\mathrm{d}m}{\mathrm{d}v}\,\mathrm{d}v \\ &= \int_0^\infty ae^{-br}\,\mathrm{d}v \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} ae^{-br}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= 2\pi\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} ae^{-br}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 4\pi\int_{r=0}^{\infty} ae^{-br}r^2\,\mathrm{d}r \end{aligned}$$

The r integral can be done by integration by parts, or with computer software. The result is

$$m = \frac{8\pi a}{b^3}$$

14-j2

$$\begin{aligned}
-e &= \int \mathrm{d}q \\
&= \int \frac{\mathrm{d}q}{\mathrm{d}v}\,\mathrm{d}v \\
&= \int_0^\infty ae^{-br}z^2\,\mathrm{d}v \\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} ae^{-br}r^4\cos^2\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= 2\pi\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} ae^{-br}r^4\cos^2\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

Subsituting $u=\cos\theta$,

$$\begin{aligned}
-e &= 2\pi\int_{u=1}^{-1}\int_{r=0}^{\infty} ae^{-br}r^4u^2\,\mathrm{d}u\,\mathrm{d}r \\
&= \frac{4\pi}{3}\int_{r=0}^{\infty} ae^{-br}r^4\,\mathrm{d}r
\end{aligned}$$

The r integral can be done by integration by parts, or with computer software. The result is

$$\begin{aligned}
-e &= \frac{32\pi a}{b^5} \\
a &= \frac{-eb^5}{32\pi}
\end{aligned}$$

14-j3 (a) $E=2k\lambda/R$.

14-j4 (a) We know that the field of a ring, on its axis, is $kQz\left(b^2+z^2\right)^{-3/2}$, so we can get the field of the cylinder by slicing it like a salami and integrating the contributions of the rings to the total field. Each ring has a width $\mathrm{d}z$, so its charge is $\mathrm{d}Q=(\text{circumference})(\text{width})\sigma=2\pi b\sigma\,\mathrm{d}z$.

$$\begin{aligned}
E &= \int k\,\mathrm{d}Qz\left(b^2+z^2\right)^{-3/2} \\
&= \int_0^\infty 2k\pi b\sigma z\,\mathrm{d}z\left(b^2+z^2\right)^{-3/2} \\
&= 2k\pi b\sigma\int_0^\infty z\,\mathrm{d}z\left(b^2+z^2\right)^{-3/2}
\end{aligned}$$

This is a good candidate for integrals.com, but if you want to do it by hand, you can substitute $u=z^2+b^2$, which gives

$$\begin{aligned}
E &= k\pi b\sigma\int_{b^2}^{\infty} u^{-3/2}\,\mathrm{d}u \\
&= -2k\pi b\sigma\;u^{-1/2}\Big]_{b^2}^{\infty} \\
&= 2k\pi\sigma
\end{aligned}$$

(b) If you doubled the cylinder's radius, while keeping the charge density constant, you'd have a cylinder with four times the area, and therefore four times the charge. All the angles would be the same, but all the distances would be doubled. Doubling the distance makes the field four times smaller, and this cancels out the effect of the increase in charge.

14-j5 (a) The surface area of a sphere is $4\pi r^2$, so the volume of the kind of spherical shell suggested in the hint is $\mathrm{d}V=4\pi r^2\ \mathrm{d}r$. The energy is

$$\begin{aligned}
U &= \int_0^\infty \frac{\mathrm{d}V}{8\pi k}\,E^2 \\
&= \int_0^\infty \frac{\mathrm{d}V}{8\pi k}\left(\frac{kq}{r^2}\right)^2 \\
&= \frac{kq^2}{2}\int_0^\infty r^{-2}\,\mathrm{d}r \\
&= \frac{kq^2}{2}\left(\frac{1}{0}-\frac{1}{\infty}\right)
\end{aligned}$$

It's perfectly legal to take $1/\infty$, and the result is 0. However, $1/0$ gives infinity. In other words, the integral diverges near $r=0$, but not at $r=\infty$. The interpretation is that the field has infinite energy close to the charge, but finite energy in the space farther away.

(b) The energy in the exterior volume equals the result from part a, with c set to infinity, $U_{ext}=kq^2/2b$. The energy in the interior volume is

$$\begin{aligned} U_{int} &= \int_0^b \frac{\mathrm{d}V}{8\pi k}\,E^2 \\ &= \int_0^b \frac{\mathrm{d}V}{8\pi k}\,\left(\frac{kqr}{b^3}\right)^2 \\ &= \frac{kq^2}{2b^6}\int_0^b r^4\,\mathrm{d}r \\ &= \frac{kq^2}{10b} \end{aligned}$$

The total energy is $U_{int}+U_{ext}=3kq^2/5b$.

14-j6 (a) For L approaching infinity, the first term inside thw square root becomes negligible compared to the second term, and the square root approaches $\sqrt{L^2/4R^2}=L/2R$. The desired result then follows immediately.

(b) We know the field of both infinite and finite lines of charge, so we can attack the problem either by slicing the strip lengthwise or across. Let's slice it lengthwise, since the field of an infinite line of charge is particularly simple, $E=2k\lambda/R$. (This is the answer to another homework problem, given in the back of the book.) Let x be the direction across the strip, so that $\lambda=\sigma\,\mathrm{d}x$. We then have

$$E=\int_{-b/2}^{b/2}\frac{2k\sigma\,\mathrm{d}x}{R}\cdot\cos\theta \qquad ,$$

where $\theta=\cos^{-1}z/R$ is the angle between the z axis and the line connecting our point to a particular line.

$$\begin{aligned} E &= \int_{-b/2}^{b/2}\frac{2k\sigma\,\mathrm{d}x}{R}\cdot\frac{z}{R} \\ &= 2k\sigma z\int_{-b/2}^{b/2}\frac{\mathrm{d}x}{R^2} \\ &= 2k\sigma z\int_{-b/2}^{b/2}\frac{\mathrm{d}x}{x^2+z^2} \\ &= 4k\sigma\tan^{-1}\frac{b}{2z} \end{aligned}$$

(c) For z approaching zero, we have the arctangent of infinity, which is $\pi/2$, giving $E=2\pi k\sigma$, and this is the correct result for the field near a charged surface. For large z, the argument of the arctangent is small, and we have $\tan^{-1}\epsilon\approx\epsilon$, yielding $E=2kb\sigma/z=2k\lambda/z$. From far away, the strip looks infinitely thin, like a line of charge, and this is indeed the right result for the field of an infinitely long line of charge.

14-j7 We can slice the cylinder like a salami into thin slices of thickness $\mathrm{d}z$, and then integrate, because we already know that the on-axis field of a disk is $2\pi\sigma k\left(1-z/\sqrt{b^2+z^2}\right)$. Because the slices are so thin, the surface charge density of each slice is an infinitesimal number, $\mathrm{d}\sigma$. The field is

$$\begin{aligned} E &= \int \mathrm{d}E \\ &= \int 2\pi\,\mathrm{d}\sigma k\left(1-\frac{z}{\sqrt{b^2+z^2}}\right) \end{aligned}$$

The relationship between $\mathrm{d}\sigma$ and $\mathrm{d}z$ is $\mathrm{d}\sigma=\rho\,\mathrm{d}z$. (One way to see this is to consider a tiny cube of charge inside the cylinder, with dimensions $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$. The cube's charge is $\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and dividing by the area $\mathrm{d}x\,\mathrm{d}y$ gives $\rho\,\mathrm{d}z$.) We then have

$$\begin{aligned} E &= \int_0^\ell 2\pi\rho\,\mathrm{d}zk\left(1-\frac{z}{\sqrt{b^2+z^2}}\right) \\ &= 2\pi\rho k\int_0^\ell\left(1-\frac{z}{\sqrt{b^2+z^2}}\right)\,\mathrm{d}z \end{aligned}$$

The integral can be done with computer software, or by looking it up in a table, or by substituting $u=z^2$. The result is

$$E=2\pi k\rho\left(\ell+b-\sqrt{b^2+\ell^2}\right)$$

14-m6 (a) The setup is similar to the preceding problem. We again choose as our Gaussian surface a finite cylinder of length ℓ and radius R. Gauss's law gives $2\pi R\ell E=4\pi kq_{in}$, where $q_{in}=\pi R^2\ell\rho$. Solving for the field, we have $E=2\pi kR\rho$.

(b) At $R = 0$, we get $E = 0$. This makes sense, because the field has to vanish on the axis by symmetry. If it were to be nonzero, there would be no way to decide which direction it should point.
(c) The units of $E = 2\pi k R\rho$ are

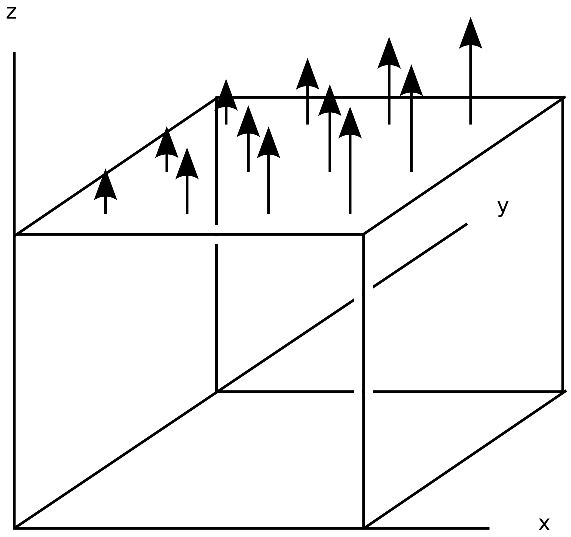
$$\frac{\mathrm{N}}{\mathrm{C}} =? \left(\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}^2}\right)(\mathrm{m})\left(\frac{\mathrm{C}}{\mathrm{m}^3}\right),$$

which does check out.
14-m7 (a) The field is parallel to four of the box's sides, and perpendicular to the other two, so only those two fluxes are nonzero. The field passes in through one of these sides and out through the other, so one flux is positive and one is negative. Because the field is constant, these two fluxes cancel exactly, and the total flux is zero. By Gauss's law, the total charge inside the box is therefore zero.

We could imagine that there were regions of positive and negative charge density, and this particular box just happened to surround a region with equal amounts of positive and negative charge. But we haven't just proved that this particular box contains zero total charge; we've proved that there's zero total charge inside *any* box. The only way for the total charge inside *any* box to be zero is if there really is zero charge density at every point in this region of space.
(b) The field must be due to some charges that are outside this region of space.
14-m8 (a) The field is parallel to all the sides of the cube except for the ones at $x = 0$ and $x = a$, so those are the only ones that might have nonzero flux through them. The field is zero at $x = 0$, so in fact the only flux is through the side at $x = a$. On this side, the field has a constant magnitude of ba, and is always perpendicular to the surface, so we don't have to do an integral to find the flux; we can just multiply the field's magnitude by the area, giving $\Phi_E = ba^3$. The charge inside the box is $q_{in} = \Phi_E/4\pi k = ba^3/4\pi k$.
(b) The field is only perpendicular to the sides of the cube at $z = 0$ and $z = a$, but because the field doesn't depend on z, the inward flux through one of these sides is the same as the outward flux through the other. The total flux is zero, so there's zero total charge inside the box.
(c) Fluxes add, so we can calculate the fluxes of the two terms separately. The first term is really no different than the field in part a, except that it's 13 times stronger, and turned around to lie along the z axis. Therefore the flux it contributes is $13ba^3$. The second term is like the field in b, and doesn't give any flux. The total charge inside the box is $13ba^3/4\pi k$.
(d) The figure below shows the only nonvanishing flux, which is through the top side of the cube. (The fluxes on the sides are zero because the field is parallel to them, and the flux through the bottom is zero because the field is zero at $z = 0$.) The flux through the top is

$$\begin{aligned}\Phi_E &= \int_{\text{top}} \mathbf{E}\cdot \mathrm{d}\mathbf{A}\\ &= \int_{\text{top}} bxa\hat{z}\cdot \mathrm{d}\mathbf{A}\\ &= \int_{x=0}^{a}\int_{y=0}^{a} bxa\ \mathrm{d}y\,\mathrm{d}x\\ &= ba\int_{x=0}^{a} x\int_{y=0}^{a}\ \mathrm{d}y\,\mathrm{d}x\\ &= ba\int_{x=0}^{a} xa\ \mathrm{d}x\\ &= \frac{1}{2}ba^4\end{aligned}$$

The charge inside is $ba^4/8\pi k$.



14-m9 The field points in the $\hat{\mathbf{x}}$ direction, but only depends on z. The reasoning is just like that in problem 44b. For any box aligned with the x, y, and z axes, the fluxes through the two sides perpendicular to x cancel out, because the fields don't differ for different values of x. If there's zero charge inside *any* such box, then there must not be any charge anywhere.

15-a1 (a) $V = Ed = 1.8 \times 10^3$ V
(b) Since $V = Ed$, a larger d would require a larger V.

15-a2 $E = V/d = 70$ mV/6.0 nm $= 1.2 \times 10^7$ V/m

15-a3 (a) Its volume is (length) × (circumference) × (thickness) = $(L)(2\pi r)(t)$. The electric field is $E = V/t$. The energy stored is

$$\begin{aligned}\frac{1}{8\pi k}|\mathbf{E}|^2 \times (\text{volume}) &= \frac{1}{8\pi k}\left(\frac{V}{t}\right)^2 (2\pi r L t)\\ &= \frac{V^2 r L}{4kt}\end{aligned}$$

(b) Making r small means that the neuron needs less energy stored in it to have voltage V across its membrane, which may explain why they evolved to be so skinny. There is a limit on how small r can be, however, because the cell can't be so skinny that it lacks room inside for its machinery. Having a thicker membrane would also reduce the energy needed (dividing by a larger number gives a smaller result), but a cell membrane can't be arbitrarily thick, because it needs to take in nutrients and expel wastes through it.

15-a4 (a) Conservation of energy gives

$$\begin{aligned}U_A &= U_B + K_B\\ K_B &= U_A - U_B\\ \frac{1}{2}mv^2 &= e\Delta V\\ v &= \sqrt{\frac{2e\Delta V}{m}}\end{aligned}$$

(b) Plugging in numbers, we get 5.9×10^7 m/s. This is about 20% of the speed of light, so the nonrelativistic assumption was good to at least a rough approximation.

15-a5 (a) The ion gains an energy eV on the way to the terminal, and gains a further energy neV on the second leg of the journey, so its total kinetic energy is $(n+1)eV$.

$$\begin{aligned}\frac{1}{2}mv^2 &= (n+1)eV\\ v &= \sqrt{\frac{2(n+1)e\Delta V}{m}}\end{aligned}$$

(b) Since the beam is hydrogen, we must have $n = 1$. Solving the equation for V, we have

$$\begin{aligned}V &= \frac{m(0.11c)^2}{2(n+1)e}\\ &= 2.8 \text{ MV}\end{aligned}$$

(c) This was the minimum velocity required for a proton to overcome the repulsion of another proton. Hydrogen is the lightest element, so the other elements in the plastic will repel the beam more strongly, and the beam will not be able to fuse with them. There will not be any extraneous nuclear reactions messing up this experiment.

15-d1 By symmetry, the field is always directly toward or away from the center. We can therefore calculate it along the x axis, where $r = x$, and the result will be valid for any location at that distance from the center. The electric field is minus the derivative of the potential,

$$\begin{aligned} E &= -\frac{\mathrm{d}V}{\mathrm{d}x} \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}e^{-x}\right) \\ &= x^{-2}e^{-x} + x^{-1}e^{-x} \end{aligned}$$

At small x, near the proton, the first term dominates, and the exponential is essentially 1, so we have $E \propto x^{-2}$, as we expect from the Coulomb force law. At large x, the second term dominates, and the field approaches zero faster than an exponential.

15-d2 (a) The voltage is the integral of the field with respect to position. We can use an indefinite integral and not worry about the constant of integration, since physically it just relates to the arbitrary choice of a reference point.

$$\begin{aligned} V &= -\int E\,\mathrm{d}r \\ &\propto -\int \frac{D}{r^3}\,\mathrm{d}r \\ &= \frac{D}{2r^2} \end{aligned}$$

(b) Let charge $+q$ be at $x = \ell/2$, and $-q$ at $x = -\ell/2$. The exact voltage is

$$V = \frac{kq}{|x-\ell/2|} - \frac{kq}{|x+\ell/2|} \qquad .$$

For large distances, it doesn't matter whether we do positive or negative x, because everything is symmetric, so let's just assume x is large and positive, in which case we can get rid of the absolute value signs:

$$V = \frac{kq}{x-\ell/2} - \frac{kq}{x+\ell/2} \qquad .$$

To get this in the form suggested in the hint, we have to divide out the common factor of kq/x:

$$\begin{aligned} V &= \frac{kq}{x}\left(\frac{1}{1-\ell/2x} - \frac{1}{1+\ell/2x}\right) \\ &\approx \frac{kq}{x}\left[\left(1+\frac{\ell}{2x}\right) - \left(1-\frac{\ell}{2x}\right)\right] \\ &= \frac{kq\ell}{x^2} \\ &= \frac{kD}{x^2} \end{aligned}$$

This is consistent with part a. (Since part a was only stated as a proportionality, it doesn't matter that the two results disagree by a factor of $k/2$.)

15-d3 This is the opposite of the previous problem. To find the field given the voltage, we take the derivative:

$$\begin{aligned} E &= -\frac{\mathrm{d}}{\mathrm{d}r}\frac{k}{r^3} \\ &= \frac{3k}{r^4} \end{aligned}$$

15-d4 The voltage is

$$\begin{aligned} V &= -\int E\,\mathrm{d}x \\ &= -ax - \frac{1}{4}bx^4 \qquad , \end{aligned}$$

where the constant of integration is irrelevant because the electrical energy is only meaningful up to an arbitrary additive constant. Conservation of energy gives

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ 0 &= \frac{1}{2}mv^2 + eV(x_2) \\ v &= \pm\sqrt{-\frac{2eV(x_2)}{m}} \\ &= \pm\sqrt{\frac{2e(ax_2 + \frac{1}{4}bx_2^4)}{m}} \qquad . \end{aligned}$$

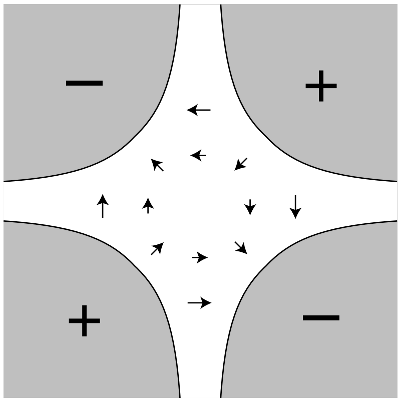
Note that for certain values of a, b, and x_2, we could end up with a negative quantity inside the square root. If so, then that would be Mother Mathematics' way of telling us that the proton would turn around before reaching position x_2.

If the quantity inside the square root is positive, then we have two square roots, a positive one and a negative one. This tells us that the proton could oscillate back and forth through position x_2 more than once.

15-g1

$$\begin{aligned} E_x &= -\frac{\mathrm{d}V}{\mathrm{d}x} \\ &= -V_o b^{-2} y \\ E_y &= -\frac{\mathrm{d}V}{\mathrm{d}y} \\ &= -V_o b^{-2} x \end{aligned}$$

The fields along the axes are particularly easy to sketch, just by looking at these equations. We can then fill in some off-axis points by making use of the fact that the electric field close to the conducting surfaces must be perpendicular to the surfaces.



15-g2 (a) The variable r depends on x, y, and z. To take the derivative of this voltage with respect to the coordinates, we need to make this explicit:

$$\begin{aligned} V &= br^2 \\ &= b(x^2+y^2+z^2) \\ E_x &= -\frac{\mathrm{d}V}{\mathrm{d}x} \\ &= -2bx \end{aligned}$$

Similarly for the other two components,

$$\begin{aligned} E_y &= -2by \\ E_z &= -2bz \end{aligned} \qquad .$$

(b) The field only depends on the derivatives, so we can add any constant onto the voltage and still get the same field, e.g. V could be $br^2 + (42\ \mathrm{V})$, and the derivatives would come out the same.

15-j1 The energy of a spherical capacitor,

$$U = \frac{kq^2}{2}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad ,$$

was derived in an example, and when b approaches infinity, we have simply $U = kq^2/2a$. The definition of capacitance is $U = q^2/2C$, so we have $C = a/k$. Plugging in the radius of the earth yields 710 μF. This is a surprisingly small value, considering how big the earth is!

15-j2

$$\begin{aligned} U &= \frac{q^2}{2C} \\ &= \frac{(CV)^2}{2C} \\ &= \frac{1}{2}CV^2 \end{aligned}$$

15-j3 (a) The difference in the new derivation comes because the modified form of Gauss's law relates the free charge to ϵE rather than E, so instead of $E = 4\pi k\sigma$, we have $\epsilon E = \sigma_{free}$, or, more for more convenient comparison, $(\epsilon/\epsilon_o)E = 4\pi k\sigma_{free}$. The result is that the capacitance is

multiplied by the factor ϵ/ϵ_o relative to its vacuum value.
(b) $U_e = (1/2)CV^2 \propto \epsilon(A/h)(Eh)^2 = \epsilon v E^2$, so the figure of merit is ϵE^2.

15-m1 (a) The answer must be of the form $\alpha q\sqrt{k/mg}$, where α is a unitless constant.
(b) We can place an image charge $-q$ above the conducting plane, at a distance 2ℓ from the real charge. The force is $(1/4)kq^2/\ell^2$. Balancing this against the gravitational force mg gives $\ell = (1/2)q\sqrt{k/mg}$.
(c) If the charge moves upward from the equilibrium position, the upward electrical force gets stronger, so there is a net upward force. Similarly, a downward displacement gives a net downward force.

15-m2 (a) Because θ is unitless, it can only depend on the unitless ratio y/x.
(b) The trick here is to use three image charges: charges $-q$ at $(x, -y)$ and $(-x, y)$, and a $+q$ at $(-x, -y)$. This is exactly analogous to the formation of images in a pair of mirrors placed at right angles to one another. If we take this arrangement of charges, flip it across the y axis, and then negate all the charges, it becomes the same arrangement again. It follows that E_y vanishes, as required, on the y axis. A similar argument holds for the x axis.

By the result of part a, we can ignore factors of q and the Coulomb constant. Vector addition gives $4F_x = -1/x^2 + x/r^3$ and $4F_y = -1/y^2 + y/r^3$. The result is

$$\tan\theta = \frac{y - r^3/y^2}{x - r^3/x^2},$$

which can also be expressed in terms of y/x, but without much apparent simplification.

16-a1 $\Delta t = \Delta q/I = e/I = 0.16\ \mu\text{s}$

16-a2 $I = \Delta q/\Delta t = (10^{10})(e)/(5\times10^{-4}\ \text{s}) = 3\ \mu\text{A}$

16-a3 $q = \int I\,\mathrm{d}t = \int bt\,\mathrm{d}t = \frac{1}{2}bt^2 + q_o$
Here q_o is a constant of integration, representing the cloud's charge at time zero.

16-a4 (a) In one revolution, the amount of charge passing by a certain angular position is 1 nC. The time required for one revolution is $1/(33.3\ \text{min}^{-1})$, so $I = \Delta q/\Delta t = 3.3\times10^{-8}\ \text{C/min} = 600\ \text{pA}$.
(b) The idea is the same. The time for one orbit is $\Delta t = 2\pi r/v$, so $I = e/\Delta t = ve/2\pi r = 1\ \text{mA}$.

16-a5

$$\begin{aligned} I &= \mathrm{d}q/\,\mathrm{d}t \\ &= 2ct \\ P &= I\Delta V \\ &= I\cdot(IR) \\ &= I^2R \\ &= 4c^2t^2 \end{aligned}$$

16-a6 $I = \mathrm{d}q/\,\mathrm{d}t = b\sin\omega t + \omega(a+bt)\cos\omega t$

16-d1 $R = \Delta V/I = (110\ \text{V})/(0.9\ \text{A}) = 120\ \Omega$

16-d2 (a) $P = I\Delta V = (\Delta V/R)\Delta V = \Delta V^2/R$
(b) The power is proportional to ΔV^2, so cutting ΔV in half reduces the power by a factor of 4, to 50 W.

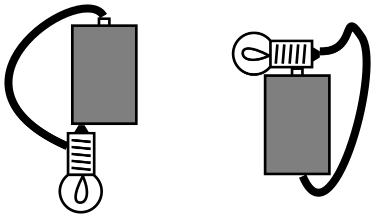
16-d3 (a) $P = I\Delta V = I(IR) = I^2R$
(b) Energy dissipated in those wires is wasted in heating the neighborhood, so we want R to be small.

16-d4 To get negative values of P, you'd have to have opposite signs for I and ΔV. But you can't control the signs of I and ΔV independently — switching the wires reverses both. The current flows in response to the electric force, whose direction is related to the direction of the voltage drop. Another way of looking at it is that the electrons collide with the atoms, causing them to vibrate, and the vibration is what heat is. There is no way that randomly vibrating atoms will all hit the electrons in the same direction, producing a current.

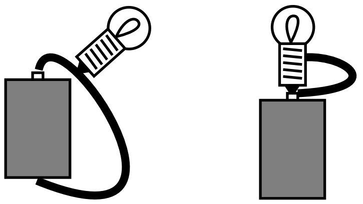
16-d5 In series, they give 11 kΩ. In parallel, they give $(1/1\ \text{k}\Omega + 1/10\ \text{k}\Omega)^{-1} = 0.9\ \text{k}\Omega$.

16-d6 Resistance is inversely proportional to the cross-sectional area. The area depends on the square of the diameter, so the ratio of resistances is $(1.2)^2 = 1.4$.

16-g1



(a) good (b) good

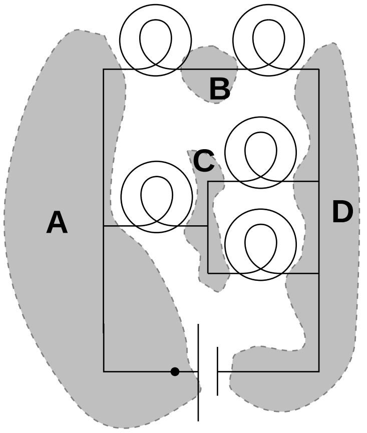


(c) no good (d) no good

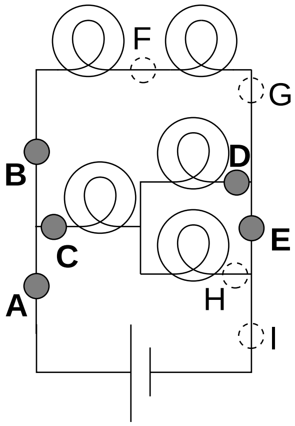
The bulb has two terminals. One is the bump on the end, and the other is the screw-threaded metal jacket. The battery also has two terminals: the bump on one end and the flat surface on the opposite end. In the two good methods, a and b, current can flow out of one terminal of the battery, into one terminal of the bulb, through the filament, out the other terminal of the bulb, and back in to the opposite terminal of the battery. In method c, there is a dead end in the bulb: the current could get in through the bump-shaped terminal, but there would be no way for it to get out of the screw-threaded terminal once it had gone through the filament. The other problem with *c* is that there is an easy way for current to go through the low-resistance wire, so the wire will get very hot and the battery will burn out. In method d, there is a complete loop of wire consisting of the bump-shaped terminal of the bulb, the filament, the screw-threaded terminal, and the wire that comes back to the bump-shaped terminal. This loop is all made out of conductors, and the voltage is equal at all points within a piece of conducting material. Therefore the voltage difference between the two ends of the filament is zero, and by Ohm's law, $V = IR$, no current will flow through the filament. Also, method d gives no way for current to get to the other terminal of the battery.)

16-g2 It's much more practical to measure voltage differences. To measure a current, you have to break the circuit somewhere and insert the meter there, but it's not possible to disconnect the circuits sealed inside the board.

16-g3 There is constant voltage throughout a conductor. The wires are good conductors, but the bulbs aren't, so all we have to do is count how many parts there are in the circuit that consist of nothing but wire. As shown in the figure, there are four such parts. For instance, if we connect one probe of the voltmeter at the dot, and one in region D, we'll get a certain voltage reading, and we'll get the *same* voltage reading even if we move the second probe to some other place in region D. We can get a total of four unique voltage readings, but region A gives zero, so there are only three unique, nonzero readings we can get.



16-g4 There are five unique currents you could measure, by inserting an ammeter at A, B, C, D, and E.

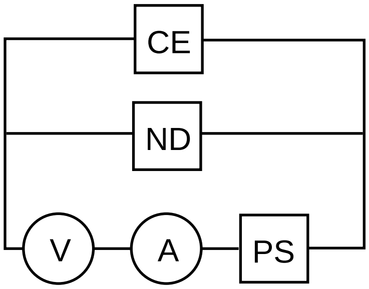


The key here is that charge is conserved. That means that, for example, you'll get exactly the same current reading at B, F, and G, because any charge that flows up through B will also have to go through F and G; it can't "drive off the road," and it can't get used up in the bulbs, because that would violate conservation of charge. Likewise, I and A give the same reading.

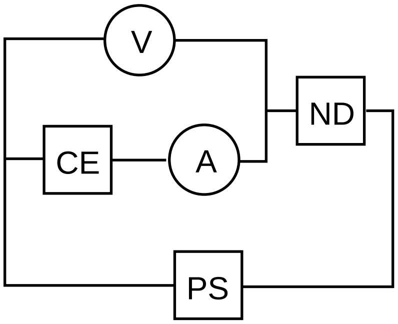
You also can't get two different results from D and H. This isn't because of conservation of charge, it's because the two bulbs are identical, and have the same voltage drop across them.

16-g5 (a) Call the voltages V_5 and V_{10}. All three currents are the same, since it's a single closed loop, and there is no place to store charge. We therefore only need one symbol for current — just call it I. Kirchoff's loop rule says $V_5 + V_{10} = 9$ volts. Ohm's law tells us $V_5 = (I)(5\ \Omega)$ and $V_{10} = (I)(10\ \Omega)$. This means that V_{10} must be twice as big as V_5, and since they have to add up to 9 volts, they must be $V_5 = 3$ V and $V_{10} = 6$ V.
(b) As explained above, all three currents are the same. One way to find the current is by solving the equation $V_5 = (I)(5\ \Omega)$ for I, giving $I = 0.6$ A. Another way would be to divide 9 V by the series resistance of 15 ohms.
(c) The 10-ohm bulb will go out, because its two sides are now connected with wire, and therefore there is no voltage difference between the two sides. The 5-ohm bulb now has the full voltage drop of the battery across it, i.e. 9 volts, and its current will be $(9\ \text{V})/(5\ \Omega) = 1.8$ A. This means it will be brighter than it was before.
(d) You'll see exactly the same thing as in the original circuit. None of the reasoning used in solving part (a) made any assumption about what order the bulbs were in. The current does not "get tired" or "get used up" by having to go through one bulb first before it gets to the other. Conservation of charge says that charge can't "get used up."

16-g6

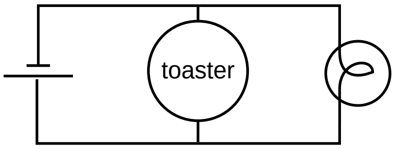


(a) They have correctly set up the CE and ND in parallel. The ammeter is measuring the total current out of the power supply, however, rather than just the current that is going to go through the ND. The voltmeter is hooked up in series, but it should be in parallel. A voltmeter has a very high resistance, so the circuit will not function when the voltmeter is hooked up this way.

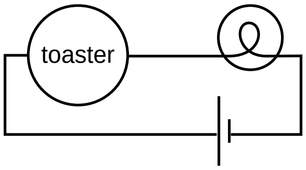


(b) The meters are hooked up correctly, but the CE and ND are in series rather than in parallel. (If they had hooked up the CE and ND in parallel, then they would have had to get the ammeter hooked up correctly to measure the current through the ND rather than the CE, but in this series circuit, both the CE and the ND have the same current through them.)

16-g7



(a)



(c)

(a) See figure.

(b) The E-shaped piece of wire across the top is one big piece of conducting material, so it must have the same voltage throughout it. Similarly, the whole E-shaped piece on the bottom is at the same voltage. Therefore, the voltage difference across each of the three components is the same. Since we know the supply is 110 V, the voltage drops across the toaster and lamp must also be 110 V. Ohm's law gives $I_{toaster} = (110\text{ V})/R_{toaster} \approx 100$ A, and $I_{lamp} = (110\text{ V})/R_{lamp} \approx 50$ A.

(c) In this setup, which is not what's really used in your house, it's not the voltage drops that are equal but the currents — whatever current goes through one component also has to go through the others. The two resistances in series add together to make a total resistance of 3 Ohms, so $I_{toaster} = I_{lamp} = V/3\ \Omega \approx 40$ A. Ohm's law can now be used to find the voltage drop across each appliance, $V_{lamp} = I_{lamp}R_{lamp} = 73$ V, and $V_{toaster} = I_{toaster}R_{toaster} = 37$ V. (I didn't round the voltages off to one sig fig because I wanted to make it clear that they add up to 110 V.)

16-g8 This setup could be made to work just fine at the extreme settings. Let R_k be the (variable) resistance of the knob, and R_h the resistance of the heater.

The "off" position would have to have $R_k = \infty$, i.e., it would be like an open circuit. The total resistance $R = R_k + R_h$ would be infinite as well, zero current would flow, and no power would be dissipated anywhere.

The full-power position would have $R_k = 0$, i.e., it would be like a piece of wire. Current would flow, but there would be no voltage drop across R_k, so the only heating would be in R_h.

The problem occurs at intermediate settings, where R_k is neither zero nor infinity. Here R_k would get hot, wasting power by heating up the knob. At $R_k = R_h$, for example, the knob would be dissipating the same amount of heat as the heating element itself! This would presumably melt the knob, or burn your hand.

16-g9 (a) This has to be true based on the junction rule, so it doesn't tell you anything about the resistance values.

(b) We already knew without even touching the circuit that the currents were going to be equal, and if we find that the voltages are also equal then $V = IR$ says that the resistances must be equal.

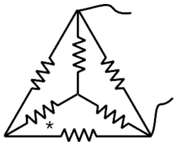
(c) The loop rule tells us that the voltage drops are equal in the parallel circuit, and if we also find out the currents are equal, then $V = IR$ says the resistances must be equal.

(d) This is not useful information for finding out about the resistors, since it has to be true based on the loop rule.

16-g10 All three in series gives one possibility. All three in parallel: one possibility. Two in series, in parallel with the third: three possibilities. Two in parallel, in series with the third: three possibilities. The total number of resistance values you can make is eight.

16-g11 The actual shape is irrelevant; all we care about is what's connected to what. Therefore, we can draw the circuit flattened into a plane. Every vertex of the tetrahedron is adjacent to every other vertex, so any two vertices to which we connect will give the same resistance.

Picking two arbitrarily, we have this:



This is unfortunately a circuit that cannot be converted into parallel and series parts, and that's what makes this a hard problem! However, we can recognize that by symmetry, there is zero current in the resistor marked with an asterisk. Eliminating this one, we recognize the whole arrangement as a triple parallel circuit consisting of resistances R, $2R$, and $2R$. The resulting resistance is $R/2$.

16-g12 The total resistance seen by the voltage source is $x+y$, so the current that flows is $I = \Delta V/(x+y)$. Using the result of problem 1 from ch. 3, the power dissipated in the lamp is $P = I^2y = \Delta V^2y(x+y)^{-2}$. Thus we want to find the value of y that maximizes the quantity $y(x+y)^{-2}$. To find a maximum, we set the derivative equal to zero:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}y}y(x+y)^{-2} \\ &= (x+y)^{-2} - 2y(x+y)^{-3} \end{aligned}$$

Multiplying both sides by $(x+y)^3$ simplifies this to

$$0 = (x+y) - 2y \qquad ,$$

which gives $x = y$.

Physical explanation: Very large values of y are like an open circuit, e.g. like opening a switch. Very little current flows, so no power is dissipated in the lamp. Very small values of y result in more current, but the current is still limited by the resistance of the cord, and this current, passing through a small resistance y, dissipates very little power. This is a specific example of something known as impedance matching: the maximum power is transmitted when the resistance of the load is matched to the resistance of the transmission line.

16-g14 An open switch in (b) is the same circuit as a closed switch in (a): the light bulb is lit, and there's no problem. When you close the switch in (b), it does cause the bulb to go out (basically because the two sides of the bulb are connected with wire and therefore have no voltage difference, but see the more detailed note below).

But huge amounts of current will now flow through the middle branch of the circuit. This is a short, which will certainly kill the battery, and may also cause enough heating to burn up the flashlight.

One subtle issue about this is that you might think closing the switch in (b) would have no effect on the bulb, since you're just adding a new parallel branch to a circuit. This would lead you to believe that although the short would kill the battery and burn things up, the bulb would stay on, contrary to the claim in the problem that it would go out. In fact, the bulb will go out in (b), because it's not a pure parallel circuit. You have to readjust your thinking a little, because ordinarily we assume that wires have negligible resistance, but in a short circuit their resistances aren't negligible anymore, because the equivalent resistance of the whole circuit is so low. That means you can't neglect the resistance of the wires going from the battery to the junctions. You basically have a series circuit now, consisting of the wire from the battery to the junction, the wire through the switch, and the wire from the switch back to the battery. This series circuit shares the voltage drop of the battery three ways, which means that the voltage across the middle piece is greatly reduced.

16-g15 This is a problem that has a trick to it. If you don't think of the trick, it's very difficult, because the circuit can't be broken down into parallel and series parts. The trick is to realize that although the circuit has been drawn in an asymmetric way, it's actually very symmetric. The two rightmost bulbs play identical roles in the circuit, as do the two leftmost one. You can see this if you redraw the circuit so that the top left bulb and the top right bulb are on one horizontal line, and similarly for the bottom left and bottom right. The bulb in the middle has zero current through it by symmetry.

16-g16 Like the preceding problem, this one can't be solved by breaking the circuit down into parallel and series parts. Unlike that one, there is no easy way to solve it by symmetry. We simply have to use the loop rule and the junction rule to write down enough equations to solve for all the unknowns. Let's label the bulbs A, B, C, ... going from top to bottom and left to right, like reading a page of text. For simplicity, we think of the bulbs at the bottom as a single bulb, E, with a resistance of two ohms. (This is the only part of the circuit that can be simplified in this way.) Let A stand for the current through bulb A, and so on. The junction rule then gives

$$\begin{aligned} D &= B+E \qquad \text{and} \\ A+B &= C \qquad . \end{aligned}$$

Since every bulb except for E has a resistance of one ohm, the voltage drop across each bulb equals the current through it. That is, we can use the symbol A interchangeable for both the current through A (in amperes) and the voltage drop across A (in volts). The only exception is the double bulb E, which has a voltage $2E$ across it. We use the loop rule three times:

$$\begin{aligned} A+C &= 1 \\ D+2E &= 1 \\ A &= D+B \qquad . \end{aligned}$$

The result is five equations in five unknowns, which we solve by eliminating one variable after another. The final results are $A = 6/13$, $B = 1/13$, $C = 7/13$, $D = 5/13$, and $E = 4/13$. The total current is $A + D = 11/13$ amps, so the power drawn from the one-volt battery is $(11/13\text{ A})(1\text{ V}) = 11/13$ W.

16-g17 Using the junction rule at the junction A feeds into, we find that the current through A is 300 mA. Using the junction rule at the junction on the left, we then have 600 mA for the total current drawn from the battery, which is also the current through D. Applying the junction rule at the junction in the middle, we find a current of 100 mA for both B and C. All our unknowns are now determined, so we can use the junction on the right side to check that our solution is correct: 500 mA plus 100 mA gives 600 mA, which is indeed what we had for D.

16-g18 (a) Let the voltage of the battery be V. In the original circuit, we have $P = IV = V^2/R$, while in the new one $P' = V^2/(R+r)$. By forming the ratio of these two quantities, we can eliminate the voltage, and $P/P' = 1 + r/R$. The result is $r = (P/P' - 1)R$.

(b) The ratio P/P' is unitless, which makes sense because we're subtracting a unitless 1 from it. The result is that ohms equal ohms, which makes sense.

(c) For $P' = P$, we have $r = 0$, which makes sense: no additional resistance was introduced, no no change occurred. For $P' = 0$, we have $r = \infty$, which also makes sense. For example if the dirt is a perfect insulator, then we have an open circuit.

(d) The current in the modified circuit is $I = V/(R+r)$, which gives $P' = I^2R = V^2R/(R+r)^2$. After a little algebra, the result is

$$r = \left(\sqrt{\frac{P}{P'}} - 1\right)R,$$

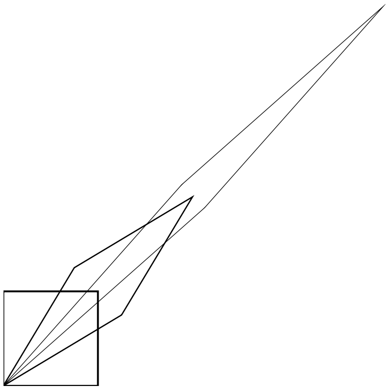
which, remarkably, differs from the result of part a only by the presence of the square root.

16-j1 An (idealized) battery is a circuit element that always maintains the same voltage difference across itself, so by the loop rule, the voltage difference across the capacitor must remain unchanged, even while the dielectric is being withdrawn. The bound charges on the surfaces of the dielectric have been attracting the free charges in the plates, causing them to charge up more than they ordinarily would have. As the dielectric is withdrawn, the capacitor will be partially discharged, and we will observe a current in the ammeter. Since the dielectric is attracted to the plates, positive work is done in extracting it, indicating that there must be an increase in the electrical energy stored in the capacitor. This may seem paradoxical, since the energy stored in a capacitor is $(1/2)CV^2$, and we are decreasing the capacitance. However, the energy $(1/2)CV^2$

is calculated in terms of the work required to deposit the free charge on the plates. In addition to this energy, there is also energy stored in the dielectric itself. By moving its bound charges farther away from the free charges in the plates, to which they are attracted, we have increased their electrical energy. This energy of the bound charges is inaccessible to the electric circuit.

16-j2 Since the circuit is open, the charge has nowhere to go, and we won't observe any current in the ammeter. Since the capacitance q/V is decreasing, the reading on the voltmeter must go up. As with the closed circuit, positive work is done to overcome the attraction between the bound and free charges. In this example, however, the work goes into the increased energy of the higher electric field betweeen the plates.

17-a1 To get good precision, you need to do this on a full-size printout, as suggested by the problem. To keep from using up a whole page on my solutions handouts, I've shown the idea below at about 25% of the actual size.



First I drew the long diagonal twice as long again, meaning about 276 mm. To keep the area the same, I had to make its "cross-bar" half as long as on the preceding parallelogram, or about 17.5 mm. The speed represented by this new parallelogram is equal to the slope of its bottom edge. To get this slope, which is the rise divided by the run, we have to draw the rise and run accurately, i.e., lined up straight with the edges of the paper. I did this by using a second piece of paper as a right angle. The result is that the slope is (rise)/(run) = 91 mm/105 mm = .86, representing 86% of the speed of light.

17-a2 (a) She says her own velocity is zero.

Time dilation happens because of relative motion. We can conclude that A and B are not in motion relative to one another, but that there is relative motion between them and C.

(b) Alice says B is at rest, but C is moving. (c) Betty says A is at rest, but C is moving. (d) Cathy says both A and B are moving, and she sees them moving in the same direction at the same speed.

17-a3 The equation for gamma is $\gamma = 1/\sqrt{1-v^2}$. If v is negative, v^2 still comes out positive. Negative and positive values of v give the same results for gamma.

This makes sense, because positive and negative values of v indicate motion in opposite directions. Relativistic effects like time dilation and length contraction shouldn't come out different if you travel west rather than east, or north rather than south. Gamma indicates how strong these relativistic effects are, so gamma shouldn't depend on the direction of motion. This relates directly to the second postulate stated in section 7.2, that all directions in space have the same properties.

Note that it *is* OK for a directional (vector) quantity like velocity to be connected to some other direction quantity, such as acceleration, by a relation such as $a = dv/dt$. Then an observer who chooses the opposite orientation for their coordinate system will simply flip all the signs of the velocities and accelerations, and there is no way to tell in experiments which choice is "special." That's different from the situation with γ, which is not directional and can be measured without even choosing a coordinate system.

17-a4 (a)

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-(v/c)^2}}\\ &= \frac{1}{\sqrt{1-\left(\frac{17000\ \text{m/s}}{3\times10^8\ \text{m/s}}\right)^2}}\\ &= 1.0000000016\end{aligned}$$

(b) Gamma gives us a comparison of the rate at which time flows on earth and on the probe. A gamma of one would indicate equal rates. Since the gamma is a tiny bit greater than one, the ratio is a tiny bit different from a one-to-one ratio. Gamma differs from one by 1.6×10^{-9}, i.e., 1.6 parts per billion. The disagreement between clocks on the probe and on earth that accumulates over each year is therefore

$$(31 \times 10^6\ \text{s})(1.6 \times 10^{-9}) = 0.050\ \text{s} \qquad .$$

This is a fairly big discrepancy. You might even be able to detect it with a consumer-grade clock. (The way NASA really notices the effect is that when the probe signals back to earth by radio, the radio waves vibrate a little more slowly than they should, i.e., the signal's frequency is shifted a tiny bit.)

17-a5 Let $r = 6.4 \times 10^6$ m be the radius of the earth, $R = 1.5 \times 10^{11}$ m the radius of its orbit, and $T = 1$ year $= 3.2 \times 10^7$ s the period of its orbit. Then the velocity is $v = (2\pi R)/T$, and we have $\gamma = 1/\sqrt{1 - v^2/c^2} = 1 + 4.8 \times 10^{-9}$. The resulting contraction of the diameter is about $(4.8 \times 10^{-9})(2r) = 6$ cm.

17-a6 (a)

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - v^2/c^2}}\\ &= \frac{1}{\sqrt{1 - 9/25}}\\ &= \frac{1}{\sqrt{16/25}}\\ &= \frac{5}{4}\end{aligned}$$

(b) The $(3/5)c$ example came out simple because 3, 4, and 5 are integers that have the relationship $3^2 + 4^2 = 5^2$. This is just like a 3-4-5 right triangle, but there's no triangle involved here! We need to find another set of numbers like this. The simplest way is to reverse the role of the 3 and 4, in which case we get

$$v = \frac{4}{5} \qquad , \qquad \gamma = \frac{5}{3}$$

If that seems like a cheap shot, then you need to find another set of integers that form a right triangle; I was lazy, and just did a Google search for "Pythagorean triple," which turned up $11^2 + 60^2 = 61^2$ as an example. This provides the examples

$$v = \frac{11}{61} \qquad , \qquad \gamma = \frac{61}{60}$$

and

$$v = \frac{60}{61} \qquad , \qquad \gamma = \frac{61}{11} \qquad .$$

18-a1 The electric force vector is

$$\begin{aligned}\mathbf{F}_E &= q\mathbf{E}\\ &= (1\ \text{N})\hat{\mathbf{y}} \qquad .\end{aligned}$$

The magnitude of the magnetic force is

$$\begin{aligned}|\mathbf{F}_B| &= |q\mathbf{v} \times \mathbf{B}|\\ &= q|\mathbf{v} \times \mathbf{B}|\\ &= q|\mathbf{v}||\mathbf{B}|\sin\theta_{vB}\\ &= 2\ \text{N} \qquad ,\end{aligned}$$

and its direction is determined by the right hand rule to be along the z axis, so

$$\mathbf{F}_B = (2\ \text{N})\hat{\mathbf{z}} \qquad .$$

The total force on the particle is

$$\mathbf{F} = (1\ \text{N})\hat{\mathbf{y}} + (2\ \text{N})\hat{\mathbf{z}} \qquad .$$

18-a2 The original rule listed flat fingers, bent fingers, and thumb in the order

$$\mathbf{v} \quad \mathbf{B} \quad \mathbf{F} \qquad .$$

Playing with the model, you'll find that the other two ways that work are

$$\begin{aligned}&\mathbf{B} \quad \mathbf{F} \quad \mathbf{v} \qquad \text{and}\\ &\mathbf{F} \quad \mathbf{v} \quad \mathbf{B} \qquad .\end{aligned}$$

These are like starting later in the original list, and then "wrapping around" when we get to the end. The three that don't work are

B **v** **F** ,
v **F** **B** , and
F **B** **v** .

These are left-handed.

18-a3 There must be either **E** or **B** or both, since some force caused the particle to accelerate. It can't be a pure magnetic field, because $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ is zero for $\mathbf{v} = 0$, so the particle would not have accelerated from rest. It can't be a pure electric field, because a uniform electric field would just have caused the particle to accelerate in a straight line. There must be both **E** and **B** present.

18-a4 (a) $\mathbf{F} = q\mathbf{v} \times \mathbf{B} = 0$, so the particle remains at rest.

(b) $\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q|\mathbf{v}||\mathbf{B}|\sin 0^\circ = 0$, so there is no force, and the particle travels in a straight line with constant speed.

(c) The force will always be perpendicular to the z axis, so since the particle initially has no velocity in the z direction, it will remain in the same plane, parallel to the x-y plane. The particle orbits in a circle in this plane, at constant speed.

18-a5 (a) $I = \lambda v$.

18-a6 (b) $2kI_1I_2L/c^2R$.

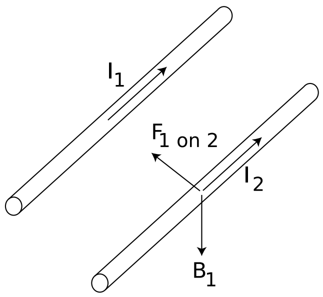
18-a7 The electric force is either in the same direction as the electric field or in the opposite direction, depending on whether the particles are positively or negatively charged. Since the magnetic force is perpendicular to both **v** and **B** and the question states that the electric field is perpendicular to both **v** and **B**, the magnetic force must be either in the same direction as the electric force or in the opposite direction. Since the goal is to make the forces cancel, we'll assume the directions of the fields are chosen so the forces are in opposite directions. That means the forces cancel if they're chosen to have equal magnitudes. The electric force equals $q\mathbf{E}$. The magnitude of the magnetic force equals $qvB\sin\theta$, and since the **v** and **B** vectors are perpendicular, $\sin\theta = 1$. Therefore we need $qE = qvB$. Canceling q from both sides and solving for v, we have $v = E/B$.

18-a8 The second data-point isn't very surprising, because multiplying charge by -2 and the velocity by -1 multiplies the magnetic force by $(-2)(-1) = 2$. The only thing the second data-point really tells us is that the field is purely magnetic. If the **F**, **v**, and **B** vectors had to be perpendicular, there would only be one possibility for this field, $\mathbf{B} = (1\ \text{kT})\hat{\mathbf{z}}$. But a magnetic field along the same direction as the motion exerts no force, so we can throw in an arbitrary x component, giving a solution such as $\mathbf{B} = (1\ \text{kT})\hat{\mathbf{z}} + (37\ \text{kT})\hat{\mathbf{x}}$.

18-a9 It can't be a purely electric field, because then the velocity would have no effect on the force. It can't be purely magnetic, either, because tripling the velocity didn't simply triple the force, as it would if the only force was $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$.

18-a10 (a) The magnitudes are the same, as required by Newton's third law.

(b)



(c) The force is in the opposite direction, again as required by Newton's third law; each wire is attracting the other.

(d) They would reoek each other.

18-a11 (a) The field is coming out of the page, perpendicular to the plane of the coils. See the answer to the self-check.

(b) The current is counterclockwise. One way to tell this is to imagine breaking down each coil into parts. For example, we could reshape them into octagons. Each of the sixteen segments would contribute the same field at the center of the loop, and the direction of the field would be given by the right-hand rule, just as if the segments were long wires.

(c) The electrons have negative charge. To produce a counterclockwise current, they need to be moving clockwise.

(d) It's a repulsion. The electrons in the beam are being pushed inward, away from the coils. This is consistent with the result of the previous problem, where we found that there is a repulsion between currents that are in opposite directions.

18-a12 (a) See the answer to the self-check.

(b)

$$\begin{aligned} a &= \frac{F}{m} \\ \frac{v^2}{r} &= \frac{qvB\sin 90°}{m} \\ \frac{v}{r} &= \frac{q}{m}B \\ r &= \frac{m}{q}\cdot\frac{v}{B} \end{aligned}$$

(c) From the definition $B = F/(qv\sin\theta)$, we see that the Tesla unit breaks down into N·s/(C·m).

$$\begin{aligned} \frac{\text{kg}}{\text{C}}\cdot\frac{\text{m/s}}{\text{T}} &= \frac{\text{kg·m}}{\text{C·s}}\cdot\frac{\text{C·m}}{\text{N·s}} \\ &= \frac{\text{kg·m}}{\text{s}^2}\cdot\frac{1}{\text{N}}\cdot\text{m} \\ &= \text{m} \end{aligned}$$

(d) The equation derived in part b has m and v on the top, so it states that increasing either of these variables will increase the radius of the circle. The opposite is true for q and B.

mass: Increasing the mass of the particles makes it harder for a given force to turn the beam away from its initial path, so the turn will be wider.

charge: Increasing the charge of the particles makes it easier for a given field to act on them, causing them to execute a tighter curve.

velocity: This is like driving a car. Freeway off-ramps have gentle curves, because a high-velocity car is harder to turn.

field: Increasing the magnetic field results in more force on each particle, making the beam curve more tightly.

(e)

$$\begin{aligned} v &= \omega r \\ \omega &= \frac{v}{r} \\ &= \frac{qB}{m} \end{aligned}$$

18-a13 (a) It can't be a purely frictional force, because then the particle wouldn't speed back up again. Magnetic fields don't do work, so it also can't be a pure magnetic field, since the particle's kinetic energy is changing. A magnetic field would make a force perpendicular to the particle's velocity vector, causing it to curve, which it isn't doing. Since there is zero magnetic force, there is probably zero magnetic field (although it is possible that the magnetic field is nonzero and parallel to the motion, causing zero force). We're left with the possibility of either a pure electrical force, or a mixture of electrical and kinetic frictional forces. If there was friction in the mix, then the friction would reverse its direction when the particle came back, while the electrical force remained constant. The forces would therefore be reinforcing for half of the motion, but at least partially canceling during the other half. The particle's acceleration, however, is the same in the two halves of the motion. We therefore conclude that an electrical force is the only one acting. The field must be down and to the left, along the direction of motion.

(b) As in part a, there is no evidence for any magnetic field. There must be friction here: if the force was purely electrical, then the particle would have constant acceleration, but it actually displays an abrupt change in acceleration from a nonzero value to zero. The simplest explanation of the observed motion is that there is no electrical field, and the particle is simply decelerated by kinetic friction. It's also possible, however, that there is an electric field along the direction of motion, either working against friction or cooperating with it in slowing the particle down. If there is an electric field, then once the particle stops, there must be a static frictional force keeping it from getting going again.

(c) The dots are equally spaced, so no work is ever being done on the particle. Therefore there can't be any electric field: the only way an electric field could avoid doing work on the particle would be if it was perpendicular to the motion, but since the particle is curving, and we're given that the electric field is uniform, it's not possible to have an electric field that is always perpendicular to the motion. Friction would also do work, so there is no friction. There must be a magnetic field into the page to produce the inward force required for circular motion.

(d) Work is being done, since the particle is slowing down and then speeding up. If we assume that the possible frictional force is exerted by some surface that is at rest with respect to the grid on the paper, then we can rule out a purely frictional force: kinetic friction would decelerate the sliding particle, but couldn't speed it up. Taking the time interval to be one and the spacing of the grid to be one, the velocity vectors are $3\hat{\mathbf{x}}+\hat{\mathbf{y}}$, $2\hat{\mathbf{x}}+2\hat{\mathbf{y}}$, etc., with the x component always decreasing by 1 and y always increasing by 1. Therefore the acceleration is a constant one up and to the left, at a 45-degree angle with respect to the grid. This force must be coming from an electric field in this direction. If there was a kinetic friction force, it would always oppose the motion, so the force wouldn't continue to be in the same direction as the direction of the motion changed. If there was a magnetic field, the magnetic force would change its strength and direction as the particle's speed changed, so again, the force wouldn't be constant.

18-a14 (a) The time for one orbit is $\Delta t = 2\pi r/v$, so $I = e/\Delta t = ve/2\pi r = 1$ mA.

(b) Using the equation for the field at the center of a current loop, and plugging in the current from the previous homework, we get about 10 T (a huge field!). (This is a one-sig-fig answer, because the radius of the orbit is only given to one sig fig.)

(c) Since the proton is not moving, it experiences no magnetic force. (Actually the proton and electron orbit around their common center of mass, but since the proton is two thousand times more massive, it essentially doesn't move. The force will be very small, but not quite zero.)

(d) No. The proton is not moving (or just barely moving), so it makes no magnetic field (or very nearly none).

(e) No. Newton's third law says that forces come in equal and opposite pairs, so an object can't exert a force on itself.

(f) Yes, using Coulomb's law, the force is 9×10^{-8} N.

(g) Yes, using Newton's law of gravity, the force is 3.6×10^{-47} N.

(h) The magnetic force is zero. The gravitational force is ridiculously small compared to the electric force. It's the electric force that keeps it in orbit.

18-a15 Particle 1 isn't moving, so it experiences only an electric force. The electric field causing this force must be

$$\frac{(1\ \text{N})\hat{\mathbf{y}}}{1\ \text{C}} = (1\ \text{N/C})\hat{\mathbf{y}} \qquad .$$

Particle 2 experiences the same force as particle 1, even though it is moving along the x axis. Therefore, the magnetic field, if any, must lie along the x axis.

The magnetic and electric forces on particle 3 cancel, so the magnetic force on it must be $(-1\ \text{N})\hat{\mathbf{y}}$. A magnetic field along the x axis that produces this force must be $(-1\ \text{T})\hat{\mathbf{x}}$.

18-a16 Four times the current makes four times the magnetic field, which makes $4^2 = 16$ times the energy.

18-a17 The field at any point on the axis can be found by adding the contributions to the field from the two loops. By symmetry, both these fields are along the axis, so we can add them without worrying about vector addition. The whole problem just deals with comparisons, so the only part of the equation in section 24.2 we need to worry about is $(b^2+z^2)^{-3/2}$. Also, we can just let b equal 1, and interpret z as z/b.

(a) Evaluating $(1+z^2)^{-3/2}$ at $z=0$ and 0.5, and

1, we get 1.0000, 0.7155, and 0.3536. The total field at the center ($z = 0.5$ relative to both coils) is 0.7155×2, and the field in the center of one of the coils is $1.0000 + 0.3536$. The percentage drop is about 5.4% (expressed as a percentage of the stronger field).

(b) By experimenting with different values of h, we find a solution at about $h/b = 1.2$, i.e., a little larger than the Helmholtz value of $h/b = 1$.

18-a18 (a) Using the equation for the magnetic field at the center of a current loop, we get 0.063 mT.

(b) Along the positive z axis.

(c) It would make a field with the same magnitude along the negative x axis.

(d) The total field is the vector sum of the two. Since they're perpendicular, the magnitude of the sum is given by the Pythagorean theorem, coming out to be 0.09 mT. It points at a 45° angle between the two, in the $x - z$ plane.

18-a19 The hard part is visualizing the directions of the four fields that are adding together. Looking along the length of the apparatus, two of the fields point at a 45-degree angle down and to the left, while the other two point at 45 degrees down and to the right. By vector addition, this means that the total field is $2\sqrt{2}$ times the individual ones. Each individual field is $2kI/c^2R = 2\sqrt{2}kI/c^2b$ so the total field is $8kI/c^2b$.

18-a20 At first it might seem possible, since we're trying to determine six numbers (E_x, E_y, E_z, B_x, B_y, and B_z), and two experiments would provide six pieces of raw data (F_{1x}, F_{1y}, F_{1z}, F_{2x}, F_{2y}, and F_{2z}). Ordinarily, we expect that six equations in six unknowns can be solved uniquely. However, this isn't true if the equations are redundant. For example, the system of equations

$$\begin{aligned} a &= 2b \\ 2a &= 4b \end{aligned}$$

can't be solved for a and b.

In fact, we can never solve these six equations for the six unknowns. Suppose we fix the design of your experiment by specifying $\mathbf{v}_1$ and $\mathbf{v}_2$. We may as well use unit charge in both cases, so these two velocities give a complete specification of the design of the experiment. But given this design, there will always be different fields that give the same results. To see this, let's find $\mathbf{E}$ and $\mathbf{B}$ that produce zero force in both measurements. We want fields such that

$$\begin{aligned} \mathbf{E} + \mathbf{v}_1 \times \mathbf{B} &= 0 \\ \mathbf{E} + \mathbf{v}_2 \times \mathbf{B} &= 0 \end{aligned} \qquad .$$

Subtracting,

$$(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{B} = 0 \qquad .$$

It's easy to show that the procedure won't work if the two velocities are the same, so let's just assume the vector $\mathbf{v}_1 - \mathbf{v}_2$ is nonzero. We can take $\mathbf{B}$ to lie along $\mathbf{v}_1 - \mathbf{v}_2$, and let $\mathbf{E} = -\mathbf{v}_1 \times \mathbf{B}$.

18-a21 If the little solenoid's current goes the same way as the big one's, then their fields reinforce and the energy density is higher than if they were separate. It's true that the fields are filling some of the same space, which would tend to reduce the total energy, but since the energy density depends on the square of the field, the increase in energy density is a bigger effect than the decrease in the amount of space. For instance, if the two fields are equal, and the volumes are nearly equal, then the combined field has an energy density four times greater, which more than compensates for the fact that the fields are occupying half the volume compared to the separate setup.

If the little solenoid's current goes the opposite way, their fields cancel at least partially, and the energy is less than when they were separate. The stable configuration is the one with the opposite currents.

18-a22 All that really matters is the cross-sectional area, a, of the smaller solenoid, and their separate fields, b and B. As the small one penetrates the big one to a depth x, we replace a volume xa from each one's interior with a new, overlapped volume xa in which the field is $B + b$.

(a) The difference in energy is

$$\frac{c^2}{8\pi k}\left[(B+b)^2 - B^2 - b^2\right] xa = \frac{c^2}{4\pi k} \cdot Bbxa \qquad .$$

(b) The energy is proportional to x, so for every millimeter deeper that we push the small one, the same amount of work is done. The force is $F = \mathrm{d}W/\,\mathrm{d}x = (c^2/4\pi k)Bba$.

18-a23 (a) Let's use the same $x-y-z$ coordinate system employed in the book's derivation of B_y for a sheet of current. Each sheet experiences a B_y and B_z due to the other one. The z component of the magnetic field produces only sideways forces, which don't contribute to the attraction or repulsion. (The total sideways force also has to cancel by symmetry, so all it does is produce a stress that tends to stretch each sheet in the y direction.) The y component, however, does produce a repulsion according to the right-hand rule. As derived in the book, we have $B_y = 2k\eta\gamma/c^2$, and because $w \gg h$, we have $\gamma \approx \pi$, giving $B_y \approx 2\pi k\eta/c^2$ everywhere except at the edges. The edges, however, don't contribute appreciably to the total force, since $w \gg h$. Since B_y is approximately constant across each sheet, the force can be computed using the same equation as for a thin wire immersed in the same field:

$$F = I\ell B = \frac{2\pi k I^2 \ell}{c^2 w} \qquad .$$

(b)

$$\begin{aligned} F &= I\ell B \\ &= I\ell \cdot \frac{2kI}{c^2 h} \\ &= \frac{2kI^2\ell}{c^2 h} \end{aligned}$$

(c) In part a, the force is proportional to $1/w$, because the width of the strips is what determines the typical distance between points on one strip and points on the other. In b, the force depends on h for similar reasons. The factor of π is because in a, many points are closer than w, but in b, no points are closer than h.

18-a24 The compass will respond to the combined fields of the magnet and the earth. There is no fundamental difference between the two- and three-dimensional cases, so for simplicity let's imagine the mystery magnet and the compass both being glued to a flat piece of wood, which we can orient at any angle in a horizontal plane. Before gluing the compass to the wood, we determine the direction of the earth's field. Then, with the compass glued down, we fiddle with the board's orientation until we get the compass to point in the direction of the earth's field again. In this orientation, the mystery magnet's field is along the same line as the earth's field. But is it north (reinforcing the earth's field) or south (opposing it)? If the magnet's field is stronger than the earth's, then we can easily tell by flipping the board 180 degrees; the compass will point in the direction of the magnet's field in both cases. If the magnet's field is weaker than the earth's, it's a little harder. The simplest technique is to turn the magnet away from the aligned direction by some small angle, say 5 degrees, and observe which way the needle is deflected. By one of these two methods, we can determine the direction of the magnet's field for once and for all, and draw it with a marking pen on the board.

Now we have to find the strength of the magnet's field. This is fairly easy. For instance, we can turn the magnet so that it's field is at a 90-degree angle with respect to the earth's, and measure the angle by which the compass deviates from magnetic north. We're now adding two field vectors that are at 90 degrees to one another, and this vector addition can be visualized in terms of a right triangle if we put the two vectors tip to tail. The tangent of the angle tells us the ratio of the two legs, which is the ratio of the magnet's field to the earth's.

18-d1 (a) The Biot-Savart law gives

$$\mathbf{B} = \frac{kI}{c^2}\int \frac{\mathrm{d}\boldsymbol{\ell} \times \mathbf{r}}{r^3}$$

Here $\mathbf{r}$ is the vector from a point on the wire to the point at which we're calculating the field. Let the wire lie on the x axis, and let the point of interest be on the y axis at $y = R$. Let θ be the angle between the $\mathbf{r}$ vector and the wire, measured on the right side so that at large negative x, $\theta \to 0$. The magnitude of the cross product occurring in the integrand is $|\,\mathrm{d}\boldsymbol{\ell}||\mathbf{r}|\sin\theta$, or

$\mathrm{d}x \cdot r \cdot \sin\theta$. We then have for the magnitude of the field

$$B = \frac{kI}{c^2}\int_{-\infty}^{\infty}\frac{\sin\theta\,\mathrm{d}x}{r^2}.$$

(b) The variables occurring in the integrand are x, r, and θ. We need to pick one of these variables to work with and reexpress the others in terms of it. Rewriting the variables x and r in terms of θ gives $x = -R\cot\theta$, $\mathrm{d}x = R\sin^{-2}\theta\,\mathrm{d}\theta$, and $r = R/\sin\theta$. These substitutions change the integral into

$$B = \frac{kI}{c^2R}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta.$$

As promised, the integral now reduces to a trivial form that equals 2.

18-d2 This is a simple generalization of the preceding problem involving a square. The contribution from each side is proportional to $(1/h)\cos\theta$, where θ is the angle between a side and a line from the center to an adjacent vertex. Let the perimeter be ℓ, so that $\cos\theta = (\ell/2n)/R$. Then the field is proportional to $\ell/hR = \ell/\rho^2$.

In the case of $n = 2$, we have $h = 0$, so the field is infinite, which makes sense. For $n \rightarrow \infty$, we have $B = (Ik/c^2)\ell/\rho^2 = (Ik/c^2)2\pi r/r^2$, which is correct.

18-d3 Note that in the Biot-Savart law, the variable $\mathbf{r}$ is defined as a vector that points from the current to the point at which the field is being calculated, whereas in the polar coordinates used to express the equation of the spiral, the vector more naturally points the opposite way. This requires some fiddling with signs, which I'll suppress, and simply identify $\mathrm{d}\boldsymbol{\ell}$ with $\mathrm{d}\mathbf{r}$.

$$\mathbf{B} = \frac{kI}{c^2}\int\frac{\mathrm{d}\boldsymbol{\ell}\times\mathbf{r}}{r^3}$$

The vector $\mathrm{d}\mathbf{r}$ has components $\mathrm{d}x = w(\cos\theta - \theta\sin\theta)$ and $\mathrm{d}y = w(\sin\theta + \theta\cos\theta)$. Evaluating the vector cross product, and substituting θ/w for r, we find

$$\begin{aligned}
\mathbf{B} &= \frac{kI}{c^2w}\int\frac{\theta(\cos\theta\sin\theta - \theta\sin^2\theta - \cos\theta\sin\theta - \theta\cos^2\theta)\,\mathrm{d}\theta}{\theta^3}\\
&= \frac{kI}{c^2w}\int\frac{\mathrm{d}\theta}{\theta}\\
&= \frac{kI}{c^2w}\ln\frac{\theta_2}{\theta_1}\\
&= \frac{kI}{c^2w}\ln\frac{b}{a}
\end{aligned}$$

18-g2 The field of a wire is $B = 2kI/c^2R$, in the direction tangent to the circle centered on the wire and passing through the point of interest. As suggested in the problem, the field approaches zero as R approaches infinity, so we can neglect the contributions from the short, infinitely distant pieces of wire at the ends of the rectangle. Of the two infinite wires, each makes an equal contribution to the circulation, so we only need to integrate over one of them, and then double that result. Let the wire lie along the z axis, and let the wires be parallel to the x axis. Then

$$\begin{aligned}
\Gamma_B &= 2\int_{-\infty}^{+\infty}\mathbf{B}\cdot\hat{\mathbf{x}}\,\mathrm{d}x\\
&= 2\int_{-\infty}^{+\infty}|\mathbf{B}||\hat{\mathbf{x}}|\cos\theta_{Bx}\,\mathrm{d}x\\
&= 2\int_{-\infty}^{+\infty}\frac{2kI}{c^2R}\cdot 1\cdot\frac{h}{R}\,\mathrm{d}x\\
&= \frac{4kIh}{c^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{R^2}\\
&= \frac{4kIh}{c^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{h^2+x^2}\\
&= \frac{4kIh}{c^2}\cdot\frac{1}{h}\tan^{-1}\frac{x}{h}\Big]_{-\infty}^{+\infty}\\
&= \frac{4\pi kI}{c^2}
\end{aligned}$$

This is the result expected from Ampère's law.

19-a1 (a) Since the wire is moving through a magnetic field, the charges in the wire are also moving through a magnetic field, and they feel a force. The electrons in in the metal are free to flow, so a current is established. This current dissipates energy into heat, and by conservation of energy, that heat energy has to come from somewhere. The only place it can come from is the kinetic energy of the wire, so we conclude that the wire is decelerating. Therefore the force on the wire must be to the left. (b) The electrons have a lot of random thermal motion, but let's just talk about their motion on the average. On the average, an electron in the wire is moving to the right. Using the right-hand rule, we find that the vector cross product $\mathbf{v} \times \mathbf{B}$ points out of the page, but since q is negative, the force acting on the electron, $q\mathbf{v} \times \mathbf{B}$, points into the page. This tells us that the electrons flow counterclockwise as seen from above. Once this current is established, the velocity vector of the electrons in the rolling wire acquires a new component, into the page. This creates an additional magnetic force on the electrons, with $\mathbf{v} \times \mathbf{B}$ to the right and $q\mathbf{v} \times \mathbf{B}$ to the left, in agreement with the answer to part a. (c) Since we've already established the direction of the force in excruciating detail, I'll completely ignore signs in everything that follows. Faraday's law tells us

$$-\frac{\partial \Phi_B}{\partial t} = \Gamma_E \qquad .$$

We're told that the resistance of the circuit is dominated by the resistance of the rolling wire. In a series circuit, the current is the same everywhere, and by Ohm's law, we find that the voltage drop across the high-resistance rolling wire must be much greater than the voltage drop around the U. This means that the voltage drop V across the wire is essentially the same as Γ_E. Letting R be the resistance of the rolling wire, we have

$$\begin{aligned} I &= \frac{\Gamma_E}{R} \\ &= \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}t}(BA) \\ &= \frac{1}{R}B\frac{\mathrm{d}A}{\mathrm{d}t} \end{aligned}$$

Let the wire's velocity be v, and let its length be ℓ. We then have $\mathrm{d}A/\,\mathrm{d}t = v\ell$, so

$$I = Bv\ell/R \qquad .$$

The problem now boils down to finding the force on a current-carrying wire in a perpendicular magnetic field. As worked out in problems 9 and 10 in this chapter (whose answers are in the back of the book), the force is $I\ell B$, so our final result for the force on the wire is

$$F = B^2\ell^2 v/R \qquad .$$

Let's interpret this result. Note that B is squared. This means that if we reversed the direction of the field, the force would still be in the same direction. That makes sense, because the answer to part a never required us to use the direction of the field. We also find that v is *not* squared. That also makes sense. If the wire was rolling in the other direction, then the logic of part a would still require that kinetic energy be converted into heat, so the wire would still have to decelerate, and the force acting on it would have to flip in order to be opposite to its motion.

19-a2 (a) The changing magnetic field induces an electric field, which causes a current to run around the loop. The resistance of the wire causes energy to be turned into heat. This energy can only be coming from the mechanical energy (KE and gravitational PE) of the loop, so it must be making the loop fall with an acceleration less than g. The force must be upward.

(b) An electron in the wire has negative charge, so the force it feels in a magnetic field is in the direction of $-\mathbf{v} \times \mathbf{B}$. This is into the page for an electron in the top, and out of the page for an electron in the bottom. Because the field is stronger at the top, that force dominates, and a clockwise current will be established. Once the current is established, an electron at the top has a nonvanishing v_y, and this new component gives rise to a new component of $-\mathbf{v} \times \mathbf{B}$, pointing up. The force at the bottom of the loop is downward, but it's weaker, so the total force is upward.

(c) As worked out in problems 9 and 10 in this chapter (whose answers are in the back of the book), the force exerted on a wire by a magnetic field is $I\ell B$. Let subscripts t and b stand for the top and bottom of the loop. Ignoring signs except where they're necessary, we have:

$$\begin{aligned}F &= F_t - F_b\\ &= I\ell k(z_t - z_b)\\ &= IAk\\ &= Ak\Gamma_E/R\\ &= \frac{Ak}{R}\frac{\partial\Phi_B}{\partial t}\\ &= \frac{Ak}{R}\frac{\partial}{\partial t}\int kz\,\mathrm{d}y\,\mathrm{d}z\\ &= \frac{Ak^2(\text{width})}{R}\frac{\partial}{\partial t}\frac{z_t^2 - z_b^2}{2}\\ &= \frac{Ak^2(\text{width})}{R}\left(z_t\frac{\mathrm{d}z_t}{\mathrm{d}t} - z_b\frac{\mathrm{d}z_b}{\mathrm{d}t}\right)\\ &= \frac{Ak^2(\text{width})(\text{height})v}{R}\\ &= \frac{A^2k^2v}{R}\end{aligned}$$

The behavior of this expression is sensible. The argument in part a didn't depend on the sign of k, so it makes sense that the final answer depends on k^2, and is therefore also independent of the sign of k. It also makes sense that the final answer has v to the first power. If we threw the loop upward, the original argument would still imply that mechanical energy was being taken out and dumped into ohmic heating, but the force required would be downward. Having R in the denominator makes sense. For instance, if we cut the wire, then it would be an open circuit, and there would be no way to establish a current; this corresponds to the case of $R = \infty$, which would give $F = 0$ in our expression.

19-a3 (a) If your socks come out of the dryer clinging together, the electric fields around them are caused by charges, $4\pi kq_{in}$. Your car's alternator is a generator that runs based on the principle that a curly electric field is caused by a changing magnetic field, $-\partial\Phi_B/\partial t$. You're reading right now using light, which is an electromagnetic wave; such a wave could not have its curly magnetic field unless such a field could be caused by a time-varying electric field, $-\partial\Phi_E/\partial t$. Electromagnets are used at the junkyard to pick up cars and scrap metal. The magnetic field is caused by a current, $4\pi kI_{through}$.

(b) Wherever k occurs, we replace it with $1/4\pi\epsilon_o$, and c^2 can be replaced with $4\pi k/\mu_o = 1/\mu_o\epsilon_o$. The results are

$$\begin{aligned}\Phi_E &= \frac{1}{\epsilon_o}q_{in}\\ \Phi_B &= 0\\ \Gamma_E &= -\frac{\partial\Phi_B}{\partial t}\\ \frac{1}{\mu_o\epsilon_o}\Gamma_B &= \frac{\partial\Phi_E}{\partial t} + \frac{1}{\epsilon_o}I_{through}\end{aligned}$$

19-a4 The relationship between the field and the voltage is

$$E = \frac{V}{d} \qquad ,$$

so for the flux we have

$$\Phi_E = \frac{\pi R^2 V}{d} \qquad .$$

Based on the symmetry of the capacitor, we clearly need to choose an Amperian surface that is a circle centered on the axis. To extract any information about the magnetic field at radius R, we're going to have to make it a circle of radius R. Using Maxwell's equation for the circulation of the magnetic field,

$$\begin{aligned}c^2\Gamma_B &= \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}\\ &= \frac{\pi R^2 s}{d}\end{aligned}$$

Now the magnetic field could have components in various directions, but only the component tangent to the circle will contribute to this circulation. That means we can't use this technique to get the components of the field in the radial direction and in the direction parallel to the axis. Let's compute the tangential component, B_t, first, and then worry about the other two.

$$c^2 \cdot 2\pi R B_t = \frac{\pi R^2 s}{d}$$
$$B_t = \frac{sR}{2c^2 d}$$

The radial component must vanish because if it didn't, the magnetic field would be diverging from or converging on the capacitor, and Gauss's law for magnetism forbids this.

The axial component is a little trickier to rule out. If there was an axial component, then the field pattern would have to point in some direction along the axis, let's say up. Since the field cannot diverge or converge, the field pattern would then have to come back down in the region outside the capacitor. But then the magnetic field would have a circulation around a loop lying in an up-down plane. This would violate Maxwell's equation for the circulation of B, because there is no electric flux through such a surface (the electric field doesn't twirl around the axis), and there are no currents through such a surface (the currents are all axial and radial currents in the wires and the plates of the capacitor).

19-d1 Because $E = cB$ in an electromagnetic wave, and $v < c$ for a material particle, it follows that $F_B < F_E$. Therefore if we want to minimize the total force, the best we can do is to arrange for the magnetic force to be as strong as possible and opposite in direction to the electric force, so as to get as much cancellation as possible.

Let the wave have its electric field along the x axis, its magnetic field along the y axis, and therefore its direction of propagation along the z axis. If the particle is positively charged, then the electric force is in the positive x direction. We'd like to arrange for a magnetic force in the negative x direction. We can accomplish that by having the particle move in the positive z direction, i.e., along with the wave. Since this is perpendicular to the magnetic field, the magnetic force is maximized, which is what we want.

Nothing really changes if it's a negative charge; it's just that the directions of the two forces are reversed.

19-d2 (a) We have

$$\mathrm{d}U_e = \frac{1}{8\pi k} E^2 \, \mathrm{d}v \qquad \text{and}$$
$$\mathrm{d}U_m = \frac{c^2}{8\pi k} B^2 \, \mathrm{d}v \qquad .$$

In an electromagnetic wave, $B = E/c$, so

$$\mathrm{d}U_m = \frac{1}{8\pi k} E^2 \, \mathrm{d}v$$
$$= \mathrm{d}U_e \qquad .$$

(b)

$$|\,\mathrm{d}\mathbf{p}| = \frac{\mathrm{d}U}{c}$$
$$= \frac{2\,\mathrm{d}U_e}{c}$$

The electric and magnetic fields are perpendicular to each other in an electromagnetic wave, so

$$|\mathbf{E} \times \mathbf{B}| \, \mathrm{d}v = EB \, \mathrm{d}v$$
$$= \frac{E^2}{c} \, \mathrm{d}v$$
$$= \frac{8\pi k}{c} \, \mathrm{d}U_e \qquad .$$

Finally,

$$|\,\mathrm{d}\mathbf{p}| = \frac{EB}{4\pi k} \, \mathrm{d}v \qquad .$$

(The Poynting vector is conventionally defined as this momentum density multiplied by an additional factor of c^2.)

19-d3 (a) The definition of power gives $U = Pt$, and in time t the beam travels a distance $x = ct$ and fills a volume $v = Ax = Act$, so $U/v = Pt/Act = P/Ac$. (b) In time t, the beam travels a distance ct, and fills a volume Act. We need to relate the energy in this volume to the amplitudes of the fields. Using the result of the previous problem, we know that half the energy is due to the electric field, and half is due to the magnetic field, so as a shortcut we can find the energy due to the electric field, and double it:

$$U = 2U_e \tag{24.10}$$

The electric field is not constant throughout this volume; it's a sine wave. That means we need to do an integral over the volume,

$$\begin{aligned}
U &= 2U_e \\
&= 2\int \mathrm{d}U_e \\
&= 2\int \frac{1}{8\pi k}E^2\,\mathrm{d}v \\
&= \frac{1}{4\pi k}\int E^2\,\mathrm{d}v \\
&= \frac{1}{4\pi k}\int \tilde{E}^2 \sin^2(\text{stuff})\,\mathrm{d}v \\
&= \frac{\tilde{E}^2}{4\pi k}\int \sin^2(\text{stuff})\,\mathrm{d}v
\end{aligned}$$

But when you square a sine wave that goes from -1 to 1, you get a sine wave that goes from 0 to 1. The average of such a sine wave is 1/2, so we have

$$\begin{aligned}
U &= \frac{\tilde{E}^2}{8\pi k}V \\
&= \frac{\tilde{E}^2 Act}{8\pi k}
\end{aligned}$$

Now to relate this to the power,

$$\begin{aligned}
P &= \frac{U}{t} \\
&= \frac{\tilde{E}^2 Ac}{8\pi k}
\end{aligned}$$

$$\begin{aligned}
\tilde{E} &= \sqrt{\frac{8\pi kP}{Ac}} \\
\tilde{B} &= \frac{\tilde{E}}{c} \\
&= \sqrt{\frac{8\pi kP}{Ac^3}}
\end{aligned}$$

(c) Plugging in, we get $\tilde{E} = 1.01 \times 10^3$ V/m and $\tilde{B} = 3.36 \times 10^{-6}$ T. The electric field is about a thousand times weaker than the field in a thunderstorm. This sort of makes sense, because a beam of sunlight isn't enough to rip apart the atoms in the air as in a bolt of lightning. The magnetic field is also quite weak compared to everyday fields like the Earth's field.

19-d4 Suppose we have such a wave, with the fields lying along the y axis and the wave propagating along the x axis. Then we could construct an Ampèrian loop lying in the x-y plane such that Γ_E would be nonzero. According to Maxwell's equations, this would require there to be a time-varying magnetic flux through the loop. But if the magnetic field itself lies in the x-y plane, then the magnetic flux through a loop in the x-y plane is identically zero, since the field vector is parallel to the area vector.

19-d5 (a)

$$\begin{aligned}
a_x &= E\frac{q}{m} \\
&= \frac{\tilde{E}q}{m}\sin\omega t \\
v_x &= \frac{\tilde{E}q}{m\omega}(1-\cos\omega t) \\
a_z &= \frac{q}{m}v_x B \\
&= \left(\frac{q\tilde{B}}{m}\right)\left(\frac{\tilde{E}q}{m\omega}\right)(1-\cos\omega t)\sin\omega t \\
&= \left(\frac{q^2\tilde{E}\tilde{B}}{m^2\omega}\right)(-\frac{1}{2}\sin 2\omega t + \sin\omega t) \\
v_z &= \left(\frac{q^2\tilde{E}\tilde{B}}{m^2\omega}\right)\left(-\frac{1}{2\omega}\cos 2\omega t + \frac{1}{\omega}\cos\omega t\right) \\
z &= \left(\frac{q^2\tilde{E}\tilde{B}}{m^2\omega}\right)\left(-\frac{1}{4\omega^2}\sin 2\omega t + \frac{1}{\omega^2}\sin\omega t\right) \\
&= \left(\frac{q^2\tilde{E}\tilde{B}}{m^2\omega^3}\right)\left(-\frac{1}{4}\sin 2\omega t + \sin\omega t\right)
\end{aligned}$$

For me, this is a counterintuitive result: I would have expected the particle to oscillate at the frequency of the wave, rather than at a combination ω and 2ω.

(b) We need to find the maximum of the expression $-(1/4)\sin 2x+\sin x$. Setting the derivative equal to zero, we have

$$\begin{aligned}-\frac{1}{2}\cos 2x+\cos x&=0\\ \cos^2 x-\cos x-\frac{1}{2}&=0\end{aligned}$$

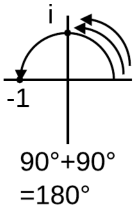
$$\begin{aligned}x&=\cos^{-1}\frac{1-\sqrt{3}}{2}\\ &=111.5^\circ\end{aligned}$$

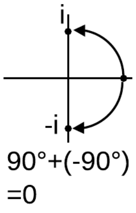
At this value of x, the expression has a value of 1.101.

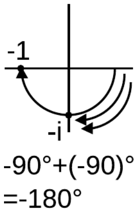
19-d6 (a) For a material object, $\mathbf{p}=m\mathbf{v}$. The velocity vector reverses itself, but mass is still positive, so the momentum vector is reversed.
(b) In the forward-time universe, conservation of momentum is $\mathbf{p}_{1,i}+\mathbf{p}_{2,i}=\mathbf{p}_{1,f}+\mathbf{p}_{2,f}$. In the backward-time universe, all the momenta are reversed, but that just negates both sides of the equation, so momentum is still conserved.

20-a1 $\arg i=90^\circ$, $\arg(-i)=-90^\circ$, $\arg 37=0$

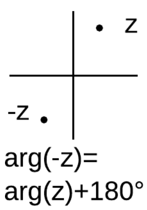
20-a2

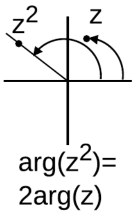


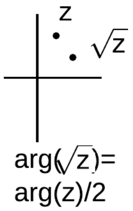




20-a3







20-a4 1, i, -1, $-i$

20-a5

$$\begin{aligned}&|1+i|=\sqrt{1^2+1^2}=\sqrt{2}\\ &\arg(1+i)=45^\circ\\ &\left|\frac{1}{1+i}\right|=\frac{1}{|1+i|}=1/\sqrt{2}\\ &\arg\left(\frac{1}{1+i}\right)=-\arg(1+i)=-45^\circ\end{aligned}$$

The real part is $(1/\sqrt{2})\cos(-45^\circ)=1/2$. The imaginary part is $(1/\sqrt{2})\sin(-45^\circ)=-1/2$.

20-d1 (a) As suggested in section 10.5.9, let's identify sinusoidal functions with points in the complex plane. The function $4\sin\omega t$ corresponds to the real number 4, and $3\cos\omega t$ to $3i$. Their sum is the function corresponding to $4+3i$. Now we have to go backwards and find what sinusoidal function $4+3i$ represents. The magnitude of this complex number represents the amplitude, A, of the result, so $A=\sqrt{4^2+3^2}=5$. The argument represents the phase angle $\delta=\tan^{-1}(3/4)=37^\circ$.
(b)

$$\begin{aligned}5\sin(\omega t+37^\circ)&=5[\sin\omega t\cos 37^\circ+sin37^\circ\cos\omega t]\\ &=3.9\sin\omega t+3.0\cos\omega t\end{aligned}$$

To within rounding errors, this verifies that the result was correct.

20-d2 (a) Multiplication adds the arguments, so raising something to the third power triples its

argument. The result is 3θ (or $3\theta + 2\pi n$, where n is any integer).
(b) If $a^3 = z$, then the argument of a should be 1/3 that of z. (There would be additional possibilities such that $\arg 3a = \arg z + 2\pi n$, but we're only talking about the first quadrant, so that can't happen.) The only one that looks like it has 1/3 the argument of z is x.
(c) The argument has to be 1/3 that of $a + bi$, i.e., $(1/3)\tan^{-1}(b/a)$.
(d) The easy way to do the cube root is to use the fact that if $y = x^3$, then $|y| = |x|^3$, and $\arg y = 3 \arg x$. Therefore if y is the quantity inside the exponent in the denominator, we have $|x| = (\sqrt{8})^{1/3} = \sqrt{2}$, and the easiest possibility for the argument is $\arg x = (1/3) \arg y = \pi/4$. (There are other possibilities such as $11\pi/12$, because arguments that differ by a multiple of 2π represent the same thing.) Division divides the magnitudes, giving a magnitude of 1. Division subtracts the arguments, giving 0. Therefore the result of the calculation is 1.

20-d3 When you have an exponent with a funky base, the standard method for reexpressing it in terms of base e is to apply the identity $\ln(a^b) = b \ln a$, which gives $a^b = \exp(b \ln a)$. This gives $i^i = e^{i \ln i}$. This reduces the problem to one of finding the natural log of i. Euler's formula tells us that raising e to the power $(\pi/2)i$ gives i. (Actually there is more than one solution, but we'll come back to that.) Taking $\ln i = (\pi/2)i$, we find

$$i^i = e^{-\pi/2} \qquad .$$

Euler's formula tells us that e^x is the same as $e^{x+2\pi i}$, $e^{x+4\pi i}$, etc. That means that $\ln i$ has more than one possible value. It can be $(\pi/2 + 2\pi n)i$, where n is any integer. This means that i^i could be any real number of the form $e^{-\pi/2-2\pi n}$. It shouldn't be too surprising that an exponent such as i^i can have more than one value. For example, $4^{1/2}$ can equal either 2 or -2.

20-g1 (a) By definition, one henry is 1 J/A^2. The quantity $L\,\mathrm{d}I/\,\mathrm{d}t$ therefore has units of

$$\begin{aligned}\left(\frac{\text{J}}{\text{A}^2}\right)\left(\frac{\text{C/s}}{\text{s}}\right) &= \left(\frac{\text{J}}{(\text{C/s})^2}\right)\left(\frac{\text{C/s}}{\text{s}}\right)\\ &= \frac{\text{J}}{\text{C}}\\ &= \text{V}\end{aligned}$$

(b) A farad is defined as 1 C^2/J, so RC has units of

$$\begin{aligned}\Omega\left(\frac{\text{C}^2}{\text{J}}\right) &= \left(\frac{\text{V}}{\text{A}}\right)\left(\frac{\text{C}^2}{\text{J}}\right)\\ &= \left(\frac{\text{J/C}}{\text{C/s}}\right)\left(\frac{\text{C}^2}{\text{J}}\right)\\ &= \text{s}\end{aligned}$$

(c) L/R has units of

$$\begin{aligned}\left(\frac{\text{J}}{\text{A}^2}\right)\Omega^{-1} &= \left(\frac{\text{J}}{\text{A}^2}\right)\left(\frac{\text{A}}{\text{V}}\right)\\ &= \frac{\text{J}}{\text{A}\cdot\text{V}}\\ &= \frac{\text{J}}{\text{W}}\\ &= \frac{\text{J}}{\text{J/s}}\\ &= \text{s}\end{aligned}$$

20-g2 The definition of inductance is $U = LI^2/2$. If we think of the pair of inductors as one big inductor, then applying this definition, I has to be the total current that goes in. This gets split in half, so the energy stored in a single inductor is $U_{one} = L_{one}(I/2)^2/2$, and the total energy is $U = 2U_{one} = L_{one}I^2/4$. Equating this to $LI^2/2$, we get $L = L_{one}/2$. You get half the inductance by putting two in parallel.

20-g3 We normally think of wires as having zero resistance. If the wire has some inductance, then the added impedance is higher at high frequencies, so stray inductance will be most important at high frquencies. The impedance of a capacitor, however, varies inversely with frequency, so stray capacitance will be most important at low frequencies.

20-g4 The reasoning is very similar to the reasoning by which we derived the impedance of a capacitor. Suppose an inductor has a current $I = \tilde{I}\sin\omega t$ passing through it. The voltage difference across the inductor is

$$\begin{aligned} V &= L\,\frac{\mathrm{d}I}{\mathrm{d}t} \\ &= L\omega\tilde{I}\cos\omega t \qquad . \end{aligned}$$

The amplitude of this voltage is $L\omega\tilde{I}$. The impedance is defined as the amplitude of the voltage divided by the amplitude of the current, so we get $L\omega$.

20-g5 Solving $\omega \approx 1/\sqrt{LC}$ for C, we get $C = 1/L\omega^2 = 1/4\pi^2L^2f^2$. Looking at the nearest handy FM radio dial, the range of frequencies is about 88-108 MHz, so the capacitor needs to cover the range from about 2.2 to 3.3 pF.

20-g6 (a)

$$\begin{aligned} Z &= (1/Z_R + 1/Z_C)^{-1} \\ &= [1/R + 1/(-i/\omega C)]^{-1} \\ &= (2.70 + 6.28i)^{-1} \times 10^5\ \Omega \end{aligned}$$

The magnitude of a complex number's inverse is the inverse of its magnitude, so

$$\begin{aligned} |Z| &= \frac{1}{\sqrt{2.70^2 + 6.28^2}} \times 10^5\ \Omega \\ &= 1.5 \times 10^4\ \Omega \end{aligned}$$

(rounding the final result to two sig figs). The argument of a complex number's inverse is minus the argument of the original number, so $\arg Z = -\tan^{-1}(6.28/2.70) = -69°$.

(b) The amplitude of the current is

$$\begin{aligned} \tilde{I} &= \tilde{\Delta V}/|Z| \\ &= 68\ \text{nA} \qquad . \end{aligned}$$

Suppose the current has the phase of a sine wave. Then we associate the current with a point on the real axis, and multiplying it by the impedance will give a voltage that has an argument of $-69°$, i.e., closer to the negative imaginary axis. The negative imaginary axis represents something with the phase of $-\cos$, and if you sketch the graphs of sin and $-\cos$, you'll see that $-\cos$ is behind by 90 degrees in phase. Therefore the voltage is behind the current, by 69 degrees.

Another method for doing this, which avoids having to memorize what represents what in the complex plane, is to note that an impedance with an argument of $-90°$ (on the negative imaginary axis) is capacitive, so our parallel RC circuit, with its impedance having an argument of $-69°$, is dominated by its capacitor. For a purely capacitive circuit, the current leads the voltage, so we know that will also be true in our circuit. The mnemonic "eVIL" may come in handy here. It tells you that the voltage (V) leads the current (I) in an inductive (L) circuit, whereas it's the other way around in a capacitive one. (Some people know this as "ELI the ICE man," with E as a notation for emf, which is a concept, closely related to voltage, that we'll encounter later.)

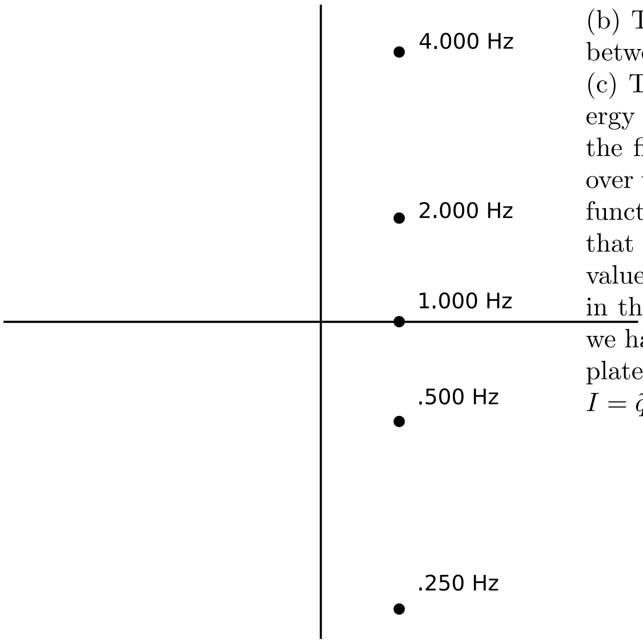
20-g7 (a) Impedances add in series, so

$$\begin{aligned} Z &= R + i\omega L - \frac{i}{\omega C} \\ &= 1 + i(\omega - 1/\omega) \qquad , \end{aligned}$$

where, for ease of writing, I've omitted the units and sig figs. The results are:

ω (Hz)	Z (Ω)
0.250	$1.000 - 3.750i$
.500	$1.000 - 1.500i$
1.000	1.000
2.000	$1.000 + 1.500i$
4.000	$1.000 + 3.750i$

Here's what it looks like in the complex plane:



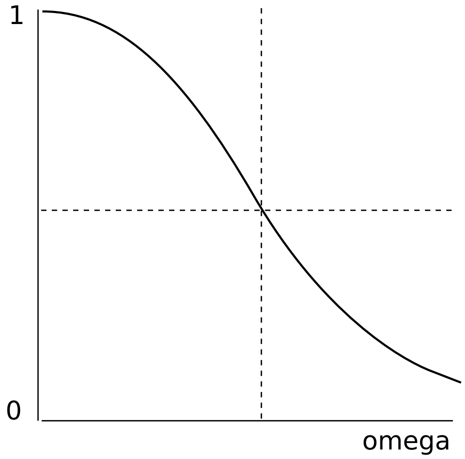
(b) The resonant frequency is $\omega \approx 1/\sqrt{LC} =$ 1.000 Hz. This is the frequency at which the magnitude of the impedance is the smallest, so the largest current will flow for a given voltage.
(c) $f = \omega/2\pi = 0.1592$ Hz

20-g8 In parallel, it will be the inverse impedances that add. The total inverse impedance is $1/Z = (0.4 - 0.3i)\ \text{k}\Omega^{-1}$. The real part of this can be contributed by the inverse resistance of a resistor with a resistance of 2.5 kΩ. For the imaginary part, we need a component whose inverse impedance $1/Z$ will be a negative imaginary value. That means that its impedance Z has to be a positive imaginary value, so it has to be an inductor, just as in the original series setup. We then have $L = Z/(i\omega) = Z/(2\pi i f) = 10$ mH.

20-g9 (a) A DC current can't flow through a capacitor, so the steady-state behavior of the circuit is that the capacitor is charged up, and there is no current flowing through the inductor. That means there's an electric field in the capacitor, but no magnetic field in the inductor. All of the energy is stored in the capacitor, and none in the inductor.
(b) The energy is being shuffled back and forth between the capacitor and the inductor.
(c) The fields oscillate sinusoidally, and the energy in a field is proportional to the square of the field, so the variation in each field's energy over time has the form of a square of a sinusoidal function. The square of a sine wave is a function that oscillates between 0 and 1, and its average value is 1/2, so in each case, the average energy in the field equals half the maximum energy. If we have $q = \tilde{q}\sin\omega t$ for the charge on one of the plates of the capacitor, then differentiatoin gives $I = \tilde{q}\omega\cos\omega t$. Then

$$\begin{aligned}
u &= \frac{\overline{U_C}}{\overline{U_L}+\overline{U_C}} \\
&= \frac{U_{C,max}}{U_{L,max}+U_{C,max}} \\
&= \left(\frac{U_{L,max}}{U_{C,max}}+1\right)^{-1} \\
&= \left(\frac{(1/2)L\tilde{I}^2}{(1/2C)\tilde{q}^2}+1\right)^{-1} \\
&= \left(\frac{LC\tilde{q}^2\omega^2}{\tilde{q}^2}+1\right)^{-1} \\
&= \frac{1}{LC\omega^2+1}
\end{aligned}$$

The figure shows the behavior of this expression as a function of frequency. The capacitor's share is 100% at zero frequency, in agreement with part a. It drops to 50% at resonance ($\omega = 1/\sqrt{LC}$), and at twice the resonant frequency, it equals 20%.



21-a1 (a) According to the ideal gas law, $PV = nkT$, standard values of pressure and temperature give $V \propto n$, since k is also a constant.
(b)

$$
\begin{aligned}
V &= nkT/P \\
&= (6.0 \times 10^{23})(1.38 \times 10^{-23}\ \mathrm{J/K})(273\ \mathrm{K}) \\
&\quad /(1.01 \times 10^{5}\ \mathrm{Pa}) \\
&= 2.2 \times 10^{-2}\ \mathrm{J/Pa} \\
&= 2.2 \times 10^{-2}\ \mathrm{N{\cdot}m/(N/m^2)} \\
&= 2.2 \times 10^{-2}\ \mathrm{m^3} \\
&= 2.2 \times 10^{4}\ \mathrm{cm^3} \\
&= 22\ \text{liters}
\end{aligned}
$$

21-a2 The motion of the piston through an infinitesimal distance $\mathrm{d}x$ will increase the interior volume of the cylinder by $A\,\mathrm{d}x$, where A is the cylinder's cross-sectional area. The work done is $\mathrm{d}W = F\,\mathrm{d}x = (PA)(\mathrm{d}V/A) = P\,\mathrm{d}V$.

Note that the result is not true in general for finite rather than infinitesimal changes. If we attempt to carry through the same proof for finite changes, it fails because $W \neq F\Delta x$ when F depends on x.

21-a3 (a) We have

$$
\begin{aligned}
\mathrm{d}P &= \rho g\,\mathrm{d}y \\
\Delta P &= \int \rho g\,\mathrm{d}y,
\end{aligned}
$$

and since we're taking water to be incompressible, and g doesn't change very much over 11 km of height, we can treat ρ and g as constants and take them outside the integral.

$$
\begin{aligned}
\Delta P &= \rho g \Delta y \\
&= (1.0\ \mathrm{g/cm^3})(9.8\ \mathrm{m/s^2})(11.0\ \mathrm{km}) \\
&= (1.0 \times 10^{3}\ \mathrm{kg/m^3})(9.8\ \mathrm{m/s^2})(1.10 \times 10^{4}\ \mathrm{m}) \\
&= 1.0 \times 10^{8}\ \mathrm{Pa} \\
&= 1.0 \times 10^{3}\ \mathrm{atm}.
\end{aligned}
$$

The precision of the result is limited to a few percent, due to the compressibility of the water, so we have at most two significant figures. If the change in pressure were exactly a thousand atmospheres, then the pressure at the bottom would be 1001 atmospheres; however, this distinction is not relevant at the level of approximation we're attempting here.
(b) Since the air in the bubble is in thermal contact with the water, it's reasonable to assume that it keeps the same temperature the whole time. The ideal gas law is $PV = nkT$, and rewriting this as a proportionality gives

$$V \propto P^{-1},$$

or

$$\frac{V_f}{V_i} = \left(\frac{P_f}{P_i}\right)^{-1} \approx 10^3.$$

Since the volume is proportional to the cube of the linear dimensions, the growth in radius is about a factor of 10.

21-a4 (a) This can't be done simply by changing the units of k. The zero of the Celsius scale is not a temperature at which the pressure in an ideal gas goes to zero.
(b) This won't work, because the conversion between mass and number of atoms isn't universal. It depends on the element.
(c) Yes, in this system the unit of energy would be 1 $\mathrm{g{\cdot}cm^2/s^2} = 10^{-7}$ J. The Boltzmann constant has SI units of joules per kelvin. Therefore the value of the Boltzmann constant would be greater in the cgs system by a factor of 10^7.

21-d1 (a) Looking up the relevant masses, the result is 6.7×10^{-27} kg. (The electrons' masses are nearly negligible.)
(b) n=1 kg/(6.7×10^{-27} kg)= 1.5×10^{26}
(c) Approximating helium as an ideal gas, we can ignore the interactions between atoms, so there is no electrical energy to take into account, only kinetic. The thermal energy of the gas is $E = n\bar{K} = (3/2)nkT$. A change in temperature of one degree requires a change in energy of $(3/2)nk(1\text{ K}) = 3.1\times10^3\text{ (J/K)K} = 3.1\times10^3$ J.
21-d2 The ideal gas law says $PV = nkT$. The volume is the same as before, and the temperature is the same, since the gas is back in thermal equilibrium with the room. Since n is doubled, P must be doubled as well.
21-d3

$$\begin{aligned}P &= \frac{n}{V}kT\\ &= \left(10^{-5}\text{ cm}^{-3}\right)\left(1.38\times10^{-23}\text{ J/K}\right)\left(10^{3}\text{ K}\right)\\ &= \left(10\text{ m}^{-3}\right)\left(1.38\times10^{-23}\text{ J/K}\right)\left(10^{3}\text{ K}\right)\\ &\sim 10^{-19}\text{ J}\cdot\text{m}^{-3}\\ &= 10^{-19}\text{ N}\cdot\text{m}\cdot\text{m}^{-3}\\ &= 10^{-19}\text{ Pa}\end{aligned}$$

21-d4 (c) $n\approx16$
21-d5 (a) Roughly speaking, the thermal energy is $\sim k_BT$ (where k_B is the Boltzmann constant), and we need this to be on the same order of magnitude as ke^2/r (where k is the Coulomb constant). For this type of rough estimate it's not especially crucial to get all the factors of two right, but let's do so anyway. Each proton's average kinetic energy due to motion along a particular axis is $(1/2)k_BT$. If two protons are colliding along a certain line in the center-of-mass frame, then their average combined kinetic energy due to motion along that axis is $2(1/2)k_BT = k_BT$. So in fact the factors of 2 cancel. We have $T = ke^2/k_Br$.
(b) The units are K = (J·m/C^2)(C^2)/((J/K)·m), which does work out.
(c) The numerical result is $\sim10^{10}$ K, which as suggested is much higher than the temperature at the core of the sun.

21-d6 The heat required in order to raise the temperature of a monoatomic gas by ΔT is $(3/2)nk\Delta T$. The rate of temperature increase is therefore

$$\begin{aligned}\frac{\Delta T}{\Delta t} &= \frac{2P}{3nk}\\ &= 1.5\times10^{-8}\text{ K/s.}\end{aligned}$$

I was surprised by how small this was. It would take several years to heat by one degree.

21-d7 (a) By equipartition, the average kinetic energy is $(3/2)kT$.
(b) Equating this to $(1/2)mv^2$ gives $\overline{v^2} = 3kT/m$. This comes out large because m is so small for an electron compared to an atom.

21-d8 The mean translational kinetic energy per molecule is $(1/2)kT$ in both cases. Therefore the typical speeds are related by $mv^2\propto$ const., or $v\propto m^{-1/2}$. We then have $v_D/v_H = (m_D/m_H)^{-1/2} = \sqrt{2/3}$.

21-g1 If the full-sized brick A undergoes some process, such as heating it with a blowtorch, then we want to be able to apply the equation $\Delta S = Q/T$ to either the whole brick or half of it, which would be identical to B. When we redefine the boundary of the system to contain only half of the brick, the quantities ΔS and Q are each half as big, because entropy and energy are additive quantities. T, meanwhile, stays the same, because temperature isn't additive — two cups of coffee aren't twice as hot as one. These changes to the variables leave the equation consistent, since each side has been divided by 2.

21-g2 The energy of 1.3 J equals the amount Q gained by the spoon and minus the amount $-Q$ lost by the coffee. The change in entropy is

$$\Delta S = \frac{Q}{T_\text{spoon}} - \frac{Q}{T_\text{coffee}} = 7.2\times10^{-4}\text{ J/K.}$$

21-g3 (a)

$$\begin{aligned}
\text{eff} &= \frac{Q_L}{W} \\
&= \frac{Q_L}{Q_H - Q_L} \qquad \text{[cons. of energy]} \\
&= \left(\frac{Q_H}{Q_L} - 1\right)^{-1} \\
&= \left(\frac{T_H}{T_L} - 1\right)^{-1} \qquad \text{[Carnot engine]} \\
&= \left(\frac{308\ \text{K}}{300\ \text{K}} - 1\right)^{-1} \\
&= 38 \\
\text{EER} &\approx (10/3)(\text{eff}) \\
&\approx 150
\end{aligned}$$

(b)

$$\begin{aligned}
\Delta S &= \frac{Q_H}{T_H} - \frac{Q_L}{T_L} \\
&= \frac{W + Q_L}{T_H} - \frac{Q_L}{T_L} \qquad \text{[cons. of energy]} \\
&= W\left(\frac{1 + \text{eff}}{T_H} - \frac{\text{eff}}{T_L}\right) \\
&= 3 \times 10^4\ \text{J/K}
\end{aligned}$$

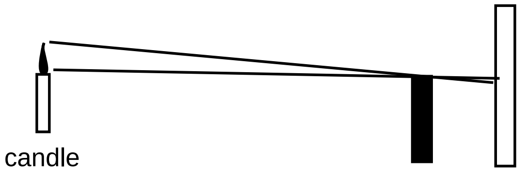
The result is positive, so it's consistent with the second law.

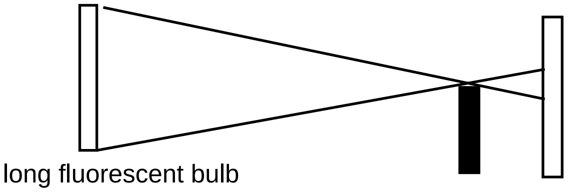
21-g4 (a) $\sim 2 - 10\%$ (b) 5% (c) The high end for the body's actual efficiency is higher than the limit imposed by the laws of thermodynamics. However, the high end of the 1-5 watt range quoted in the problem probably includes large people who aren't just lying around. Still, it's impressive that the human body comes so close to the thermodynamic limit.

21-g5 (a) The equipartition theorem says that we get $(1/2)kT$ worth of energy for every term in the energy that is the square of a variable. The variables here are the n momenta of the particles, so we have $E = (1/2)nkT$. That is, the heat capacity is one third that of a gas in three dimensions.
(b) The number of dimensions matter because in three dimensions, if you want to heat the gas, you have to supply enough energy to create more motion along all three axes.
(c) The energy does not break down into terms that involve squares of variables. Therefore the equipartition theorem does not hold. The heat capacity will not be the same as in part a, and the equipartition theorem doesn't give us any easy way to calculate it.

21-g6 (a) Looking up the relevant density for air, and converting everything to mks, we get a frequency of 730 Hz. This is on the right order of magnitude, which is promising, considering the crudeness of the approximation. (b) This brings the result down to 400 Hz, which is amazingly close to the observed frequency of 300 Hz.

22-a1 The fuzzy part of the shadow is the part that is only illuminated by some of the rays from the light source. This is the area between the extreme rays shown in the figures. The fuzzy area is larger in the second figure.

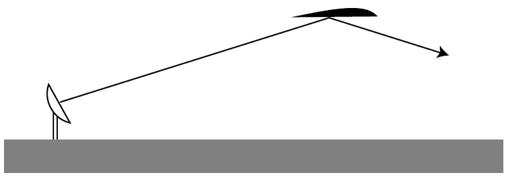




22-a2 The time it takes light to travel a few meters, say 3 m, is $(3\ \text{m})/(3 \times 10^8\ \text{m/s}) = 10^{-8}$ s.

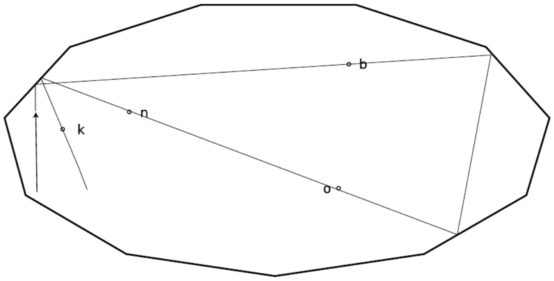
22-a3 $f = c/\lambda = (3 \times 10^8\ \text{m/s})/(10^{-9}\ \text{m}) = 3 \times 10^{17}$ Hz, or call it 10^{17} Hz, since it's just an estimate. It would be UV or an x-ray.

22-a4 Because the surfaces are flat, you get specular reflection. In specular reflection, all the reflected rays go in one direction. Unless the plane is directly overhead, that direction won't be the right direction to make the rays come back to the radar station.

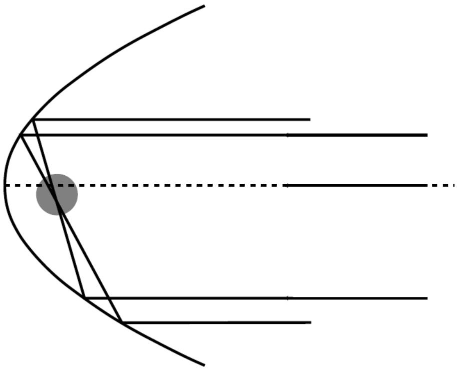


This is different from a normal plane, which has complicated, bumpy surfaces. These surfaces give diffuse reflection, which spreads the reflected rays randomly in more or less every possible direction.

22-a5 It spells “bonk.”



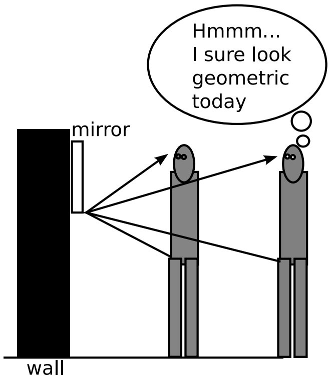
22-a6 (a) The rays all cross at pretty much the same place, given the accuracy with which we can draw them.
(b) It could be used to cook food, for instance. All the sunlight is concentrated in a small area.
(c) Put the lightbulb at the point where the rays cross. The outgoing rays will then form a parallel beam going out to the right.



22-d1 The image is at an equal distance behind the mirror, so the distance between the man and his image is twice his distance from the mirror, and his separation from his image is reducing at a rate of 2.0 m/s.

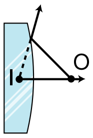
22-d2 If you’re going to see a particular part of your body, then rays from that part of your body have to be able to go to the mirror, be reflected, and go into your eye. If we’re trying to find the lowest part you can see, then we’re interested in the case where the ray hits the very bottom of the mirror. Backing up doesn’t help, because the triangles are isoceles by the law of specular reflection, and the top of the base is fixed in height.

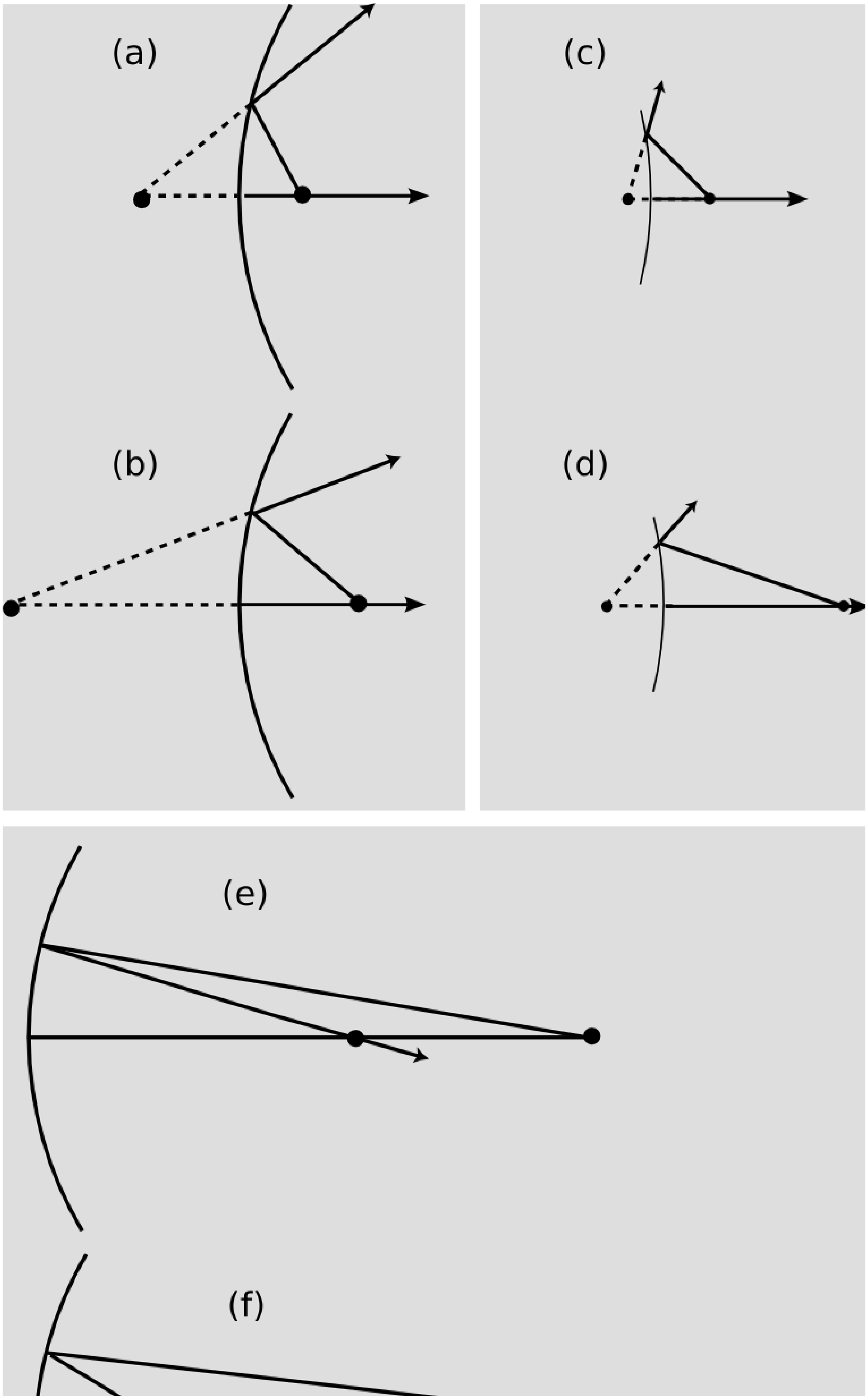
Another way of understanding the result is that if you go twice as far away, the mirror subtends half the angle, but your image also subtends half the angle, so the mirror still encompasses the same amount of your image.



22-g1 (a) The image is closer to the surface of the mirror, so the magnification is less than one.

(b) It’s virtual, because the rays only appear to have passed through point I. There is no way it could make a real image, because in order to make a real image it would have to bring the rays back together.



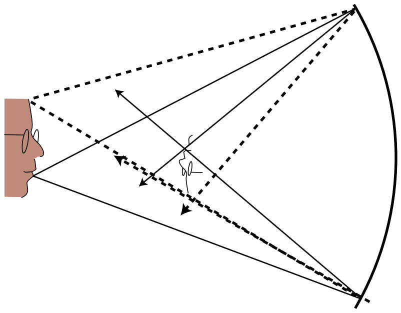

(a)
(b)
(c)
(d)
(e)
(f)

22-g2 See figures a-f.
a and b: converging mirror, virtual image; increased object distance gives increased image distance
c and d: diverging mirror, virtual image: increased object distance gives increased image distance
e and f: converging mirror, real image: increased object distance gives decreased image distance

22-g3 The magnification is the ratio of the image's size to the object's size. It has nothing to do with the person's location. The angular magnification, however, does depend on the person's location, because things farther away subtend smaller angles. The distance to the actual object is not changed significantly, since it's zillions of miles away in outer space, but the distance to the image does change if the observer's point of view changes. If you can get closer to the image, the angular magnification is greater.

22-g4 Usually you need to draw in a normal in order to get an accurate drawing of the reflected rays. Here, however, we already have one set of rays, and it's pretty easy to do the new rays so that they fan in or fan out by the correct angle compared to the old ones.

As claimed, the image of the forehead (where the dashed rays cross) is below the image of the lips.



22-g5 An image is where *all* the rays that started from a certain point come together for one big family reunion. Only 3 represents an image. Points like 1, 2, and 4 look special, but only because of the particular rays we chose to trace, out of the infinite number of rays fanning out from the point on the object. If we had chosen a different set of rays, they would all have crossed at 3 as before, but not at 1, 2, and 4.

22-g6 The problem can be simplified conceptually by replacing the hypothetical square watch with a square hole in a piece of paper. This version is also easy to construct and test experimentally. The bundle of rays transmitted through the hole is identical to the bundle that would be reflected by a specular-reflecting square of the same size. Let A be the aperture size, i.e., a linear dimension of the hole, and let d be the distance from the aperture to the surface on which the spot is being projected. The results depend on how A and d compare. When d is small compared to A, the spot is square, but when d is large compared to A, we get a pinhole camera, which forms a circular spot shaped like the sun.

22-g7 For a flat mirror, d_i and d_o are equal, so the magnification is 1, i.e., the image is the same size as the object.

22-g8 We can tell that the object in the drawing is infinitely far away, because the rays are parallel. The focal length is defined in terms of the image distance in the case where the object is infinitely far away, so the focal length equals the image distance. Tracing the rays, we find that they cross at about 11 mm from the mirror, so that's its focal length.

22-g9 The diameters of the mirrors are irrelevant. The focal length simply relates to the curvature of the mirror, a more strongly curved mirror having a shorter focal length. From shortest focal length to longest, the ranking of the mirrors is C, B, A.

22-g10 (a) The object distance is less than the focal length, so the image is virtual: because the object is so close, the cone of rays is diverging too strongly for the mirror to bring it back to a focus. (b) Now the object distance is greater than the focal length, so the image is real. (c),(d) A diverging mirror can only make virtual images.

22-g11 (a) In problem #2 we found that the equation relating the object and image distances was of the form $1/f = -1/d_i + 1/d_o$. Let's make $f = 1.00$ m. To get a virtual image we need

$d_o < f$, so let $d_o = 0.50$ m. Solving for d_i, we find $d_i = 1/(1/d_o - 1/f) = 1.00$ m. The magnification is $M = d_i/d_o = 2.00$. If we change d_o to 0.55 m, the magnification becomes 2.22. The magnification changes somewhat with distance, so the store's ad must be assuming you'll use the mirror at a certain distance. It can't have a magnification of 5 at all distances.
(b) Theoretically yes, but in practical terms no. If you go through a calculation similar to the one in part a, you'll find that the images of both planets are formed at almost exactly the same d_i, $d_i = f$, since $1/d_o$ is pretty close to zero for any astronomical object. The more distant planet has an image half as big ($M = d_i/d_o$, and d_o is doubled), but we're talking about *angular* magnification here, so what we care about is the angular size of the image compared to the angular size of the object. The more distant planet has half the angular size, but its image has half the angular size as well, so the angular magnification is the same. If you think about it, it wouldn't make much sense for the angular magnification to depend on the planet's distance — if it did, then determining astronomical distances would be much easier than it actually is!

22-g12 (a) This occurs when the d_i is infinite. Let's say it's a converging mirror creating a virtual image, as in problems 2 and 3. Then we'd get an infinite d_i if we put $d_o = f$, i.e., the object is at the focal point of the mirror. The image is infinitely large, but it's also infinitely far away, so its angular size isn't infinite; an angular size can never be more than about 180° since you can't see in back of your head!.
(b) It's not possible to make the magnification infinite by having $d_o = 0$. The image location and object location are related by $1/f = 1/d_o - 1/d_i$, so $1/d_i = 1/d_o - 1/f$. If d_o is zero, then $1/d_o$ is infinite, $1/d_i$ is infinite, and d_i is zero as well. In other words, as d_o approaches zero, so does d_i, and d_i/d_o doesn't blow up. Physically, the mirror's curvature becomes irrelevant from the point of view of a tiny flea sitting on its surface: the mirror seems flat to the flea. So physically the magnification would be 1, not infinity, for very small values of d_o.

22-g13 If the sound gets concentrated again at your ear, it must be a real image. You're at the center of the sphere formed by extending the surface. We've already found in the text that for a real image formed by a converging mirror, the equation for the image's location is $1/f = 1/d_\mathrm{o} + 1/d_i$. If the object and image distances are equal, then $1/f = 2/d_\mathrm{o}$, or $f = d_\mathrm{o}/2$. The focal length is half the distance from the surface to your head.

22-g14 (a) Let the virtual image formed by the top mirror lie at a height b above the top mirror. Then

$$\frac{1}{f} = \frac{1}{h} - \frac{1}{b} \qquad .$$

Solving for b, we have

$$b = \left(\frac{1}{h} - \frac{1}{f}\right)^{-1} \qquad .$$

The second image is at a distance h from the bottom mirror, so

$$\begin{aligned}
\frac{1}{f} &= \frac{1}{b+h} + \frac{1}{h} \\
&= \frac{1}{h + \left(\frac{1}{h} - \frac{1}{f}\right)^{-1}} + \frac{1}{h}
\end{aligned}$$

(b) Multiplying both sides by h,

$$\begin{aligned}
\frac{h}{f} &= \frac{h}{h + \left(\frac{1}{h} - \frac{1}{f}\right)^{-1}} + 1 \\
&= \frac{1}{1 + \left(1 - \frac{h}{f}\right)^{-1}} + 1 \\
x &= \frac{1}{1 + (1-x)^{-1}} + 1
\end{aligned}$$

This simplifies somewhat if we define a new variable $u = 1 - x$:

$$\begin{aligned} -u &= \frac{1}{1+1/u} \\ -u(1+1/u) &= 1 \\ -u - 1 &= 1 \\ u &= -2 \\ x &= 3 \\ h &= 3f \end{aligned}$$

Here's why this is a hard problem. The solution $h = 3f$ is unphysical. If h was greater than f, then the first image would have been real, not virtual, which would be inconsistent with the assumptions we made in the first place. The trick is in realizing that there was a hidden assumption involved in the step where we multiplied both sides by $1 + 1/u$, which was that $1 + 1/u$ was a real number. However, if $u = 0$, then $1 + 1/u$ is infinite, and we can't multiply both sides by it. That's the second solution referred to in the problem:

$$\begin{aligned} u &= 0 \\ x &= 1 \\ h &= f \end{aligned}$$

In other words, the first image is at infinity. Images at infinity are the crossing-over point between real and virtual images. That means it didn't actually matter that we assumed the first image would be virtual. We could have gotten the same answer by assuming it was real.

22-g15 (a) For a virtual image formed by a converging mirror, the extreme possibilities are $d_o \rightarrow 0$, which gives $M \rightarrow 1$ because you can't see the curvature of the mirror from up close, and $d_o = f$, which gives parallel reflected rays, $M = \infty$. The result is that $1 < M \leq \infty$.

(b) For a real image formed by a converging mirror, the extremes are obtained when either $d_o = \infty$ or $d_i = \infty$. We have $0 \leq M \leq \infty$.

For a diverging mirror, $d_o \rightarrow 0$ gives $M \rightarrow 1$ for the same reason as in part a, while $d_o \rightarrow \infty$ gives $M \rightarrow 0$. The result is $0 < M < 1$.

22-g16 (a) Initially, the object is so close to the mirror that the mirror's curvature is irrelevant — any curve looks flat if you're close enough to it. Therefore for $d_o << f$, we have $d_i \approx d_o$, and the image's velocity equals the object's. As the object's distance approaches infinity, on the other hand, the image approaches a position that is one focal length above the mirror. Therefore,the image must accelerate from rest, move upward a distance equal to the focal length, and decelerate again, approaching zero velocity.

(b) The only two variables occurring in the statement of the problem are f and g, and there is only one way to combine these to make something with units of velocuty, which is to form the expression $\sqrt{fg}$. The answer must equal $\beta\sqrt{fg}$, where β is a unitless constant.

(c) As shown in example 5, the relatonship between the image and object locations is $1/f = 1/d_i - 1/d_o$. Since we already know the form of the solution's dependence on f and g, we can arbitrarily set $f = 1$ and $g = 1$ in order to simplify the writing. We then have $d_i = 1/(1 + 1/d_o)$, where $d_o = t^2/2$, and differentiation gives $v_i = t/(t^2/2+1)^2$. To find the maximum velocity, we differentiate again, and set the derivative equal to zero. In other words, we're looking for the time when the acceleration equals zero. The acceleration is $a_i = 1/(t^2/2+1)^2 - 2t^2(t^2/2+1)^3$, which equals zero at $t = \sqrt{2/3}$. Plugging this back into the expression for v_i, we have a maximum velocity of $2^{-7/2}3^{3/2}$. This is the mysterious unitless constant β from part b. Recovering the general solution for $f \neq 1$ and $g \neq 1$, we have a maximum velocity of $2^{-7/2}3^{3/2}\sqrt{fg}$.

22-g17 f/ϵ

22-g18 For simplicity, let's assume that the cylinder has unit radius, so that its focal length is 1/2, and the relationship between the object and image distances is $1/d_i - 1/d_o = 2$. To see that the standard Peaucellier linkage doesn't do the right thing, consider the case when the object point is at infinity. Then the image distance is 1/2, whereas inversion in a circle would give a point at radius zero.

The standard definition of object and image distances is relative to the surface of the mirror, whereas the geometrically relevant quantities in this problem would clearly be measured from the center of the cylinder, $u = d_o + 1$ and $v = 1 - d_i$. The relationship between these two variables is $1/u + 1/v = 2$, which is the equation of a hyperbola with perpendicular asymptotes at $u = 1/2$ and $v = 1/2$.

This gives us strong reason to hope that the a construction based on the Peaucellier can be done, since the Peaucellier linkage calculates the inverse of a number, and the graph of the function $y = 1/x$ is also a hyperbola with perpendicular asymptotes. To make the connection, let $p = u - 1/2$ and $q = v - 1/2$, so that $p = 1/4q$. This shows that we need to make a Peaucellier linkage with $k = 1/2$. Positioning it so that when the rhombus is flattened, points P and P' lie on the surface of the cylinder. The point O should not be located at the center of the cylinder; we need to modify the linkage by adding a seventh rod, with length 1/2, to tie O to the center of the cylinder. We also need to add constraints to keep the axis of the Peaucellier linkage lined up with the seventh rod.

22-j1 Snell's law is

$$n_a \sin\theta_a = n_d \sin\theta_d \qquad ,$$

where a refers to air and d to diamond, and $n_a \approx 1$. Total internal reflection occurs when there is no solution for θ_a, which occurs when the value required for $\sin\theta_a$ is greater than 1. The critical angle is $\theta_d = \sin^{-1}(1/n_d) = 24.4$ deg. It's measured from the normal, since that's how the angles in Snell's law are always measured. It's a minimum, because making θ_d bigger would require $\sin\theta_a$ to be more than 1, which is impossible. In a substance with a smaller index of refraction, $1/n$ would be bigger, and $\sin^{-1}(1/n)$ would also get bigger, since the inverse sine increases as you make its input bigger; the minimum angle for total internal reflection would be bigger, meaning that it would be harder to get total internal reflection.

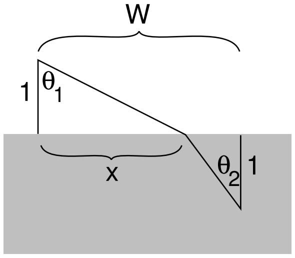
22-j2 Snell's law gives $\sin\theta_1 = (1+\epsilon)\sin\theta_2$. To get the greatest deflection, we want the limit in which θ_1 approaches 90 °. Expanding $\sin\theta_2$ in a Taylor series around 90 °, we have $\sin\theta_2 = 1 - \phi^2/2 + \ldots$, where $\theta_2 = \pi/2 - \phi$. The result is $\phi = \sqrt{2\epsilon}$.

22-j3 We have $n = \sin\phi/\sin\theta$. Doing implicit differentiation, we find $\mathrm{d}n = -\sin\phi(\cos\theta/\sin^2\theta)\,\mathrm{d}\theta$, which can be rewritten as $\mathrm{d}n = -n\cot\theta\,\mathrm{d}\theta$. This can be minimized by making θ as big as possible. To make θ as big as possible, we want ϕ to be as close as possible to 90 degrees, i.e., almost grazing the surface of the tank.

This result makes sense, because we're depending on refraction in order to get a measurement of n. At $\phi = 0$, we get $\theta = 0$, which provides no information at all about the index of refraction — the error bars become infinite. The amount of refraction increases as the angles get bigger.

22-j4 $P = (1/2)(n^2 - 1)$

22-j5 If the principle of least time is going to work at all, it must work between any two points, so without loss of generality, we assume the points to lie at unit distance from the boundary between the two media. We also simplify things by letting $c = 1$ (which can be achieved, for example, by the proper choice of units).



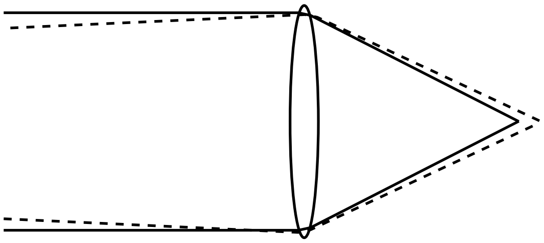
The time taken by the ray along the path shown in the figure is

$$T = n_1\sqrt{1+x^2} + n_2\sqrt{1+(W-x)^2} \qquad .$$

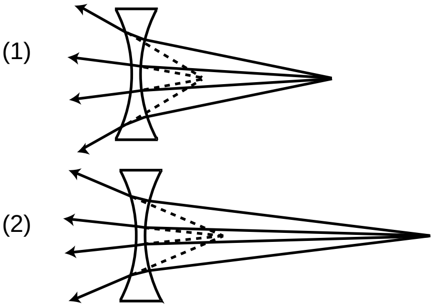
The minimum occurs when the derivative with respect to x equals zero:

$$0 = \frac{\mathrm{d}T}{\mathrm{d}x}$$
$$0 = n_1\left(1+x^2\right)^{-1/2}x - n_2\left[1+(W-x)^2\right]^{-1/2}(W-x)$$
$$0 = n_1\sin\theta_1 - n_2\sin\theta_2$$

22-m1 The rays from a point on a very distant object are parallel, like the solid lines in the figure. The rays from a point on a nearer object are slightly diverging, like the dashed lines. The lens cannot bend the rays from the nearer object strongly enough to make them converge as close to the lens, so the lens-to-film distance has to be greater when the object is nearer.



22-m3 (a) See figure (1).
(b) Decreasing θ_o, as in figure (2), makes a cone of rays that is less strongly diverging, which the lens cannot bend outward as far, so θ_i becomes smaller. Since decreasing θ_o makes θ_i smaller as well, the signs in the equation equation $\pm\theta_o \pm \theta_i = \theta_f$ must be opposite. Since θ_i is greater than θ_o, we must have $-\theta_o + \theta_i = \theta_f$ in order to get a positive θ_f. The equation relating the image and object locations must be $1/f = -1/d_o + 1/d_i$. Solving for d_i and plugging in, we get 30.8 cm.



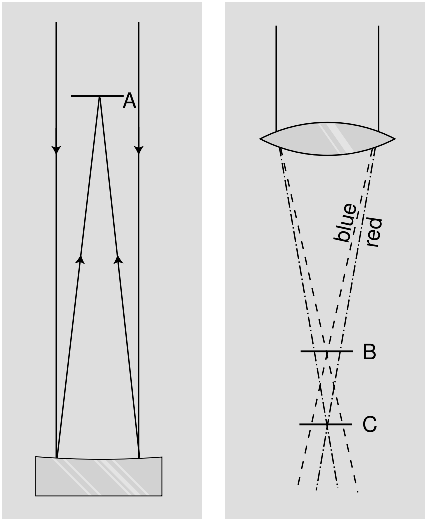
22-m4 The refracted ray that was bent closer to the normal in the plastic when the plastic was in air will be bent farther from the normal in the plastic when the plastic is in water. It will become a diverging lens.

22-m5 The figure shows the simplest possible reflecting and refracting telescopes, with no diagonal mirror or eyepiece.

In both cases, parallel rays enter from the star, which is infinitely far away. Each of these white rays is really a mixture of rays of all the different colors.

With the reflecting telescope, all the colors are reflected at the same angle and form a single image at one place. If we insert a piece of paper or a digital camera chip at A, only one spot is illuminated with the light from the star.

With the refracting telescope, it is never possible to get all the colors in focus at once. Each color has its own image, at a different distance from the lens. If the paper or chip is at B, where the image occurs for a specific wavelength of blue light, we get a blue dot surrounded by a fuzzy halo consisting of other colors such as red. If we move it to C, where the image occurs for a certain wavelength of red, we get a red dot surrounded by the other colors.



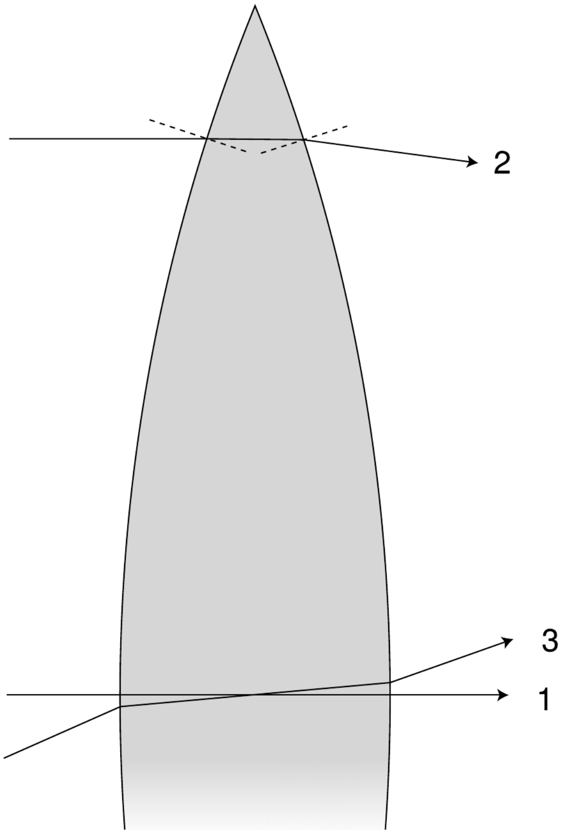
22-m6 Refraction occurs only at the boundary between two substances, which in this case means the surface of the lens. Light doesn't get bent at all inside the lens, so the thickness of the lens isn't really what's important. What matters is the angles of the lens' surfaces at various points.

Ray 1 makes an angle of zero with respect to the normal as it enters the lens, so it doesn't get bent at all, and likewise at the back.

At the edge of the lens, 2, the front and back are not parallel, so a ray that traverses the lens at the edge ends up being bent quite a bit.

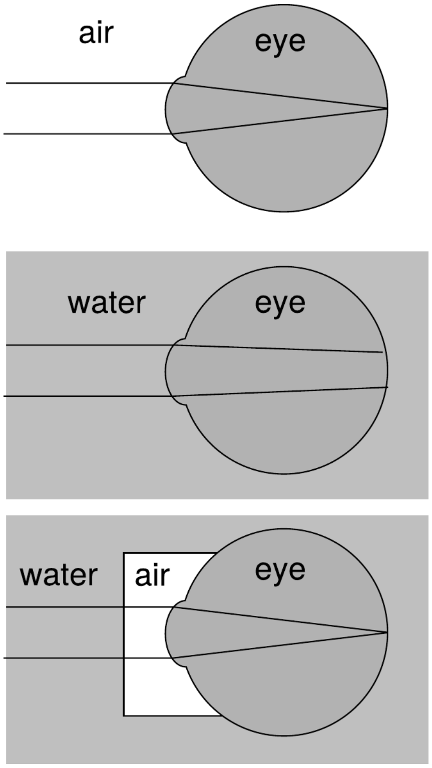
Although I drew both ray 1 and ray 2 coming in along the axis of the lens, it really doesn't matter. For instance, ray 3 bends on the way in, but bends an equal amount on the way out, so it still emerges from the lens moving in the same direction as the direction it originally had.

Summarizing and systematizing these observations, we can say that for a ray that enters the lens at the center, where the surfaces are parallel, the sum of the two deflection angles is zero. Since the total deflection is zero at the center, it must be larger away from the center.

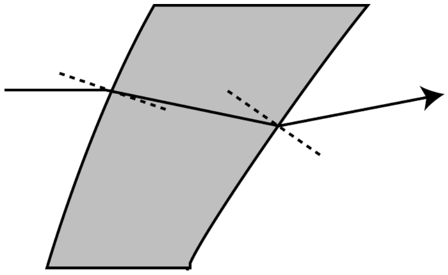


22-m7 Normally, in air, your eyes do most of their focusing at the air-eye boundary. When you swim without goggles, there is almost no difference in speed at the water-eye interface, so light is not strongly refracted there (see figure), and the image is far behind the retina.

Goggles fix this problem for the following reason. The light rays cross a water-air boundary as they enter the goggles, but they're coming in along the normal, so they don't get bent. At the air-eye boundary, they get bent the same amount they normally would when you weren't swimming.



22-m8 (a) See the figure below. The first refraction clearly bends it inward. However, the back surface of the lens is more slanted, so the ray makes a bigger angle with respect to the normal at the back surface. The bending at the back surface is therefore greater than the bending at the front surface, and the ray ends up being bent *outward* more than inward.



(b) Lens 2 must act the same as lens 1. It's diverging. One way of knowing this is time-reversal symmetry: if we flip the original figure over and then reverse the direction of the ray, it's still a valid diagram.

Lens 3 is diverging like lens 1 on top, and diverging like lens 2 on the bottom. It's a diverging lens.

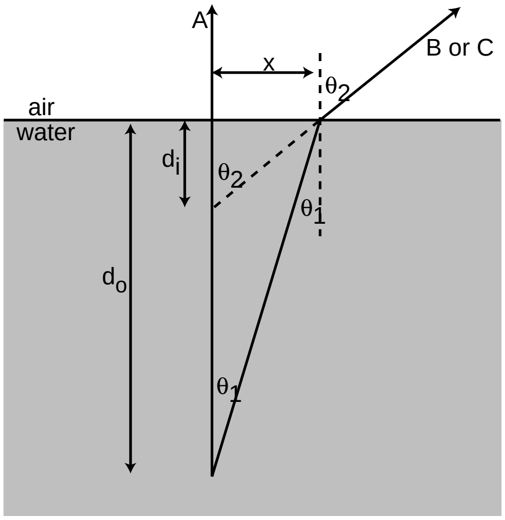
As for lens 4, any close-up diagram we draw of a particular ray passing through it will look exactly like the corresponding close-up diagram for some part of lens 1. Lens 4 behaves the same as lens 1.

22-m9 I tried various circular household objects and found one whose radius of 1.5 cm seemed to make it coincide with the cornea's curvature. Because the drawing is two times bigger than life, the radius of curvature of the cornea is 1.5 cm/2=0.75 cm. The lensmaker's formula is $f = [(n-1)\,|1/r_1 \pm 1/r_2|]^{-1}$, but the air-cornea interface is just a single surface, so we just need to use $f = [(n-1)\,|1/r_1|]^{-1} = |r_1|/(n-1) = 2.5$ cm.

22-m10 (a) Doing trig on the diagram below gives $\tan\theta_1 = x/d_o$, and $\tan\theta_2 = x/d_i$. Eliminating the unknown x and solving for d_i gives $d_i = d_o \tan\theta_1/\tan\theta_2$. Ray B has $\theta_1 = 1.0°$, and applying Snell's law with $n = 1.3$ gives $\theta_2 = 1.3°$, so rays A and B intersect at a depth $d_i = 0.769$ m. (Note that although we only know n to 2 significant figures, the point of the problem is to compare image depths for various rays, so we need to keep more sig figs.) Similarly, rays A and C intersect at $d_i = 0.733$ m.

(b) It might make it a little blurry if you were looking at it from close enough so that widely divergent rays like B and C could both get into your eye. Another effect is usually far more noticeable, however: if you try looking at an object on the bottom of a swimming pool from various angles, it appears to shift around.

(c) For small angles, $\sin\theta = \tan\theta$, so $d_i = d_o \tan\theta_1/\tan\theta_2 = d_o \sin\theta_1/\sin\theta_2 = d_o/n$, which is the same number regardless of θ.



22-m11 Since d_o is much greater than d_i, the lens-film distance d_i is essentially the same as f.

(a) Splitting the triangle inside the camera into two right triangles, straightforward trigonometry gives

$$\theta = 2\tan^{-1}\frac{w}{2f}$$

for the field of view. This comes out to be 39° and 64° for the two lenses. (b) For small angles, the tangent is approximately the same as the angle itself, provided we measure everything in radians. The equation above then simplifies to

$$\theta = \frac{w}{f}$$

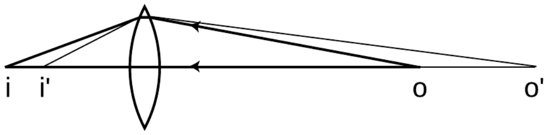
The results for the two lenses are .70 rad = 40°, and 1.25 rad = 72°. This is a decent approximation.

(c) With the 28-mm lens, which is closer to the film, the entire field of view we had with the 50-mm lens is now confined to a small part of the film. Using our small-angle approximation

$\theta = w/f$, the amount of light contained within the same angular width θ is now striking a piece of the film whose linear dimensions are smaller by the ratio 28/50. Area depends on the square of the linear dimensions, so all other things being equal, the film would now be overexposed by a factor of $(50/28)^2 = 3.2$. To compensate, we need to shorten the exposure by a factor of 3.2.

22-m14 One surface is curved outward and one inward. Therefore the minus sign applies in the lensmaker's equation. Since the radii of curvature are equal, the quantity $1/r_1 - 1/r_2$ equals zero, and the resulting focal length is infinite. A big focal length indicates a weak lens. An infinite focal length tells us that the lens is infinitely weak — it doesn't focus or defocus rays at all.

22-m15 (a) The situation being described requires a real image, since the rays need to converge at a point on Becky's neck. See the ray diagram drawn with thick lines, showing object location o and image location i.



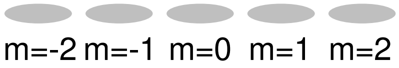
If we move the object farther away, to o′ the cone of rays intercepted by the lens (thin lines) is less strongly diverging, and the lens is able to bring it to a closer focus, at i′. In the diagrams, we see that a smaller θ_o leads to a larger θ_i, so the signs in the equation $\pm\theta_o \pm \theta_i = \theta_f$ must be the same, and therefore both positive, since θ_f is positive by definition. The equation relating the image and object locations must be $1/f = 1/d_o + 1/d_i$.

(b) The case with $d_i = f$ is not possible, because then we need $1/d_o = 0$, i.e., $d_o = \infty$. Although it is possible in principle to have an object so far away that it is practically at infinity, that is not possible in this situation, since Zahra can't take her lens very far away from the fire. By the way, this means that the *focal length* f is not where the *focus* happens — the focus happens at d_i.

For similar reasons, we can't have $d_o = f$.

Since all the variables are positive, we must have $1/d_o$ and $1/d_i$ both less than $1/f$. This implies that $d_o > f$ and $d_i > f$. Of the nine logical possibilities in the table, only this one is actually possible for this real image.

22-q1 The natural variable to use for labeling the fringes is m, as shown in the figure.



The angle θ is also different for the different fringes. The wavelength λ is a property of all the light being used, not a property of a particular fringe. For instance, if it's blue light, it might have a wavelength of 300 nm or so, and that would be the wavelength of all the light in every fringe. The variable d refers to a dimension of the physical object the light is diffracting around, e.g., the distance between two slits, so it's also not different for the different fringes. In particular, the light and dark pattern of fringes is *not* just a shadow of the slits — you could have hundreds of fringes made by only one or two slits.

22-q2 The center-to-center spacing of patterns 2 and 3 are the same. The one with the small center-to-center spacing, pattern 1, corresponds to the grating with the large slit spacing, C. Grating A has more repetition, so its pattern is the one with narrower lines, 3. To summarize, 1=C, 2=B, 3=A.

22-q3 Red light is at the long-wavelength end of the spectrum. Longer wavelengths give larger diffraction angles, so if the slits had been identical, the red pattern would have been the more widely spaced one. We conclude that different slits were used.

22-q4 Let's write x for the distance from the slits to the wall, and y for the spacing between fringes. Three fringes cover about 5.1 cm in the photo, so the spacing is about $y = 1.70$ cm. An example in the text worked out the case where the angles are small, and we want to relate the wavelength to the angular spacing. The result (using radians, not degrees!) is

$$\begin{aligned}\lambda &\approx d\Delta\theta \\ &= d\tan^{-1}(y/x) \\ &\approx d\cdot(y/x) \\ &= 580 \text{ nm}\end{aligned}$$

As a check on the precision of my result, I measured across five fringes and got 8.3 cm, which gives $y = 1.66$ cm, and $\lambda = 570$ nm. Typically, measuring across more fringes would reduce the relative error in the measurement and give a better result. However, the picture only shows five maxima, so measuring across them required me to estimate the position of the minima on the ends without being able to see the next bright fringe. Roughly, then, I'd say the two results of 580 and 570 nm are equally trustworthy, and it might make sense to average them to get about 575 nm as my best value, with error bars of five or ten nm (one or two percent).

22-q5 You don't want the wave properties of light to cause all kinds of funny-looking diffraction effects. You want to see the thing you're looking at in the same way you'd see a big object. Diffraction effects are most pronounced when the wavelength of the light is relatively large compared to the size of the object the light is interacting with, so red would be the worst. Blue light is near the short-wavelength end of the visible spectrum, which would be the best.

22-q6 (a) You can tell it's a single slit because of the double-width central fringe.

(b) Four fringes on the top pattern are about 23.5 mm, while five fringes on the bottom one are about 14.5 mm. The spacings are 5.88 and 2.90 mm, with a ratio of 2.03. A smaller d leads to larger diffraction angles, so the width of the slit used to make the bottom pattern was almost exactly twice as wide as the one used to make the top one.

22-q7 There is not actually a sharp edge to the visible rainbow; there is a range of wavelengths at each end of the spectrum over which your eye's sensitivity drops off. However, as stated in the problem, the range is roughly 400 to 700 nm, which is slightly less than a factor of two. You get a series of rainbows, each broader than the previous one, but with a constant center-to-center angular spacing. The angle of a given color within the rainbow of order m is determined by the equation $m\lambda = d\sin\theta$, or $\theta = \sin^{-1}(m\lambda/d)$. Since the red end is very nearly twice the wavelength of the violet end, the $m = 1$ red has very nearly the same value of θ as the $m = 2$ violet, so the first and second order spectra almost overlap, but not quite. The $m = 2$ and $m = 3$ spectra will definitely overlap, because the quantity $m\lambda/d$ is greater for $m = 2$ red than for $m = 3$ violet.

22-q8 For the size of the diffraction blob, we have:

$$\begin{aligned}\frac{\lambda}{d} &\sim \sin\theta \\ &\approx \theta \\ \theta &\sim \frac{700 \text{ nm}}{10 \text{ m}} \\ &\approx 10^{-7} \text{ radians}\end{aligned}$$

For the actual angular size of the star, the small-angle approximation gives

$$\begin{aligned}\theta &\sim \frac{10^{9} \text{ m}}{10^{17} \text{ m}} \\ &= 10^{-8} \text{ radians}\end{aligned}$$

The diffraction blob is ten times bigger than the actual disk of the star, so we can never make an image of the star itself in this way.

22-q9 (a) The patterns have two structures, a coarse one and a fine one. You can look up in the book which corresponds to w and which to d, or just use the fact that smaller features make bigger diffraction angles. The top and middle patterns have the same coarse spacing, so they have the same w. The fine structure in the top pattern has 7 fringes in 12.5 mm, for a spacing of 1.79 mm, while the middle pattern has 11 fringes in 41.5 mm, giving a spacing of 3.77 mm. The value of d for the middle pattern is therefore $(0.50 \text{ mm})(1.79/3.77) = 0.23$ mm.

(b) This one has about the same d as the top one (it's difficult to measure accurately because each group has only a small number of fringes), but the coarse spacing is different, indicating a different value of w. It has two coarse groupings in 23 mm, i.e., a spacing of 12.5 mm. The coarse groupings in the original pattern were about 23 mm apart, so there is a factor of two between the $w = 0.04$ mm of the top pattern and the $w = 0.08$ mm of the bottom one.

22-q10 (a) The grating's spacing is the number of centimeters per line, which is one over the number of lines per centimeter, $d = (1/2000\ \text{lines/cm}) = 5 \times 10^{-4}$ cm/line. "Lines" are a fake unit that we can get rid of at will. Converting to meters, we have $d = 5 \times 10^{-6}$ m. The angles of the $m = -1$ and $m = 1$ spots are given by $\theta = \sin^{-1}(m\lambda/d) = +8.0$ degrees. The spacing of the spots on the screen is related to this by simple trig: $(2.0\ \text{m})(\tan\theta) = 28$ cm.

(b) Note that the approximation $\sin\theta \approx \theta$ only works if we use units of radians. With it, we get $\theta = m\lambda/d = .139$ radians, which works out to be almost exactly the same as the answer we got without the approximation.

22-q12 The equation, solved for θ, is $\theta = \sin^{-1}(m\lambda/d)$. The sine function only ranges from -1 to $+1$, so the inverse sine is undefined for $|m\lambda/d| > 1$, i.e., $|m| > d/\lambda$. Physically, we only get fringes out to angles of 90 degrees (the inverse sine of 1) on both sides, corresponding to values of m less than d/λ.

22-q13 High frequencies have short wavelengths, which leads to small diffraction angles. This is a good thing, because we want the image to be sharply defined. Ultimately, you can't pick out any details that are smaller than one wavelength. (This only explains why they would use the highest possible frequency for imaging, not why they would use a lower one for heating. The most likely explanation is that it's just more expensive to get a speaker to vibrate really fast.)

23-a1 (a) In units of 10^{-27} kg, the amount of mass lost is

$$1.67495 - 1.67265 - 0.00091 = 0.00139 \qquad ,$$

i.e., 0.00139×10^{-27} kg. This mass has been converted into energy:

$$\begin{aligned} E &= mc^2 \\ &= (0.00139 \times 10^{-27})(3 \times 10^8)^2 \\ &= 1.25 \times 10^{-13}\ \text{joules} \qquad . \end{aligned}$$

(In the meter-kilogram-second system of units, kilograms multiplied velocity squared give units of joules.)

(b) For this process, the total mass changes from 1.67265 to 1.67586 (again in units of 10^{-27} kg). This is an *increase* in mass, which means that we would have to have a source of energy to make the reaction happen. A free proton has no such source of energy; it's all by itself, so there's nothing for it to get energy from.

23-a2 (a)

$$\begin{aligned} p &= m\gamma v \\ &= \frac{mv}{\sqrt{1-v^2}} \\ p^2\left(1-v^2\right) &= m^2v^2 \\ p^2 &= \left(m^2+p^2\right)v^2 \\ v &= \frac{p}{\sqrt{m^2+p^2}} \end{aligned}$$

(b) At low velocities, the second term inside the square root is negligible, and we have

$$\begin{aligned} v &\approx \frac{p}{\sqrt{m^2}} \\ &= \frac{p}{m} \qquad , \end{aligned}$$

which is the nonrelativistic result.

(c) For very large momenta, the m^2 term is negligible compared to the p^2 term, so

$$\begin{aligned} v &= \frac{p}{\sqrt{p^2}} \\ &= \pm 1, \end{aligned}$$

which is the speed of light in natural units.

(d) Converting to SI units, we have

$$v = \frac{p}{\sqrt{m^2+p^2/c^2}}.$$

23-a4 Really this boils down to expanding γ in a Taylor series, and that task can be simplified a lot by letting $x = v^2$, and thinking of γ as a function of x. Then

$$\begin{aligned}\gamma &= (1-x)^{-1/2}\\ \gamma' &= \frac{1}{2}(1-x)^{-3/2}\\ \gamma'' &= \frac{3}{4}(1-x)^{-5/2}\end{aligned}$$

Evaluating these at $x = 0$, we get 1, 1/2, and 3/4, so the Taylor series is

$$\begin{aligned}\gamma &= 1 + \frac{1}{1!}\cdot\frac{1}{2}x + \frac{1}{2!}\cdot\frac{3}{4}x^2 + \ldots\\ &= 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots\\ K &= \frac{1}{2}mv^2 + \frac{3}{8}mv^4 + \ldots \qquad ,\end{aligned}$$

which in ordinary units would be

$$K = \frac{1}{2}mv^2 + \frac{3}{8c^2}mv^4 + \ldots \qquad .$$

The odd terms vanish, because kinetic energy is a scalar that doesn't depend on the direction of motion. The first term is the nonrelativistic expression, as claimed.

23-a5 We have $p = m\gamma v = m(1-v^2)^{-1/2}v$. A brute-force evaluation of the Taylor series for this expression can be done, but it's messy. One way to simplify the calculation is by evaluating the Taylor series for γ, and then simply multiplying by v. Now that we've reduced the problem to evaluating γ as a Taylor series, a second simplification is to let $x = v^2$, and think of γ as a function of x. Then

$$\begin{aligned}\gamma &= (1-x)^{-1/2}\\ \gamma' &= \frac{1}{2}(1-x)^{-3/2}\\ \gamma'' &= \frac{3}{4}(1-x)^{-5/2}\end{aligned}$$

Evaluating these at $x = 0$, we get 1, 1/2, and 3/4, so the Taylor series is

$$\begin{aligned}\gamma &= 1 + \frac{1}{1!}\cdot\frac{1}{2}x + \frac{1}{2!}\cdot\frac{3}{4}x^2 + \ldots\\ &= 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots\end{aligned}$$

Applying this to the expression for momentum, we have

$$\begin{aligned}p &= m\gamma v\\ &= mv\left(1 + \frac{1}{2}v^2 + \ldots\right)\\ &= mv + \frac{1}{2}mv^3 + \ldots \qquad ,\end{aligned}$$

or, in ordinary units,

$$p = mv + \frac{1}{2c^2}mv^3 + \ldots \qquad .$$

The even-order terms vanish, because when v flips its sign, p must flip its sign as well. The first nonvanishing term is the newtonian expression mv.

23-a6 The first derivative of the function $D(v)$ is

$$D' = \frac{1}{2}\left(\frac{1-v}{1+v}\right)^{-1/2}\left[-\frac{1}{1+v} + (1-v)(-1)(1+v)^{-2}\right] \qquad .$$

Evaluating this at $v = 0$ gives $D'(0) = -1$. The first two terms of the Taylor series are

$$\begin{aligned}D(v) &\approx D(0) + D'(0)v\\ &= 1 - v \qquad .\end{aligned}$$

This is identical to the nonrelativistic Doppler shift for the wavelength detected from a source receding at velocity v relative to the medium (if v is expressed as a fraction of the wave's speed). (The nonrelativistic version is actually more complicated, because in principle both the source and the observer could be moving relative to the medium. For Doppler shifts of light, there is no medium, so clearly this can't be relevant. In the nonrelativistic case we get shifts of $1 - v$ and $1/(1 + v)$ for a moving source and a moving observer, but the latter's Taylor series is $1 - v + \ldots$, which still matches the relativistic expression in its first two terms.)

23-a7 (a) In units with $c = 1$, we have

$$\begin{aligned} v &= L/t \\ K &= m(\gamma - 1) \\ \tau &= t/\gamma. \end{aligned}$$

The solution of these equations is

$$K = m\left[\sqrt{1+\left(\frac{L}{\tau}\right)^2} - 1\right],$$

or, reinserting factors of c,

$$K = mc^2\left[\sqrt{1+\left(\frac{L}{c\tau}\right)^2} - 1\right].$$

(b) For the given data, the result is something like 10^{12} megatons of TNT. Your ultrarelativistic friend's body has so much kinetic energy that if he collided with the earth, it would be the end of the world, so I think Congress should pass a law prohibiting him from doing this.

24-a1 [Note: An old version of the problem listed the parent isotope incorrectly as ^{234}Pu. It should be ^{241}Pu.]

Plutonium is atomic number 94. In alpha decay, we're removing two protons and two neutrons, so the mass number will be reduced by four, and the atomic number (number of protons) by 2. The nucleus left over after alpha decay therefore has atomic number 92 (uranium), and mass number 237; it's ^{237}U.

In electron decay, we're converting one neutron to a proton. The mass number is unchanged, but the atomic number is bumped up by one, to 95, americium. The product is ^{241}Am.

24-a2 (a) $+e - e = 0 + 0$

(b) The mass of a neutron is greater than the mass of a proton plus the mass of an electron, so this is a process that increases mass, and a hydrogen atom would therefore have to have a source of energy in order to do it.

24-d1 After two years, the substance will be 75% decayed, not 100%.

24-d2 (a) The activity falls off by a factor of two as you go from $t = 5$ min to $t = 32$ min, so its half-life is about 27 min. It looks like it's ^{214}Pb.

(b) If it's a single isotope with a single well-defined half-life, then the ratio of the activity at 20 min to the activity at 10 min should be the same as the ratio of the activity at 30 min to the activity at 20 min. Only the stuff produced by the second ion speed gives ratios equal to three significant figures, which is the accuracy of the data.

24-d3 (a) To a good approximation, the mass of a nucleus is proportional to the number of protons plus the number of neutrons, which is the mass number. The total change in mass number from 238 to 206 is 32, so if mass is to be conserved, the number of alpha particles, with mass number four, emitted must be 32/4=8.

(b) The charge of the original nucleus was 92 (in units of the fundamental charge), and the nuclei produced have a total charge of $82 + 8 \times 2 = 98$. To conserve charge, 6 electrons must have been produced, so 6 beta decays must have occurred.

(c) The mass of a uranium atom is about 238 times the mass of a neutron or proton, so the number of surviving U atoms is

$$\frac{1230 \times 10^{-6}\ \text{kg}}{238 m_p} = 3.09 \times 10^{21} \quad .$$

(The masses of protons and neutrons are nearly the same, so we don't have to add their masses separately.) Similarly, the number of helium atoms is

$$\frac{2.3 \times 10^{-6}\ \text{kg}}{4 m_p} = 3.44 \times 10^{20} \quad .$$

Each of the decayed U atoms produced 8 helium atoms, so the number of decayed U atoms is $3.44 \times 10^{20}/8 = 4.30 \times 10^{19}$. The probability of decay since the rock was formed equals (number that decayed)/(number that decayed + number that survived) = 0.0137, so the probability of survival was 0.9863, and $0.9863 = 0.5^{t/t_{1/2}}$. Taking natural logarithms of both sides, $\ln 0.9863 = (t/t_{1/2}) \ln 0.5$, yielding $t = 90$ million years.

24-g1 The probabilities are independent, so $(1/6) \times (1/6) = 1/36$.

24-g2 Suppose we record the results of a long string of games played one after another. One method of testing for independence would be to use the data to estimate the probability of winning, $P(W)$, and also break down the list into sequential pairs (1st and 2nd game, 3rd and 4th, etc.) and use these pairs to measure the probability of winning twice in a row, $P(W \text{ and } W)$. The definition of independence then gives $P(W \text{ and } W) = P(W)^2$. We can test whether the left-hand and right-hand sides of this equation are in fact equal, to within the statistical sampling error imposed by the number of games observed.

Another approach is to look for extremely long runs of wins. If independence holds, then the probability of n wins in a row is $P(W)^n$, which falls off exponentially. Any observation of a very large n would therefore prove that the rolls were not statistically independent. For a statistical analysis of data from players who ran up world-record winning streaks, see Ethier and Hoppe, arxiv.org/pdf/0906.1545. The results are described in terms of the number of rolls, which is greater than the number of games.

24-g3 (a) The variable r varies continuously from 0 to b — it's not as though we have rings of finite width. However, suppose we imagine an extremely narrow ring extending from r to $r + \mathrm{d}r$. Since the person is blindfolded, the probability of hitting a given region depends only on its area. This ring has a circumference of $2\pi r$, so its area is $2\pi r\, \mathrm{d}r$. This is proportional to r, so the probability of getting a result between r and $r + \mathrm{d}r$ is proportional to r.

(b)

$$\begin{aligned} 1 &= \int_0^b D(r)\, \mathrm{d}r \\ 1 &= \int_0^b kr\, \mathrm{d}r \\ 1 &= \frac{1}{2}kb^2 \\ k &= \frac{2}{b^2} \\ D(r) &= \frac{2r}{b^2} \end{aligned}$$

(c)

$$\begin{aligned} \bar{r} &= \int_0^b rD(r)\, \mathrm{d}r \\ &= \int_0^b \frac{2r^2}{b^2}\, \mathrm{d}r \\ &= \frac{2}{3}b \end{aligned}$$

(d) The average value of r is greater than $b/2$. This makes sense, because the area between $r = 0$ and $r = b/2$ is only one fourth of the total area.

24-g4 (a) The distribution has to look like the ordinary probability distribution for radioactive decay, but with the infinite tail chopped off, and scaled back up to restore its normalization. Chopping off the tail at one half-life cuts out half the total area, since by definition half the atoms decay after that time. Therefore, the distribution needs to be scaled up by a factor of 2:

$$D(t) = \begin{cases} \frac{2\ln 2}{t_{1/2}} 0.5^{t/t_{1/2}} & 0 \le t \le t_{1/2} \\ 0 & t > t_{1/2} \end{cases},$$

(b)

$$\begin{aligned} \bar{t} &= \int_0^{t_{1/2}} \frac{2\ln 2}{t_{1/2}} 0.5^{t/t_{1/2}}\, t\ \mathrm{d}t \\ &= -2c \int_0^{t_{1/2}} t\, e^{ct}\ \mathrm{d}t \end{aligned}$$

where $c = -\ln 2/t_{1/2}$ as in the examples in section 12.1. Then

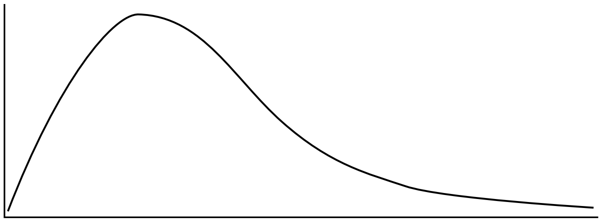
$$\begin{aligned}\bar{t} &= (-2c)\left(\frac{t}{c}e^{ct} - \frac{1}{c^2}e^{ct}\right)\Bigg|_0^{t_{1/2}} \\ &= (-2c)\left\{\left[\frac{t_{1/2}}{2c} - \frac{1}{2c^2}\right] - \left[-\frac{1}{c^2}\right]\right\} \\ &= -\frac{1}{c} - t_{1/2} \\ &= \left(\frac{1}{\ln 2} - 1\right)t_{1/2} \\ &\approx 0.44t_{1/2}\end{aligned}$$

(c) The variable t can only lie between 0 and $t_{1/2}$, so it makes sense that the average is between those two numbers. It also makes sense that the average is less than half of $t_{1/2}$, because the shorter times have higher probabilities. However, since the calculus was fairly complicated, I decided to follow up this plausibility check with a check by numerical integration. Suppressing all units, setting $t_{1/2} = 1$, and dividing the interval from $t = 0$ to $t = 1$ into five equal subintervals of width $\Delta t = 0.2$, I got this:

t	$D(t)$	$D(t) \cdot t$
.1	1.29	0.13
.3	1.13	0.34
.5	.98	0.49
.7	.85	0.60
.9	.74	0.67

The sum of $D(t)\Delta t$ comes out to be 1 to within rounding errors, and the sum of $D(t) \cdot t\Delta t$ is 0.45, which agrees with the result from symbolic calculus to within the accuracy to be expected from the crude approximation to the integral.

24-g5 (a) Without the factor of v in front, it would have the form of the classic bell curve. Since it's only defined for positive v, we chop off the left half of the bell. Then we multiply by v, which makes the curve drop down to zero at the origin. The result looks like this:



(b) We want to integrate D from 0 to ∞. The subsitution $u = cv^2$ gives, $\mathrm{d}u = 2cv\,\mathrm{d}v$, turning the integrand into $(b/2c)e^{-u}\,\mathrm{d}u$. The integral of $e^{-u}\,\mathrm{d}u$ from 0 to ∞ is 1, so the result is $1 = (b/2c) \cdot 1$, or $b = 2c$.

(c) I warned you not to try it by hand! (It requires a trick.) Using computer software, the result is $\int_0^\infty 2cv^2 e^{-cv^2}\,\mathrm{d}v = (1/2)\sqrt{\pi/c}$.

24-j1 The amount of energy lost by the electrons as they leave the surface, E_s, depends on the material. With a material with a high E_s, you wouldn't be sensitive to the red end of the spectrum, which has low-energy photons.

24-j2 Hertz noticed that the cutoff for producing the photoelectric effect in metals was somewhere in the visible-light spectrum. We're assuming that the energy required to ionize DNA is similar to the energy required to rip an electron out of a metal, so again, the cutoff should at a frequency that's of roughly the same order of magnitude as the frequency of visible light. The types of waves are listed in order of increasing frequency, so the only ones that can cause cancer are the ones farther down the list than the cutoff. Ultraviolet and x-rays are certainly capable of doing it. It might be possible to get cancer from visible light — our analysis is too crude to tell exactly where the cutoff is. You are not going to get cancer from power lines, FM radio, cell phones, or microwave ovens.

24-j3 $E = hf$, and frequency corresponds to color, so if each photon was losing 20% of its energy you'd notice a pronounced change in color.

24-j4 In both case, we're talking about 100 joules worth of light. Consulting the chart of the electromagnetiv spectrum, we find that the frequency of visible light is greater than that of radio waves by a factor of about

$(10^{14}\text{ Hz})/(10^8\text{ Hz}) = 10^6$. (The radio spectrum covers a lot of ground, but near the middle of it we have FM radio at about 100 MHz = 10^8 Hz. The visible spectrum only spans about a factor of two in frequency, so that's less of an issue.) Since a photon's energy is given by $E = hf$, that means a visible photon has a million times more energy than a radio photon. There are a million times more photons in the radio signal than in the visible light. If you assumed a different value for the frequency of the radio waves, you might have gotten an answer that differed from this one by a factor of 10 or 100.

24-j5 Visible light has a wavelength on the order of 10^{-6} m, so a typical photon of visible light is about $E = hf = hc/\lambda \approx 2 \times 10^{-19}$ J, so the number of photons emitted per second is about $(100\text{ J/s})/(2 \times 10^{-19}\text{ J/photon}) = 5 \times 10^{20}$ photons/s. The time each photon spends on its voyage is $(3\text{ m})/c \approx 10^{-8}$ s. The number of photons in the beam equals the number of photons emitted during the time required for a photon to get across, $(5\times 10^{20}\text{ photons/s})\times(10^{-8}\text{ s}) = 5\times 10^{12}$ photons. Given the crudeness of the initial assumptions, this is only an order-of-magnitude result, so let's call it 10^{13} photons.

24-j6 (a)

$$\begin{aligned}\text{power} &= (1\text{ mW/m}^2)(\text{cross-sectional area})\\ &= (1\text{ mW/m}^2)(\pi r^2)\\ &\sim 10^{-32}\text{ W}\end{aligned}$$

(rounding to a power of 10).

(b) power $= \Delta E/\Delta t$, so

$$\begin{aligned}\Delta t &= \Delta E/\text{power}\\ &\sim 10^{13}\text{ s}\end{aligned}$$

The classical theory says it should take thousands of years for an electron to soak up enough energy to be ejected, which contradicts experiment, so the classical theory must be wrong.

24-j7 (a) The surface area of the spherical surface is $A = 4\pi r^2$. During a time interval t, the energy from the transmitter travels a distance $w = ct$, and by the time this energy reaches the distance r, it has been spread out over a spherical shell of volume Aw. The energy emitted during this time is Pt, so the energy density is $E/V = PT/(Aw) = P/4\pi cr^2$.

(b) The number of photons n in a volume $V = \lambda^3$ is

$$\begin{aligned}n &= \frac{E}{hf}\\ &= \frac{(E/V)\lambda^3}{hc/\lambda}\\ &= \frac{P\lambda^4}{4\pi hc^2r^2}\end{aligned}$$

(c) Let's take $P = 10$ kW and $r = 10$ km. The result then comes out to be about 10^{21}. This huge number is consistent with the correspondence principle.

24-m1 Momentum is related to kinetic energy by $E = p^2/2m$, so

$$\begin{aligned}\lambda &= h/p\\ &= h/\sqrt{2mE}\\ &= h/\sqrt{2mq\Delta V}\\ &= 1 \times 10^{-11}\text{ m}\end{aligned}$$

24-m2 It's conceivable that the Δx is actually *less* than the diameter of the nucleus, but it can't be *more*, since the proton stays inside the nucleus. The maximum possible value of Δx is therefore roughly the same as the diameter of the nucleus. A maximum value of Δx leads to a minimum value of Δp, since they're inversely related: $\Delta p \gtrapprox h/\Delta x$. Plugging in, we get $\Delta v = \Delta p/m \gtrapprox h/m\Delta x \gtrapprox 3 \times 10^7$ m/s. Note the two inequalities in a row: one because the maximum uncertainty is the diameter of the nucleus, and one because the uncertainty principle is an inequality. It's perfectly possible that they're moving much faster than this; what we've found is only a minimum.

Our result is 10% of the speed of light, so the nonrelativistic equations are good enough for this kind of estimate.

24-m3 The wavelength equals $2L/n$, so we have

$$\begin{aligned} KE &= \frac{p^2}{2m} \\ &= \frac{(h/\lambda)^2}{2m} \\ &= \frac{h^2 n^2}{8mL^2} \end{aligned}$$

24-m4 (a) $\lambda = h/p = h/mv = 10^{-8}$ m $= 10$ nm.
(b) The area of one circuit would be 10^{-8} cm^2, which might be a square 10^{-4} cm on a side, i.e. 1000 nm.
(c) No. Diffraction is only an important effect when the size of the object the wave is diffracting around is comparable to the wavelength.
(d) If the linear size of a circuit was of the same order of magnitude as the wavelength, then we'd definitely have a lot of diffraction. A 10 nm × 10 nm circuit would have an area of 100 nm^2 = 10^{-12} cm^2, so you could fit 10^{12} of them on the chip.
24-m5 (a)

$$\begin{aligned} \frac{d^2\Psi}{dx^2} &= \frac{d^2}{dx^2} e^{-bx^2} \\ &= \frac{d}{dx}\left(-2bxe^{-bx^2}\right) \\ &= -2be^{-bx^2} + (-2bx)(-2bx)e^{-bx^2} \\ &= 2b\left(2bx^2-1\right)e^{-bx^2} \end{aligned}$$

$$\begin{aligned} Ee^{-bx^2} &= -\frac{\hbar^2}{2m}\left(2b\left(2bx^2-1\right)e^{-bx^2}\right) + \frac{1}{2}kx^2e^{-bx^2} \\ E &= -\frac{\hbar^2 b}{m}\left(2bx^2-1\right) + \frac{1}{2}kx^2 \\ 0 &= \left(\frac{1}{2}k - \frac{2\hbar^2 b^2}{m}\right)x^2 + \left(\frac{\hbar^2 b}{m} - E\right) \end{aligned}$$

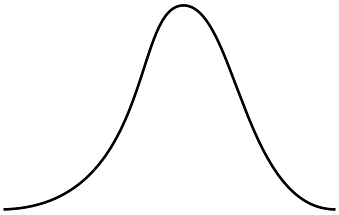
If this equation is going to be true for all values of x, then both coefficients in the polynomial have to be zero,

$$k = \frac{4\hbar^2 b^2}{m}$$

and

$$E = \frac{\hbar^2 b}{m}$$

If the problem is specified by giving the potential U, then we can find the unknowns E and b that complete the specification of the wave that is the solution. The result for b is $\sqrt{km}/2\hbar$. (b) It's a bell curve, like this:



(c) If b was bigger, then we'd get the same value of the wavefunction for smaller values of x. In other words, increasing b causes the bell to get skinnier.
(d) Making k greater makes b greater, which makes it skinnier. That makes sense, because a skinnier wavefunction means that the atom is unlikely to be found very far from the equilibrium point.
24-m6 Say the basketball has a mass of 1 kg, and in general its mass is about 0.1 m from the axis of rotation. If it's spinning at a speed of 1 m/s, then its angular momentum is on the order of 0.1 J·s. This is roughly $10^{32}\hbar$. The fact that this number is so huge is an example of the correspondence principle. Angular momentum is quantized in units of $\hbar$, so the basketball can't continuously speed up or slow down its rotation; it has to do it in quantum leaps from one value of L to another. But because L is so huge when expressed in units of $\hbar$, the process appears smooth and continuous. Therefore quantum mechanics doesn't contradict the results of the centuries of previous experiments in which rotation appeared to be smoothly variable.
24-m7 The energies are inversely proportional to the wavelengths, so in terms of energies the equation would be $E_1 = E_2 + E_3$. This would happen for sets of transitions like G, D, and E

in the previous problem, where the atom can get from one state to another either by a single hop or by two shorter hops.

24-m8 (a) The size of the diagram is comparable to a, but b is so small that you wouldn't be able to see it on that scale.

(b) The stuff inside the exponential is $-r/a$, which is essentially zero for all values of r between 0 and b. In other words, the proton is super-tiny compared to the size of the electron cloud, and the electron's probability density therefore won't change appreciably from one part of the proton to the other. We don't need to do an integral at all. We can just multiply:

$$\begin{aligned}P &= \Psi^2 V\\ &= \left(\pi^{-1/2}a^{-3/2}e^{-r/a}\right)^2\left(\frac{4\pi}{3}b^3\right)\\ &\approx \left(\pi^{-1/2}a^{-3/2}\right)^2\left(\frac{4\pi}{3}b^3\right)\\ &= \frac{4b^3}{3a^3}\end{aligned}$$

(c) 1×10^{-15}

(d) The probability density never dies down to zero, but it gets very very close to zero. The atom is fuzzy. It doesn't have a sharply defined boundary. There is theoretically a finite probability for the electron to be a centimeter away from the proton, but it's extremely small.

24-m9 The requirement of normalization is

$$\int \Psi^2\,\mathrm{d}v = 1 \qquad ,$$

where $\mathrm{d}v$ is an infinitesimal volume element, and the integral is over all space. The method used for carrying out this integral is similar to the one used in problem 13-8. The problem has spherical symmetry, so we can break space down into a series of concentric spherical shells, each one stretching from r to $r+\mathrm{d}r$. The volume of such a shell equals its area multiplied by its thickness, i.e., $\mathrm{d}v = 4\pi r^2\,\mathrm{d}r$.

$$\int_0^\infty \Psi^2\cdot 4\pi r^2\,\mathrm{d}r = 4a^{-3}\int_0^\infty e^{-2r/a}r^2\,\mathrm{d}r \qquad .$$

The change of variable $u = 2r/a$ transforms this into

$$\frac{1}{2}\int_0^\infty e^{-u}u^2\,\mathrm{d}u \qquad .$$

The definite integral can be done by integration by parts, but I just used computer software (the open-source program Maxima) to find out that it equals 2. The result of the entire calculation is 1, as required by normalization.

24-m10 The result of problem 13-21, for a one-dimensional box of length L, was $K = \frac{h^2n^2}{8mL^2}$. In three three dimensions, we have some number n_x of half-wavelengths that fit in the x direction, along length a, and similarly for b and c. The three contributions to the energies add, since $K \propto p^2 = p_x^2 + p_y^2 + p_z^2$ is additive. The result is

$$K = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}+\frac{n_z^2}{c^2}\right)$$

24-m11 All of the constants cancel out, so we only need the n-dependent part of the expressions. The result is

$$\frac{\lambda_E}{\lambda_F} = \frac{E_F}{E_E} = \frac{2^{-2}-5^{-2}}{2^{-2}-4^{-2}} = \frac{28}{25}$$

24-m12 (a) The uncertainty in position is on the order of magnitude of the diameter of the nucleus, so the uncertainty principle gives $\Delta p \sim h/\Delta x \sim 10^{-18}$ kg·m/s. (b) It's not immediately obvious here if this momentum is low enough so that we could use the nonrelativistic relation $K = p^2/2m$ or if we would need to use relativity. To check, we can compare p to m in natural units, which is the same as comparing p to mc in SI units. The result is that $p \gg mc$, so the motion is not nonrelativistic. In fact it's ultra-relativistic, so the relation $m^2 = E^2 - p^2$ simplifies to $0 \approx E^2 - p^2$, or for the mass-energy $E \approx pc \sim 10^{-10}$ J. In the ultrarelativistic approximation, this is the same as the kinetic energy.

(c) $m \sim E/c^2 \sim 10^{-27}$ kg, which is about the same order of magnitude as the mass of the neutron. This makes it seem a little unlikely that you could have an electron inside the neutron.

24-m13 The probability distribution is the *square* of the wavefunction. Normalization requires that integrating the probability distribution give 1:

$$\begin{aligned} 1 &= \int_0^L \Psi^2\,\mathrm{d}x \\ &= A^2 \int_0^L \sin^2(2\pi x/L)\,\mathrm{d}x \\ &= A^2 \cdot (L/2) \qquad , \end{aligned}$$

where the final line can be performed without any techniques of integration simply because we know that the average value of $\sin^2$ over a full cycle is $1/2$. The result is $A = \sqrt{2/L}$.

24-m14 If this is to be a solution to the Schrodinger equation with a constant potential, then $\nabla^2\Psi$ has to be equal to a constant multiplied by Ψ. Calculating the partial derivatives, we have:

$$\begin{aligned} \frac{\partial^2\Psi}{\partial x^2} &= -a^2 \sin axe^{by} \\ \frac{\partial^2\Psi}{\partial y^2} &= b^2 \sin axe^{by} \\ \nabla^2\Psi &= (b^2 - a^2)\Psi \end{aligned}$$

This does equal a constant multiplied by Ψ. An easy way to tell whether this is a classically allowed region or a classically forbidden one is to test whether the kinetic energy is positive. The kinetic energy is equal to $-\nabla^2\Psi$ multiplied by positive constants. For $|b| < |a|$, the kinetic energy is positive, and the solution represents a classically allowed region.

Photo credits

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